African Journal of Mathematics and Computer Science Research Vol. 3(4), pp. 54-59, April 2010
Available online at http://www.academicjournals.org/AJMCSR ISSN 2006-9731

Full Length Research Paper

Some structural compatibilities of pre A*-algebra

A. Satyanarayana[1]*, J. Venkateswara Rao[2], K. Srinivasa Rao[3] and U. Surya Kumar[1]

[1]Department of Mathematics, ANR College, Gudivada, Krishna District, Andhra Pradesh, India.
[2]Mekelle University, Mekelle, Ethiopia.
[3]Adams Engineering College, Paloncha, Khammam District, Andhra Pradesh, India.

Accepted 18 February, 2009

A pre A*-algebra is the algebraic form of the 3-valued logic. In this paper, we define a binary operation $\oplus$ on pre A*-algebra and show that $< A, \oplus >$ is a semilattice. We also prove some results on the partial ordering $\leq_{\oplus}$ which is induced from the semilattice $< A, \oplus >$. We derive necessary and sufficient conditions for pre A*-algebra (A,∧,∨, (-)˜) to become a Boolean algebra in terms of this partial ordering and binary operation and also find the necessary conditions for (A , $\leq_{\oplus}$) as a lattice.

Key words: Pre A*- algebra, poset, semilattice, centre, Boolean algebra.

INTRODUCTION

In 1948 the study of lattice theory was first introduced by Birkhoff (1948). In a draft paper by Manes (1989), the Equational Theory of Disjoint Alternatives, E. G. Maines introduced the concept of ADA (Algebra of disjoint alternatives) $(A, \wedge, \vee, (-)', (-)_{\pi}, 0, 1, 2)$ which however differs from the definition of the ADA of his later paper (Manes, 1993); ADAs and the equational theory of if-then-else in 1993. While the ADA of the earlier draft seems to be based on extending the if-then-else concept more on the basis of Boolean algebra and the later concept was based on C-algebra $(A, \wedge, \vee, ')$ introduced by Fernando and Craig (1994). Rao (1994) was the first to introduce the concept of A*-Algebra $(A, \wedge, \vee, *, (\tilde{-}), (-)_{\pi}, 0, 1, 2)$ and also studied its equivalence with ADA, C-algebra, ADA's connection with 3-Ring, stone type representation but also introduced the concept of A*-clone, the If-Then-Else structure over A*-algebra and Ideal of A*-algebra. Venkateswara (2000) introduced the concept of pre A*-algebra $(A, \wedge, \vee, (\tilde{-}))$ analogous to C-algebra as a reduct of A*- algebra. Venkateswara and Srinivasa (2009) defined a partial ordering on a pre A*-algebra A and the properties of A as a poset are studied.

PRELIMINARIES

Definition 1

Boolean algebra is algebra $(B, \vee, \wedge, (-)', 0, 1)$ with two binary operations, one unary operation (called complementation), and two nullary operations which satisfies:

(i) $(B, \vee, \wedge)$ is a distributive lattice
(ii) $x \wedge 0 = 0,\ x \vee 1 = 1$
(iii) $x \wedge x' = 0,\ x \vee x' = 1$

We can prove that $x'' = x,\ (x \vee y)' = x' \wedge y',\ (x \wedge y)' = x' \vee y'$ for all $x, y \in B$

An algebra (A, ∧, ∨, (-)˜) satisfying

(a) $x^{\sim\sim} = x,\ \forall x \in A$,
(b) $x \wedge x = x,\ \forall x \in A$,
(c) $x \wedge y = y \wedge x,\ \forall x, y \in A$,
(d) $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim},\ \forall x, y \in A$,
(e) $x \wedge (y \wedge z) = (x \wedge y) \wedge z,\ \forall x, y, z \in A$
(f) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),\ \forall x, y, z \in A$,
(g) $x \wedge y = x \wedge (x^{\sim} \vee y),\ \forall x, y, z \in A$,

is called a Pre A*-algebra.

*Corresponding author. E-mail: asnmat1969@yahoo.in.

Abbreviation: ADA, Algebra of disjoint alternatives.

Example 1

3 = {0, 1, 2} with operations $\wedge$,$\vee$, (-) $^{\sim}$ defined below is a pre A*-algebra.

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	$x^{\sim}$
0	1
1	0
2	2

Note 1

The elements 0, 1, 2 in the above example satisfy the following laws:

(a) $2^{\sim}$ = 2; (b) 1 $\wedge$ x = x for all x $\in$ **3**; (c) 0 $\vee$ x = x for all x $\in$ **3;** (d) 2 $\wedge$ x = 2 $\vee$ x = 2 for all x $\in$ **3**.

Example 2

2 = {0, 1} with operations $\wedge$, $\vee$, (-) $^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

x	$x^{\sim}$
0	1
1	0

Note 2

(i) $\left(2,\vee,\wedge,(-\tilde{)}\right)$ is a Boolean algebra. So every Boolean algebra is a Pre A* algebra.
(ii) The identities $x^{\sim\sim}$ = x , $\forall$ x$\in$A and $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}$, $\forall$ x, y $\in$A implies that the varieties of pre A*-algebras satisfies all the dual statements of x $\wedge$ x = x , $\forall$ x$\in$A to x $\wedge$ y = x $\wedge$ ($x^{\sim}$ $\vee$ y) , $\forall x, y, z \in A$.

Note 3

Let A be a Pre A*-algebra then A is Boolean algebra iff x $\vee$ (x $\wedge$ y) = x, x $\wedge$ (x $\vee$ y) = x (absorption laws holds).

Lemma 1

Every pre A*-algebra satisfies the following laws (Venkateswara and Srinivasa, 2009).

(a) $x \vee (x^{\sim} \wedge x) = x$
(b) $(x \vee x^{\sim}) \wedge y = (x \wedge y) \vee (x^{\sim} \wedge y)$
(c) $(x \vee x^{\sim}) \wedge x = x$
(d) $(x \vee y) \wedge z = (x \wedge z) \vee (x^{\sim} \wedge y \wedge z)$

Definition 1

Let A be a Pre A*-algebra. An element x $\in$ A is called central element of A if $x \vee x^{\sim}=1$ and the set {x $\in$ $x \vee x^{\sim}=1$} of all central elements of A is called the centre of A and it is denoted by B (A). Note that if A is a pre A*-algebra with 1, then 1, 0$\in$B (A). If the centre of pre A*-algebra coincides with {0, 1} then we say that A has trivial centre.

Theorem 1

Let A be a pre A*-algebra with 1, then B (A) is a Boolean algebra with the induced operations $\wedge$, $\vee$, $(\text{-})^{\sim}$ (Venkateswara and Srinivasa, 2009).

Lemma 2

Let A be a Pre A*-algebra with 1(Venkateswara and Srinivasa, 2009),

(a) If y $\in$B (A) then $x \wedge x^{\sim} \wedge y = x \wedge x^{\sim}, \forall x \in A$
(b) $x \wedge (x \vee y) = x \vee (x \wedge y) = x$ if and only if $x, y \in B(A)$

SEMILATTICE STRUCTURE ON PRE A*-ALGEBRA

Theorem 2

Let A b e a Pre A*-algebra define a binary operation $\oplus$ on A by x $\oplus$ y= $x \vee y$ for all x, y$\in$ A then $< A, \oplus >$ is a semi lattice.

Proof

x $\oplus$ x = x $\vee$ x = x for all x$\in$ A.
For x, y$\in$ A we have x $\oplus$ y = $x \vee y$ = y $\vee$ x = y $\oplus$ x.
x$\oplus$ (y$\oplus$ z) = x$\oplus$ (y$\vee$z)
= x$\vee$ (y$\vee$z) = ($x \vee y$)$\vee$z = (x$\oplus$y) $\oplus$ z, for all x, y, z$\in$ A.
Hence $< A, \oplus >$ is a semi-lattice.

Definition 3

Let A be a pre A*-algebra define a relation $\leq_{\oplus}$ on A by x $\leq_{\oplus}$ y iff x$\oplus$y = y.

Lemma 3

Let A be a pre A*-algebra then **(A**, $\leq_{\oplus}$**)** is a poset.

Proof

Since x$\oplus$x = x$\vee$x = x, x $\leq_{\oplus}$ x, for all x$\in$ A
Therefore $\leq_{\oplus}$ is reflexive.

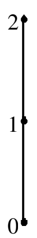
Suppose x $\leq_\oplus$ y, y $\leq_\oplus$ z, for all x, y, z$\in$ A then x $\oplus$ y = y, and y $\oplus$ z = z. Now x $\oplus$ z = x$\oplus$ (y $\oplus$z) = (x$\oplus$ y) $\oplus$ z = y$\oplus$ z = z that is x $\leq_\oplus$ z, this shows that $\leq_\oplus$ is Transitive.

Let x $\leq_\oplus$y and y $\leq_\oplus$ x for all x, y$\in$ A then x $\oplus$ y = y and y $\oplus$ x = x $\Rightarrow$ x = y. This shows that $\leq_\oplus$is anti symmetric. Therefore **(A, $\leq_\oplus$)** is a poset.

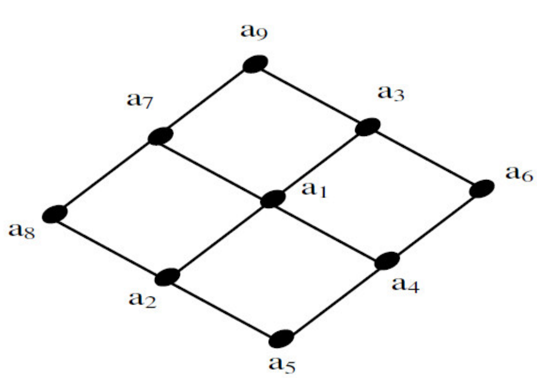
Note 4

We have x $\leq_\oplus$y iff x $\oplus$ y = y, so x $\leq_\oplus$x $\oplus$ y for all x$\in$ A. This shows that x $\oplus$y is the supremum of {x, y}.

Let A be a Pre A*-algebra with 0, 1, 2 then 0 $\leq_\oplus$x(x$\vee$0 = x for all x$\in$ A) and x $\leq_\oplus$2 (2$\vee$ x = 2 for all x$\in$ A). This gives that 2 is the greatest element and 0 is the least element of the poset (A,$\leq_\oplus$). The Hasse diagram of the poset (A,$\leq_\oplus$) is



We have $A\times A$ = { $a_1=(1,1)$, $a_2=(1,0)$, $a_3=(1,2)$, $a_4=(0,1)$, $a_5=(0,0)$, $a_6=(0,2)$, $a_7=(2,1)$, $a_8=(2,0)$, $a_9=(2,2)$} is a pre A*-algebra under point wise operation and $A\times A$ is having four central elements and remaining are non central elements, among that $a_9=(2,2)$ is satisfying the property that $a_9\tilde{}=a_9$. The Hasse diagram is of the poset ($A\times A$,$\leq_\oplus$) as shown:



Observe that x$\leq_\oplus$a$_9$ ($x\vee a_9=a_9$) and $a_5\leq_\oplus x$ ($x\vee a_5=x$) for all $x\in A\times A$. This shows that a$_9$ is the greatest element and a$_5$ is the least element of $A\times A$.

Lemma 4

The following conditions hold for any elements x and y in a pre A*-algebra A

(i) x $\leq_\oplus$ x $\vee$ y
(ii) x $\wedge$ x$\tilde{}$ $\leq_\oplus$ $x\wedge y$

Proof

(i) Consider ($x\vee y$) $\oplus$x = ($x\vee y$)$\vee$ x =$x\vee y$. Therefore, x $\leq_\oplus$ x $\vee$ y.
(ii) Consider (x$\wedge$y) $\oplus$ (x$\wedge$ x$\tilde{}$) = (x$\wedge$y) $\vee$ (x$\wedge$ x$\tilde{}$)
= x $\wedge$ (y $\vee$ x$\tilde{}$) = x $\wedge$ y (by dual of x $\wedge$ y = x $\wedge$ (x$\tilde{}$ $\vee$ y), $\forall x,y,z\in A$)
Therefore x $\wedge$ x$\tilde{}$ $\leq_\oplus$ $x\wedge y$

Lemma 5

Let A be a Pre A*-algebra then $\oplus$ is distributive over $\vee$ and $\wedge$ that is (i) x$\oplus$ (y $\vee$ z) = (x $\oplus$ y) $\vee$ (x $\oplus$z). (ii) x $\oplus$ (y $\wedge$z) = (x $\oplus$ y) $\wedge$ (x $\oplus$ z)

Proof

(i) (x $\oplus$ y) $\vee$ (x $\oplus$ z)
= ($x\vee y$)$\vee$ (x$\vee$z)
= x $\vee$ (y$\vee$z)
= x$\oplus$ (y$\vee$z)
(ii) (x $\oplus$ y) $\wedge$ (x $\oplus$z)
= ($x\vee y$)$\wedge$ (x$\vee$z)
= x$\vee$ (y $\wedge$ z)
= x $\oplus$ (y $\wedge$z)

Theorem 3

Let A be a Pre A*-algebra for any x$\in$ A then the following holds in the semi-lattice$<A,\oplus>$.

(i) x$\vee$ x$\tilde{}$ is the supremum of {x , x$\tilde{}$}
(ii) x$\wedge$x$\tilde{}$ is the infimum of {x , x$\tilde{}$}

Proof

(i) x$\oplus$ (x$\vee$ x$\tilde{}$) = x$\vee$ (x$\vee$ x$\tilde{}$) = x$\vee$ x$\tilde{}$

Therefore, $x \leq_{\oplus} x \vee x^{\sim}$.
$x^{\sim} \oplus (x \vee x^{\sim}) = x^{\sim} \vee (x \vee x^{\sim}) = x \vee x^{\sim}$.
Therefore, $x^{\sim} \leq_{\oplus} x \vee x^{\sim}$.
$x \vee x^{\sim}$ is upper bound of $\{x, x^{\sim}\}$.
Let k be the upper bound of $\{x, x^{\sim}\}$
$\Rightarrow x \leq_{\oplus} k$ and $x^{\sim} \leq_{\oplus} k$ that is $x \oplus k = k$ and $x^{\sim} \oplus k = k$
$\Rightarrow x \vee k = k$ and $x^{\sim} \vee k = k$.
Now $k \oplus (x \vee x^{\sim}) = k \vee (x \vee x^{\sim}) = (k \vee x) \vee x^{\sim}$
$= k \vee x^{\sim} = k$
$\therefore x \vee x^{\sim} \leq_{\oplus} k$.
Therefore, $x \vee x^{\sim}$ is least upper bound of $\{x, x^{\sim}\}$.
Sup $\{x, x^{\sim}\} = x \vee x^{\sim}$

(ii) $x \oplus (x \wedge x^{\sim}) = x \vee (x \wedge x^{\sim}) = x$
Therefore $x \wedge x^{\sim} \leq_{\oplus}$.
$x^{\sim} \oplus (x \wedge x^{\sim}) = x^{\sim} \vee (x \wedge x^{\sim}) = x^{\sim}$.
Therefore, $x \wedge x^{\sim} \leq_{\oplus} x^{\sim}$.
$x \wedge x^{\sim}$ is lower bound of $\{x, x^{\sim}\}$.
Let l be the lower bound of $\{x, x^{\sim}\}$
$\Rightarrow l \leq_{\oplus} x$ and $l \leq_{\oplus} x^{\sim}$ that is $x \oplus l = x$ and $x^{\sim} \oplus l = x^{\sim}$
$\Rightarrow l \vee x = x$ and $l \vee x^{\sim} = x^{\sim}$
Now $l \oplus (x \wedge x^{\sim}) = l \vee (x \wedge x^{\sim}) = (l \vee x) \wedge (l \vee x^{\sim})$
$= x \wedge x^{\sim} \Rightarrow l \leq_{\oplus} x \wedge x^{\sim}$
$\therefore x \wedge x^{\sim}$ is greatest lower bound of $\{x, x^{\sim}\}$.
Inf $\{x, x^{\sim}\} = x \wedge x^{\sim}$.

Lemma 6

In the poset $(A, \leq_{\oplus})$ and $x, y \in A$. If $x \leq_{\oplus} y$ then for $a \in A$.
(i) $a \wedge x \leq_{\oplus} a \wedge y$
(ii) $a \vee x \leq_{\oplus} a \vee y$

Proof

If $x \leq_{\oplus} y$ then $x \oplus y = y \Rightarrow x \vee y = y$
(i) $(a \wedge x) \oplus (a \wedge y) = (a \wedge x) \vee (a \wedge y) = a \wedge (x \vee y) = a \wedge y$
$\therefore a \wedge x \leq_{\oplus} a \wedge y$
(ii) $(a \vee x) \oplus (a \vee y) = (a \vee x) \vee (a \vee y) = a \vee (x \vee y) = a \vee y$
$\therefore a \vee x \leq_{\oplus} a \vee y$

Lemma 7

Let A be a Pre A*-algebra for any $x, y \in A$, $x \leq_{\oplus} x \vee y$ then, $x \vee y$ is the upper bound of $\{x, y\}$.

Proof

Suppose $x \leq_{\oplus} x \vee y$ then $x \vee y$ is upper bound of x
Now $(x \vee y) \oplus y = (x \vee y) \vee y = x \vee y \Rightarrow y \leq_{\oplus} x \vee y$
Therefore $x \vee y$ is upper bound of y.
$\therefore x \vee y$ is the upper bound of $\{x, y\}$.

Theorem 4

Let A be a Pre A*-algebra for any $x, y \in A$ then sup $\{x, y\} = x \vee y$ in the semilattice $< A, \oplus >$.

Proof

$(x \vee y) \oplus x = (x \vee y) \vee x = x \vee y$
$\therefore x \leq_{\oplus} x \vee y$.
$(x \vee y) \oplus y = (x \vee y) \vee y = x \vee y$
$\Rightarrow y \leq_{\oplus} x \vee y$
Therefore $x \vee y$ is upper bound of y
$\therefore x \vee y$ is the upper bound of $\{x, y\}$
Suppose m is the upper bound of $\{x, y\}$
$\Rightarrow x \leq_{\oplus} m$ and $y \leq_{\oplus} m$ that is $m \oplus x = m$ and $m \oplus y = m$
$\Rightarrow m \vee x = m$ and $m \vee y = m$
Now $m \oplus (x \vee y) = m \vee (x \vee y) = (m \vee x) \vee y = m \vee y = m$
$\Rightarrow x \vee y \leq_{\oplus} m$
$\therefore x \vee y$ is the least upper bound of $\{x, y\}$
$\therefore$ sup $\{x, y\} = x \vee y$

Note 5

In general for a pre A*-algebra with 1, $x \vee y$ need not be the greatest lower bound of $\{x, y\}$ in $(A, \leq_{\oplus})$. For example $2 \vee x = 2 \wedge x = 2, \forall x \in A$ is not a greatest lower bound. However we have the following theorem.

Theorem 5

In a semi lattice $<A, \oplus>$ with 1, for any $x, y \in B(A)$ then inf $\{x, y\} = x \wedge y$

Proof

If $x, y \in B(A)$, then by lemma 2b, $x \vee (x \wedge y) = x$ and $y \vee (x \wedge y) = y$
$\Rightarrow x \oplus (x \wedge y) = x$ and $y \oplus (x \wedge y) = y$
This shows that $x \wedge y \leq_{\oplus} x$ and $x \wedge y \leq_{\oplus} y$

Hence $x \wedge y$ is an lower bound of $\{x, y\}$.
Suppose k is an lower bound of $\{x, y\}$, then $k \leq_{\oplus} x$, $k \leq_{\oplus} y$
$\Rightarrow k \oplus x = x$ and $k \oplus y = y$
$\Rightarrow x \vee k = x$, $y \vee k = y$

Now k $\oplus$ (x $\wedge$ y) = k $\vee$ (x $\wedge$ y) = (k $\vee$ x) $\wedge$ (k $\vee$ y) = x $\wedge$ y.
Therefore, k $\leq_{\oplus}$ x $\wedge$ y
x $\wedge$ y is the greatest lower bound of {x , y}. Hence, inf {x, y} = x $\wedge$ y

Theorem 6

If A is a Pre A*-algebra and $x \vee (x \wedge y) = x$, for all x, y$\in$ A then (A, $\leq_{\oplus}$) is a lattice.

Proof

By theorem 4 every pair of elements have supremum. If $x \vee (x \wedge y) = x$ for all x, y$\in$ A then by theorem 5 every pair of elements have infimum. Hence (A, $\leq_{\oplus}$) is a lattice.

Lemma 8

Let A be a pre A*-algebra then:

(i) x$\vee$ (x $\oplus$ y) = x $\vee$ y.
(ii) (x $\oplus$ y) $\vee$ x= x $\oplus$ y.

Proof

(i) x$\vee$ (x $\oplus$ y) = x $\vee$ (x$\vee$ y) = x $\vee$ y
(ii) (x $\oplus$ y) $\vee$ x = (x$\vee$ y) $\vee$ x = x $\vee$ y= x$\oplus$ y

Now we present a number of equivalent conditions for a pre A*-algebra become a Boolean algebra.

Theorem 7

The following conditions are equivalent for any pre A*-algebra (A, $\wedge$, $\vee$ $(-)^{\sim}$)

(1) A is Boolean algebra
(2) x $\wedge$ y $\leq_{\oplus}$ x for all x, y$\in$ A
(3) x $\wedge$ y $\leq_{\oplus}$ y for all x, y$\in$ A
(4) x $\wedge$ y is a lower bound of $\{x, y\}$ in (A , $\leq_{\oplus}$) for all x, y$\in$ A
(5) x $\wedge$ y is a infimum of $\{x, y\}$ in (A , $\leq_{\oplus}$) for all x, y$\in$ A
(6) x $\vee$ $x^{\sim}$ is the least element in (A , $\leq_{\oplus}$) for every x$\in$ A

Proof

(1) $\Rightarrow$(2) Suppose A is a Boolean algebra.
Now x$\oplus$ (x$\wedge$ y) = x$\vee$ (x $\wedge$ y) = x (by absorption law)
$\therefore$ x $\wedge$ y $\leq_{\oplus}$ x
(2) $\Rightarrow$(3) Suppose x $\wedge$ y $\leq_{\oplus}$ x then x$\oplus$ (x$\wedge$ y) = x. Therefore x$\vee$ (x $\wedge$ y) = x
Now y$\oplus$ (x $\wedge$ y) = y$\vee$ (x$\wedge$ y) = y. Therefore, x $\wedge$ y $\leq_{\oplus}$ y
(3) $\Rightarrow$(4) Suppose that x$\wedge$y$\leq_{\oplus}$y $\Rightarrow$ y$\oplus$ (x$\wedge$y) = y therefore y $\vee$ (x $\wedge$ y) = y
Since x $\wedge$ y $\leq_{\oplus}$ y then x $\wedge$ y is lower bound of y
Now x $\oplus$ (x$\wedge$ y) = x$\vee$ (x$\wedge$ y) = x (by supposition)
$\therefore$ x $\wedge$ y $\leq_{\oplus}$ x
$\Rightarrow$ x $\wedge$ y is a lower bound of x.
$\therefore$ x $\wedge$ y is a lower bound of $\{x, y\}$.
(4) $\Rightarrow$ (5) Suppose x $\wedge$ y is a lower bound of $\{x, y\}$ Suppose z is a lower bound of $\{x, y\}$ then z $\leq_{\oplus}$x, z $\leq_{\oplus}$ y that is x $\oplus$ z = x and y$\oplus$z = y
$\Rightarrow$ x $\vee$ z = x , y $\vee$z = y
Now z $\oplus$ (x $\wedge$ y) = z$\vee$ (x $\wedge$ y) = (z $\vee$ x) $\wedge$ (z $\vee$ y) = x $\wedge$ y
Therefore, z $\leq_{\oplus}$ x $\wedge$ y
x $\wedge$ y is the greatest lower bound of $\{x, y\}$
Hence Inf $\{x, y\}$ = x $\wedge$ y
(5) $\Rightarrow$(6) Suppose Inf $\{x, y\}$ = x $\wedge$ y then $x, y \in B(A)$
Now Inf {x $\wedge$ $x^{\sim}$, y} = x $\wedge$$x^{\sim}$ $\wedge$ y = x $\wedge$ $x^{\sim}$ (by lemma 2a)
$\Rightarrow$ x $\wedge$ $x^{\sim}$ $\leq_{\oplus}$ y. Therefore x $\wedge$ $x^{\sim}$ is the least element in (A, $\leq_{\oplus}$).
(6) $\Rightarrow$ (1) Suppose x $\wedge$ $x^{\sim}$ is the least element in A then x $\wedge$ $x^{\sim}$ $\leq_{\oplus}$ y, for y$\in$ A
$\Rightarrow$ (x $\wedge$ $x^{\sim)}$ $\oplus$ y = y $\Rightarrow$ (x $\wedge$ $x^{\sim}$) $\vee$ y = y
Now y $\wedge$ (x $\vee$ y) = [(x $\wedge$ $x^{\sim}$) $\vee$ y] $\vee$ (x $\vee$ y)
= [(x $\wedge$ $x^{\sim}$)$\vee$ x] $\vee$ y= (x $\wedge$ $x^{\sim}$) $\vee$y = y (by supposition).

Therefore by Note 3 we have B is Boolean algebra.

Theorem 8

Let A be a pre A*-algebra x$\vee$ $x^{\sim}$ is the greatest element in (A, $\leq_{\oplus}$) for every x$\in$ A then A is Boolean algebra.

Proof

Suppose x$\vee$ $x^{\sim}$ is the greatest element in (A,$\leq_{\oplus}$) then y $\leq_{\oplus}$ x $\vee$ $x^{\sim}$
$\Rightarrow$(x $\vee$ $x^{\sim)}$ $\oplus$ y = x$\vee$ $x^{\sim}$
$\Rightarrow$ (x$\vee$ $x^{\sim)}$ $\vee$y = x $\vee$ $x^{\sim}$
Now x$\vee$ (x $\wedge$ y) = [x$\wedge$ ($x^{\sim}$ $\vee$ x)] $\vee$ (x $\wedge$ y)
= x$\wedge$ [(x$\vee$ $x^{\sim}$)$\vee$ y] = x $\wedge$ (x$\vee$ $x^{\sim}$) = x (by supposition)
$\therefore$ x$\vee$ (x$\wedge$ y)=x, absorption law holds

By Note 3 we have B is Boolean algebra.

REFERENCES

Birkhoff G (1948). Lattice theory, American Mathematical Society, Colloquium, publishers, New York.
Fernando G, Craig C (1994). Squir: The Algebra of Conditional logic, Algebra Universalis 27: 88-110.
Rao KP (1994). A*-algebra and If-Then-Else structures (thesis), Nagarjuna University, A.P., India.
Manes EG (1989). The Equational Theory of Disjoint Alternatives, personal communication to Prof. N.V.Subrahmanyam.
Manes EG (1993). ADA and the Equational Theory of If-Then-Else, Algebra Universalis 30: 373-339.
Venkateswara RJ (2000). On A*- algebras (thesis), Nagarjuna University, A.P., India.
Venkateswara RJ, Srinivasa RK (2009). Pre A*-algebra as a Poset, Afr. J. Mathe. Comput. Sci. Res. 2: 73-80.