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Full Length Research Paper

Some affine connexions in a generalized structure manifold

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In this paper we have studied some affine connexions in a generalised structure manifold. Certain theorems are have also been proved, which are of great geometrical importance.

Key words: C^∞ -manifold, generalised structure, generalised metric structure, F- structure, π - structure, AMS Subject Classification Number: 53.

INTRODUCTION

We consider a differentiable manifold V_n of differentiability class C^∞ and of dimension n. Let there exist in V_n a tensor field F of the type (1, 1), s linearly independent vector fields U_i, i = 1, 2… s and s linearly independent 1-forms u^i such that for any arbitrary vector field X, we have

$$\overline{\overline{X}} = b^2 X + cu^i(X)U_i, \tag{1.1}$$

$$\overline{U_i} = p_i^j U_j \tag{1.2}$$

Where

F(X) $\overset{def}{=}$ $\overline{X}$ and b^2 , c are constants

Then the structure {F, u^i , U_i , p_i^j ; i ,j=1,2,……., s} will be known as generalised structure and V_n will be known as generalised structure manifold of order s where s < n.

Agreement 1.1

All the equations which follow hold for arbitrary vector fields X, Y, Z, …….,etc.

Now replacing X by $\overline{X}$ in (1.1), we get

$$\overline{\overline{\overline{X}}} = b^2 \overline{X} + cu^i(\overline{X})U_i \tag{1.3}$$

Operating F in (1.1), we get

$$\overline{\overline{\overline{X}}} = b^2 \overline{X} + cu^i(X)\overline{U_i}$$

Using (1.2) in above, we get

$$\overline{\overline{\overline{X}}} = b^2 \overline{X} + cu^i(X)\, p_i^j U_j \tag{1.4}$$

From (1.3) and (1.4), we have

$$u^i(\overline{X}) = u^j(X)\, p_j^i \tag{1.5}$$

Further, operating F in (1.2) and using (1.1) and (1.2), we get

$${}^{(2)}p\,{}_i^j = b^2 \delta_i^j + cu^j(U_i) \tag{1.6}$$

Where

$${}^{(r)}p\,{}_j^i = {}^{(r-1)}p\,{}_k^i\, p_j^k$$

On generalised structure manifold V_n, let us introduce a metric tensor g such that $'F$ defined by

$'F(X,Y) \overset{def}{=} g(\overline{X},Y)$ is skew-symmetric, then V_n is

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called generalised metric structure manifold.

We have on a generalised metric structure manifold $g(\overline{X},Y)+g(X,\overline{Y})=0$. Replacing Y by $\overline{Y}$ in above equation and using (1.1), we obtain

$$g(\overline{X},\overline{Y})+b^2 g(X,Y)+cu^i(X)u^i(Y)=0 \qquad (1.7)$$

Where

$$u^i(X)=g(U_{i,}X) \qquad (1.8)$$

Then V_n satisfying (1.7), (1.8) is called generalised metric structure manifold (Mishra, 1984).

Agreement 1.2: The generalised metric structure manifold will always be denoted by V_n.

Definitions: (Boothby, 1975; Kobayasi and Nomizu, 1996)

Almost tangent metric manifold: A differentiable manifold M_n on which there exists a tensor field F of the type (1, 1) such that

$$F^2 = 0 \qquad (1.9)$$

is called an almost tangent manifold and {F} is called an almost tangent structure on M_n.

On almost tangent manifold, let us introduce a metric g such $'F$ defined by $'F(X,Y) \overset{def}{=} g(\overline{X},Y)$ is alternating. Then M_n is called an almost tangent metric manifold and structure {F, g} is called an almost tangent metric structure.

Almost Hermite manifold: A differentiable manifold M_n on which there exists a tensor field F of the type (1, 1) such that

$$F^2 = -I_n \qquad (1.10)$$

is called an almost complex manifold and {F} is called an almost complex structure.

An almost complex manifold endowed with an almost complex structure and a metric g such that

$$g(\overline{X},\overline{Y}) = g(X,Y) \qquad (1.11)$$

is called an almost Hermite manifold and structure {F, g} is called an almost Hermite structure.

Metric π-structure manifold: (Mishra and Singh, 1975; Duggal, 1969). A differentiable manifold M_n on which there exists a tensor field F of the type (1, 1) such that

$$F^2 = -\lambda^2 I_n \qquad (1.12)$$

where λ is a non zero complex constant. Then {F} is called a π- structure or G-F structure and M_n is called π- structure manifold or G-F structure manifold.

On almost tangent manifold, let us introduce a metric g such that $'F$ defined by $'F(X,Y) \overset{def}{=} g(\overline{X},Y)$ is alternating. Then {F, g} is called metric π- structure or H-structure and M_n is called metric π- structure manifold or H- structure manifold.

F-Structure Manifold: (Yano, 1963). Let M_n be an n dimensional differentiable manifold of class C^∞ and let there be a tensor field of the type (1, 1) and rank r (1≤r≤n) everywhere such that

$$F^3 + F = 0 \qquad (1.13)$$

Then {F} is called an F-structure and M_n is called F-structure manifold.

Almost Grayan manifold: (Sasaki, 1960): If on an differentiable manifold M_n (n = 2m+1) of differentiability class C^{r+1}, there exist a tensor field F of the type (1, 1), a 1- form u and a vector field U, satisfying

$$F^2 = -I_n + u \otimes U \qquad (1.14)$$

and

$$\overline{U} = 0 \qquad (1.15)$$

Then M_n is called an almost contact manifold and the structure {F, U, u} is said to give an almost contact structure to M_n.

On an almost contact manifold, let us introduce a metric g such that $'F$ defined by $'F(X,Y) \overset{def}{=} g(\overline{X},Y)$ is skew symmetric. Then M_n is called an almost Grayan manifold and the structure {F, g, U, u} is called an almost Grayan structure. In this manifold it can be easily calculated

$$g(\overline{X},\overline{Y}) = g(X,Y) - u\ (X)u\ (Y) = 0 \qquad (1.16)$$

Torsion tensor A vector valued, skew–symmetric, bilinear function S defined by

$$S(X,Y) \overset{def}{=} D_X Y - D_Y X - [X,Y] \tag{1.17}$$

is called torsion tensor of a connexion D in a C^∞ manifold V_n.

For the symmetric or torsion free connexion D, the torsion tensor vanishes.

Curvature tensor: The tensor K of the type (1, 3) defined by

$$K(X,Y,Z) \overset{def}{=} D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \tag{1.18}$$

is called the curvature tensor of the connexion D.

Remark 1.1

It may be noted that V_n gives an almost tangent metric manifold ,an almost Hermite manifold, metric π-structure manifold, F- structure manifold, an almost Grayan manifold and {F, g , u^1, u^2, U_1, U_2} structure manifold according as (b^2 = 0,c = 0) ; (b^2 = -1, c = 0) ; ($b^2 = \lambda^r$, c = 0) ;(b^2 = -1, p_i^j = 0) ;(b^2 = -1, c = 1, i , j = 1, p_1^1 = 0) and (b^2 = -1, c = 1, $p_i^j + p_j^i$ = 0 ; i , j = 1,2) respectively.

Affine Connexion D: Let us consider in V_n an affine connexion D satisfying (Duggal, 1971; Mishra, 1984)

$$(D_X F)Y = 0 \tag{2.1)a}$$

and we call it as F-connexion.

(2.1)a is equivalent to

$$D_X \overline{Y} = \overline{D_X Y} \tag{2.1)b}$$

Replacing Y by $\overline{Y}$ and using (1.1), (2.1)a in above, we get

$$c[u^i(Y) D_X U_i + (D_X u^i)(Y) U_i] = 0 \tag{2.1)c}$$

Theorem (2.1): In V_n, we have

$$cu^i(Y) u^j (D_X U_i) = -(b^2 \delta_i^j - {}^{(2)}p_i^j)(D_X u^i)(Y) \tag{2.2}$$

$$cu^j (D_X U_i) U_j = -(b^2 \delta_i^j - {}^{(2)}p_i^j)(D_X U_j) \tag{2.3}$$

Proof

Operating u^j in (2.1)c and using (1.6), we get (2.2). Putting U_i for Y in (2.1)c and using (1.6), we obtain (2.3).

Theorem 2.2

In V_n, we have

$$S(\overline{X},\overline{Y}) + b^2 S(X,Y) + cu^i(S(X,Y))U_i - \overline{S(\overline{X},Y)} - \overline{S(X,\overline{Y})}$$
$$= -[\overline{X},\overline{Y}] - b^2[X,Y] - cu^i([X,Y])U_i + \overline{[\overline{X},Y]} + \overline{[X,\overline{Y}]} \tag{2.5}$$

Proof

From (2.1)b, we get

$$D_{\overline{X}} \overline{Y} = \overline{D_{\overline{X}} Y}, \quad D_{\overline{Y}} \overline{X} = \overline{D_{\overline{Y}} X}, \quad \overline{D_X \overline{Y}} = \overline{\overline{D_X Y}},$$
$$\overline{D_Y \overline{X}} = \overline{\overline{D_Y X}} \tag{2.6}$$

Now in view of (1.1), we have

$$S(\overline{X},\overline{Y}) + b^2 S(X,Y) + cu^i(S(X,Y))U_i - \overline{S(\overline{X},Y)} - \overline{S(X,\overline{Y})}$$
$$= S(\overline{X},\overline{Y}) + \overline{\overline{S(X,Y)}} - \overline{S(\overline{X},Y)} - \overline{S(X,\overline{Y})}$$

Using (1.17) and (2.6) in right hand side of above, we get (2.5). Now, we consider in V_n a scalar valued bilinear function μ, vector valued linear function υ and a 1-form σ given by,

$$\mu(X,Y) \overset{def}{=} (D_Y u^i)(\overline{X}) - (D_X u^i)(\overline{Y}) + (D_{\overline{Y}} u^i)(X) - (D_{\overline{X}} u^i)(Y) \tag{2.7}$$

$$\upsilon(X) \overset{def}{=} (D_{U_i} F)(X) - (D_X F)(U_i) - D_{\overline{X}} U_i \tag{2.8}$$

and

$$\sigma(X) \overset{def}{=} (D_X u^j)(U_i) - (D_{U_i} u^j)(X) \tag{2.9}$$

i , j=1, 2,……., s.

Theorem (2.3)

In V_n, we have

$$(b^2\delta_i^j - {}^{(2)}p^j_{\ i})\mu(X,Y) = c[u^i(X)u^j(D_{\overline{Y}}U_i) - u^i(Y)u^j(D_{\overline{X}}U_i) - u^i(\overline{X})(D_Y u^j)U_i + u^i(\overline{Y})(D_X u^j)U_i] \quad (2.10)a$$

$$(b^2\delta_i^j - {}^{(2)}p^j_{\ i})\mu(X,Y) = -c[u^i(X)u^j(\upsilon(Y)) - u^i(Y)u^j(\upsilon(X)) + u^i(\overline{X})(D_Y u^j)U_i - u^i(\overline{Y})(D_X u^j)U_i] \quad (2.10)b$$

and

$$(b^2\delta_i^j - {}^{(2)}p^j_{\ i})\mu(X,Y) = -c[u^i(X)\{\sigma(\overline{X}) + (D_{U_i}u^j)(\overline{X})\} - u^i(Y)\{\sigma(\overline{Y}) + (D_{U_i}u^j)(\overline{Y})\} - u^i(\overline{X})(D_Y u^j)U_i + u^i(\overline{Y})(D_X u^j)U_i] \quad (2.10)c$$

Proof

Replacing Y by $\overline{Y}$ in (2.2), we get

$$cu^i(\overline{Y})u^j(D_X U_i) = -(b^2\delta_i^j - {}^{(2)}p^j_{\ i})(D_X u^i)(\overline{Y}) \quad (2.11)$$

Replacing X by $\overline{X}$ in (2.2), we get

$$cu^i(Y)u^j(D_{\overline{X}}U_i) = -(b^2\delta_i^j - {}^{(2)}p^j_{\ i})(D_{\overline{X}}u^i)(Y) \quad (2.12)$$

Further by using (2.11), (2.12) in (2.7), we get (2.10)a. Using (2.1)a in (2.8), we get

$$\upsilon(X) = -(D_{\overline{X}}U_i) \quad (2.13)$$

Using (2.13) in (2.10)a, we get (2.10)b. Replacing X by $\overline{X}$ in (2.9), we get

$$-u^j(D_{\overline{X}}U_i) = \sigma(\overline{X}) + (D_{U_i}u^j)(\overline{X}) \quad (2.14)$$

Using (2.14) in (2.10)a, we get(2.10)c.

Theorem 2.4

In V_n, we have

$$\overline{K(X,Y,\overline{Z})} = b^2K(X,Y,Z) + cu^i(K(X,Y,Z))U_i \quad (2.16)a$$

$$p_i^j u^i(K(X,Y,\overline{Z})) = b^2u^j(K(X,Y,Z)) + ({}^{(2)}p^j_{\ i} - b^2\delta_i^j)u^i(K(X,Y,Z)) \quad (2.16)b$$

and

$$b^2[K(\overline{X},\overline{Y},Z) + K(\overline{Y},\overline{Z},X) + K(\overline{Z},\overline{X},Y)] = -c[u^i(Z)K(\overline{X},\overline{Y},U_i) + u^i(X)K(\overline{Y},\overline{Z},U_i) + u^i(Y)K(\overline{Z},\overline{X},U_i)] \quad (2.16)c$$

Proof

Replacing Z by $\overline{Z}$ in (1.18) and using (2.1)b, we get

$$K(X,Y,\overline{Z}) = \overline{K(X,Y,Z)} \quad (2.17)$$

Operating F in (2.17) and using (1.1), we obtain (2.16)a. Operating u^j on both sides of (2.16)a and using (1.5) and (1.6), we get(2.16)b. Bianchi's first identity of symmetric connexion D is given by

$$K(X,Y,Z) + K(Y,Z,X) + K(Z,X,Y) = 0 \quad (2.18)$$

Operating F in (2.18), we get

$$\overline{K(X,Y,Z)} + \overline{K(Y,Z,X)} + \overline{K(Z,X,Y)} = 0 \quad (2.19)$$

Using (2.17) in (2.19), we get

$$K(X,Y,\overline{Z}) + K(Y,Z,\overline{X}) + K(Z,X,\overline{Y}) = 0 \quad (2.20)$$

Replacing X by $\overline{X}$, Y by $\overline{Y}$ & Z by $\overline{Z}$ in (2.20) and using (1.1), we get (2.16)c.

Affine connexion $\tilde{D}$: Let us consider in V_n an affine connexion $\tilde{D}$ satisfying

$$u^i(Y)(\tilde{D}_X U_i) + (\tilde{D}_X u^i)(Y)U_i = 0 \quad (3.1)$$

Theorem 3.1

In V_n, we have

$$u^i(Y)[b^2(\tilde{D}_X U_i) + cu^j(\tilde{D}_X U_i)U_j] + (\tilde{D}_X u^i)(Y)p_i^j p_j^k U_k = 0 \quad (3.2)a$$

$$({}^{(2)}p^j_{\ i} - b^2\delta_i^j) \text{ div } U_j = cu^j(\tilde{D}_{U_j}U_i) \quad (3.2)b$$

Where

$$\text{div } X \overset{def}{=} (C_1^1\nabla X) \quad (3.3) \quad \text{and}$$

$$(\nabla X)Y \overset{def}{=} (\tilde{D}_Y X) \quad (3.4)$$

Proof

Operating F^2 in (3.1) and using (1.1) and (1.2), we get (3.2)a. Now contracting (3.1) with respect to X and using (3.3) and (3.4), we get

$$u^i(Y) div U_i + (\tilde{D}_{U_i} u^i)(Y) = 0 \quad (3.5)$$

Replacing i by j, then Y by U_i in (3.3) and using (1.6), we get (3.2)b.

Theorem 3.2

In V_n, we have

$$c u^i(Y) u^j (\tilde{D}_X U_i) + ({}^{(2)}p_i^j - b^2 \delta_i^j)(\tilde{D}_X u^i)(Y) = 0 \quad (3.6)a$$

$$({}^{(2)}p_i^j - b^2 \delta_i^j)(\tilde{D}_X u^i)(Y) u^j (\tilde{D}_Z U_i) = c u^i(Y) u^j (\tilde{D}_Z U_j)(\tilde{D}_X u^j)(U_j) \quad (3.6)b$$

Proof

By operating u^j on (3.1) and using (1.6), we obtain (3.6)a.

Multiplying $(3.2)_c$ with $u^j(\tilde{D_Z} U_j)$, we get (3.2)d.

Affine connexion $\overset{o}{D}$

Let us consider in V_n an affine connexion $\overset{o}{D}$ satisfying

$$u^i(Y)(\overset{o}{D}_X U_i) + (\overset{o}{D}_X u^i)(Y) U_i = 0 \quad (4.1)a$$

And

$$(\overset{o}{D}_X F)(Y) + (\overset{o}{D}_Y F)(X) = 0 \quad (4.1)b$$

It may be noted that all the results of the section above hold for $\overset{o}{D}$. In addition we have the following results:

Theorem 4.1

In V_n, we have

$$\overline{\overset{o}{D}_X \overline{Y}} + \overline{\overset{o}{D}_Y \overline{X}} - b^2(\overset{o}{D}_X Y + \overset{o}{D}_Y X) = c[u^i(\overset{o}{D}_X Y) + u^i(\overset{o}{D}_Y X)] U_i \quad (4.2)a$$

$$\overset{o}{D}_{\overline{Y}} \overline{X} - b^2(\overset{o}{D}_Y X) = \overline{\overset{o}{D}_{\overline{Y}} X} - \overline{\overset{o}{D}_Y \overline{X}} + c u^i(\overset{o}{D}_Y X) U_i \quad (4.2)b$$

And

$$\overset{o}{D}_{\overline{Y}} \overline{X} + b^2(\overset{o}{D}_Y \overline{X} - \overline{\overset{o}{D}_Y X} - \overset{o}{D}_{\overline{Y}} X) = c[\{u^i(\overset{o}{D}_{\overline{Y}} X) - u^i(\overset{o}{D}_Y \overline{X})\} U_i + u^i(\overset{o}{D}_Y X)\overline{U_i}] \quad (4.2)c$$

Proof

The equation (4.1)b is equivalent to

$$\overset{o}{D}_X \overline{Y} + \overset{o}{D}_Y \overline{X} = \overline{\overset{o}{D}_X Y} + \overline{\overset{o}{D}_Y X} \quad (4.3)$$

Operating F in (4.3) and using (1.1), we get (4.2)a. Replacing Y by $\overline{Y}$ in (4.3) and using (1.1), (4.3), we get (4.2)b. Further, Operating F (4.2)b and using (1.1), we get (4.2)c.

Affine connexion $\overset{*}{D}$:

Let us consider in V_n an affine connexion $\overset{*}{D}$ satisfying

$$u^i(Y)(\overset{*}{D}_X U_i) + (\overset{*}{D}_X u^i)(Y) U_i = 0 \quad (5.1)a$$

And

$$(\overset{*}{D}_X F)(Y) + (\overset{*}{D}_{\overline{X}} F)(\overline{Y}) = 0 \quad (5.1)b$$

It may be noted that all the results of the section three hold for $\overset{*}{D}$. In addition we have the following results:

Theorem 5.1

In V_n, we have

$$\overline{\overset{*}{D}_X Y} + \overline{\overset{*}{D}_{\overline{X}} \overline{Y}} - \overset{*}{D}_X \overline{Y} = b^2(\overset{*}{D}_{\overline{X}} Y) + c u^i(\overset{*}{D}_{\overline{X}} Y) U_i \quad (5.2)a$$

$$\overline{\overset{*}{D}_{U_j} \overline{Y}} + (\overset{*}{D}_{U_j} F)\overline{Y} = b^2(\overset{*}{D}_{U_j} Y) + c u^i(\overset{*}{D}_{U_j} Y) U_i \quad (5.2)b$$

Proof

(5.1)b is equivalent to

$$\overline{\overset{*}{D}_X Y} + \overline{\overset{*}{D}_{\overline{X}} \overline{Y}} = \overset{*}{D}_X \overline{Y} + \overset{*}{D}_{\overline{X}} \overline{\overline{Y}} \tag{5.3}$$

Using (1.1) in (5.3), we get (5.2)a. Replacing X by U_i in (5.3), we get

$$(\overset{*}{D}_{U_i} \overline{Y} - \overline{\overset{*}{D}_{U_i} Y}) + p_i^j[\overset{*}{D}_{U_j}(b^2Y + cu^i(Y)U_i] = p_i^j \overline{(\overset{*}{D}_{U_j}\overline{Y})} \tag{5.4}$$

Replacing X by U_i in (5.2)b, we get

$$(\overset{*}{D}_{U_i} \overline{Y} - \overline{\overset{*}{D}_{U_i} Y}) = -p_i^j(\overset{*}{D}_{U_i} F)\overline{Y} \tag{5.5}$$

From (5.4) and (5.5), we get

$$-p_i^j(\overset{*}{D}_{U_i}F)\overline{Y} + p_i^j[\overset{*}{D}_{U_j}(b^2Y+cu^i(Y)U_i] = p_i^j\overline{(\overset{*}{D}_{U_j}\overline{Y})} \tag{5.6}$$

Using (5.1)a in (5.6), we get (5.1)b.

Theorem 5.2

In V_n, we have

$$\overset{*}{D}_{\overline{X}}\overline{Y} - b^2\overline{(\overset{*}{D}_X\overline{Y})} + b^4(\overset{*}{D}_X Y) = \overline{\overset{*}{D}_{\overline{X}}Y} - cb^2u^i(\overset{*}{D}_X Y)U_i + cu^i(X)[b^2\{(\overset{*}{D}_{U_i}Y)+u^j(\overset{*}{D}_{U_i}Y)U_j\} + \overline{(\overset{*}{D}_{U_i}\overline{Y})}] \tag{5.7}$$

Proof

Replacing X by $\overline{X}$ in (5.3) and using (1.1), (5.1), we get (5.7).

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