

Review

Ample invertible sheaves on the irreducible components of an exceptional locus

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Resolving the n -dimensional A_n -Singularity, an exceptional locus will be obtained ($E = E_1 + E_2 + \dots + E_m$), where E_i are the irreducible components of E , and they are isomorphic to Q, R , or \mathbb{P}^3 . These are invertible sheaves. We study the conditions to be ample, using the known Kleiman's Criterion Kleiman (1966).

Key words: Simple singularity, exceptional locus, ample sheaves.

INTRODUCTION

In the affine space $A^n(x_1, \dots, x_n)$, over an algebraically closed field K with arbitrary characteristic, the n -dimensional A_n -Singularity is given by the equation:

$$f_n = x_1^{n+1} + x_2x_3 + x_4x_5, \quad \text{That is, } A_n = \text{spec}(K[x_1, \dots, x_n]/(f_n))$$

A_n has an isolated singular point in the origin Hartshorne (1977). Step by step we find that the irreducible components of the exceptional locus E_1, E_2, \dots, E_m are isomorphic to Q, R , or \mathbb{P}^3 .

The irreducible components of the exceptional locus Remark

We assume the ring $A = K[x_1, \dots, x_n]/(f)$,

Where,

$$f = \begin{cases} x_1^2 + x_2x_3 + \dots + x_{r-1}x_r, & \text{if } r \equiv 1 \pmod{2} \text{ and } n \geq r \\ x_1x_2 + x_3x_4 + \dots + x_{r-1}x_r, & \text{if } r \equiv 0 \pmod{2} \text{ and } n \geq r \end{cases}$$

Let X be the spectrum of A ($X = \text{spec } A$) and $Q = \text{proj } A$ (projective spectrum), that is, $x = c(Q)$ is the affine cone over Q .

Lemma

With the above notations:

(a) If $r = 3$, then $cl X \cong \mathbb{Z}/2\mathbb{Z}$, and $cl Q \cong \mathbb{Z}$

Where,

$cl X = \text{divisor class group of } X$

$cl Q = \text{divisor class group of } Q$

and the generator of this class is $H \cap Q$ (H is a hyperplane).

(b) If $r = 4$, then $cl X = \mathbb{Z}$, and $cl Q = \mathbb{Z} \oplus \mathbb{Z}$.

(c) If $r \geq 5$, then $cl X = 0$, and $cl Q = \mathbb{Z}$.

Proof:

(a) The proof can be obtained by the exact sequence:

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$$0 \rightarrow \mathbb{Z} \rightarrow cl Q \rightarrow cl K \rightarrow 0$$

$$1 \mapsto Q, H$$

Here, $Q = \text{proj}(K[x_1, x_2, x_3]/(x_1x_2 - x_3^2))$.

We assume $Y = v^+(x_1, x_2) \subset Q$,

that is a prime divisor and:

$$Q - Y \cong \text{spec} \left(K \left[\begin{matrix} x_2, x_3 \\ x_1, x_2 \end{matrix} \right]_{(x_3)} \right) \cong \text{spec} \left(K \left[\begin{matrix} x_2, x_3 \\ x_1, x_2 \end{matrix} \right] \right) - 0$$

because $K \left[\begin{matrix} x_2, x_3 \\ x_1, x_2 \end{matrix} \right]$ is a field.

From the exact sequence $\mathbb{Z} \rightarrow cl Q \rightarrow cl(Q - Y) \rightarrow 0$ we obtain that the sequence:

$$\mathbb{Z} \rightarrow cl Q \rightarrow 0$$

$$1 \mapsto Y$$

is exact and that means $\mathbb{Z} \rightarrow cl Q$ is injective, therefore, $cl Q \cong \mathbb{Z}$.

The rest of the lemma can be proved by taking a similar sequence.

The scheme R

We define a scheme R and determine some intersection products in the Chowring $A_*(R)$.

$$\text{Let } V = v^+(y_1y_2 + y_3y_4) \subseteq \mathbb{P}^3(y_1, \dots, y_4)$$

be a projective variety in \mathbb{P}^3 .

$$\text{Let } Q_s := v^+(y_1y_2 + y_3y_4) \subseteq \mathbb{P}^4(y_1, \dots, y_5)$$

be the

projective closure of the affine cone

$$C(V) \subseteq \mathbb{A}^4(z_1, z_2, \dots, z_4); \text{ where } z_i = y_i/y_5$$

and:

$$P = (0: 0: \dots : 0: 1) \in \mathbb{P}^4$$

be the vertex of this cone i.e. $Q_s = \overline{C(V)} \subseteq \mathbb{P}^4$,

then the projection:

$$\pi: Q_s - \{P\} \rightarrow V$$

Induces an isomorphism:

$$\pi^*: cl V \xrightarrow{\sim} cl(Q_s - \{P\}) \dots \dots (1)$$

But it is clear that:

$$Q_s - \{P\} \xrightarrow{f} Q_s,$$

Therefore, f induces an isomorphism:

$$f^*: cl Q_s \xrightarrow{\sim} cl(Q_s - \{P\}) \dots \dots (2)$$

(1) and (2) give $cl V \cong cl Q_s$.

The generator system of $cl Q_s = A_2(Q_s)$ (Chowring) can be found through a generator system of $cl V$. Now we blow-up Q_s in $P = (0: \dots : 0: 1) \in \mathbb{P}^4$, and obtain the scheme:

$$R := \begin{cases} u_1u_2 + u_3u_4 = 0 \\ x_iu_j = x_ju_i \end{cases}$$

in $\mathbb{A}^4 \times \mathbb{P}^3(x_1, \dots, x_4, u_1: \dots : u_4)$.

The exceptional locus of this blowing up ($R \xrightarrow{f} Q_s$) is:

$$K = f^{-1}(P) \cong \mathbb{P}^1 \times \mathbb{P}^1$$

The isomorphic types of the components: Alwadi and Dgheim (1988)

(i) By blowing up the A_1 singularity, in the origin of the affine space $\mathbb{A}^3(x_1, \dots, x_3)$, we obtain the exceptional locus $E = f^{-1}(P)$,

and:

$$E \cong Q := v^+(y_1^2 + y_2y_3 + y_4y_5) \subseteq \mathbb{P}^4$$

In this case E consists of one component only and:

$$cl Q = Pic Q \cong \mathbb{Z} \quad (Pic Q = \text{Picard group})$$

(ii) The exceptional locus E of the A_2 -singularity consists of two components $E = E_1 + E_2$, where, $E_1 \cong R$, and

$$E_2 \cong \mathbb{P}^3$$

We know that $A_2 = \text{spec}(K[x_1, \dots, x_5]/\langle f_2 \rangle)$, and

$$f_2 = x_1^3 + x_2x_3 + x_4x_5$$

(ii) The A_3 -singularity is given in \mathbb{A}^5 by

$$A_3 = \text{spec}(K[x_1, \dots, x_5]/\langle f_3 \rangle), \text{ where,}$$

$$f_3 = x_1^4 + x_2x_3 + x_4x_5.$$

In this case E also consists of

$$E_1, \text{ and } E_2, \text{ where, } E_1 = R, \text{ and } E_2 \cong Q.$$

(iv) In the general case,

$$A_n = \text{spec}(K[x_1, \dots, x_5]/\langle f_n \rangle), \text{ where:}$$

$$f_n = x_1^n + x_2x_3 + x_4x_5,$$

the exceptional locus $E = E_1 + \dots + E_m$:

(a) If $n \equiv 1 \pmod 2$, then $E_i \cong R$, ($i = 1, \dots, m - 1$) and $E_m \cong Q$.

(b) If $n \equiv 0 \pmod 2$, then $E_i \cong R$, ($i = 1, \dots, m - 1$)

and $E_m \cong \mathbb{P}^3$.

Note that, $m = \lfloor \frac{n+1}{2} \rfloor$.

CRITERION KLEIMAN (1966)

Let X be a nonsingular complete scheme and let $A_1 = A_1(X)$ be all curves in X , where $A_1(X)$ denotes the Chowring Fulton (2008). We assume $\text{Pic } X/\cong$ (" \cong " numerical equivalence). For an invertible sheaf $L \in \text{Pic } X$ we say $L \cong 0$ if and only if

$$L.C = 0, \quad \forall C \in A_1(X)$$

We define two real vector spaces:

$$N^1 = N^1(X) = (\text{Pic } X/\cong) \otimes_{\mathbb{Z}} \mathbb{R}$$

$$N_1 = N_1(X) = (A_1(X)/\cong) \otimes_{\mathbb{Z}} \mathbb{R}$$

We have a perfect pairing of N^1 with N_1 which will be induced by the intersection product:

$$N^1(X) \times N_1(X) \rightarrow \mathbb{R}$$

$$(L, C) \mapsto (L, C) := \text{deg}_C(L \otimes O_C)$$

We define also $\overline{NE(X)}$ to be a closed convex cone in $N_1(X)$, which can be induced by irreducible curves. With the above notations, the Kleiman's criterion take the following form:

"An invertible sheaf $L \in \text{Pic } X$ is ample if and only if the mapping:

$$L: \overline{NE(X)} - \{0\} \rightarrow \mathbb{R}$$

$$C \mapsto L.C \text{ is positive } "$$

THE AMPLE SHEAVES ON THE COMPONENTS WHICH ARE ISOMORPHIC TO R

It is known that $E_i \cong R$ are smooth projective 3-dimensional schemes. On the other hand, we have also:

$$\text{Pic } R = \mathbb{Z}.H_1 \oplus \mathbb{Z}.H_2 \oplus \mathbb{Z}.K$$

(H_1, H_2 , and K were introduced in Alexiou and Alwadi (2003))

Lemma

$$\text{Pic } R \cong \text{Pic } R/\cong$$

Proof:

If $L \in \text{Pic } R$ and $L \cong 0$, then $L \cong O_R$ and we have $L = O_R(a_1H_1 + a_2H_2 + bK)$, where O_R is a sheaf of rings on R .

$L \cong 0 \iff (L, C) = 0$, for all C and it is an irreducible curve in R . In particular this holds for the following curves:

$$C = C_{K1}, C_{K2}, C_{12} (C_{K1} = k.H_1, C_{K2} = k.H_2, C_{12} = H_1.H_2)$$

In $A_1(R)$ (the intersections in the Chowring), we have:

$$(L, C_{K1}) = a_2 - b = 0$$

$$(L.C_{K2}) = a_1 - b = 0$$

$$(L.C_{12}) = b = 0$$

Therefore, $a_1 = a_2 = b = 0$ thus, $L \cong O_{\mathbb{R}}$.

Proposition

$$\overline{NE(R)} = \mathbb{R}_+ C_{K1} + \mathbb{R}_+ C_{K2} + \mathbb{R}_+ C_{12},$$

where C_{K1}, C_{K2}, C_{12}

are as in the last lemma.

Proof:

$N_1(\mathbb{R})$ is a 3-dimensional real vector space,

and $C_{K1}, C_{K2}, C_{12} \in N_1(\mathbb{R})$. If C_{K1}, C_{K2}, C_{12}

are linearly independent, then:

$$N_1(\mathbb{R}) = \mathbb{R}_+ C_{K1} + \mathbb{R}_+ C_{K2} + \mathbb{R}_+ C_{12}$$

But if:

$$C = a_1 C_{K1} + a_2 C_{K2} + b C_{12} = 0,$$

Then:

$$0 = (H_1.C) = a_2 = 0$$

$$0 = (H_2.C) = a_1 = 0$$

$$0 = (K.C) = -a_1 - a_2 + b = 0 \Rightarrow b = 0 \Rightarrow a_1 = a_2 = b = 0$$

That is, $N_1(\mathbb{R}) = \mathbb{R}C_{K1} + \mathbb{R}C_{K2} + \mathbb{R}C_{12}$.

Now $\overline{NE(X)}$ is the closed cone in $N_1(\mathbb{R})$,

which induced by irreducible curves, that is,

$$\overline{NE(R)} = \{\sum r_i c \text{ mod } \approx : r_i \geq 0, \text{ and all } r_i = 0 \text{ except of finite number of } r_i\}$$

We note that c is irreducible over R . We show at this

$$\text{stage: } \overline{NE(R)} = \mathbb{R}_+ C_{K1} + \mathbb{R}_+ C_{K2} + \mathbb{R}_+ C_{12},$$

we first have to show:

$$\overline{NE(R)} \subseteq \mathbb{R}_+ C_{K1} + \mathbb{R}_+ C_{K2} + \mathbb{R}_+ C_{12} \text{ i.e.}$$

we have to show if:

$$\sum r_i \cdot c \in \overline{NE(R)},$$

Then:

$$\sum r_i \cdot c \in \mathbb{R}_+ C_{K1} + \mathbb{R}_+ C_{K2} + \mathbb{R}_+ C_{12}$$

Therefore, we have to show if $c \in R$ is an irreducible curve, then:

$$c \in \mathbb{R}_+ C_{K1} + \mathbb{R}_+ C_{K2} + \mathbb{R}_+ C_{12},$$

Therefore, it is sufficient to show if:

$$C \approx a_1 C_{K1} + a_2 C_{K2} + b C_{12} \quad (a_1, a_2, b \in \mathbb{R}),$$

then $a_1, a_2, b \geq 0$. For this goal we have the following cases:

(i) $C \subseteq H_1, C \subseteq H_2, C \subseteq K$ but in this case:

$$(C.H_1) = a_2 \geq 0$$

$$(C.H_2) = a_1 \geq 0, \text{ and}$$

$$(C.K) = -a_1 - a_2 + b \geq 0 \Rightarrow b \geq a_1 + a_2 \Rightarrow b \geq 0$$

(ii) $C \subseteq H_1 \Rightarrow C = \alpha.C_{K1} + \beta.C_{K2}; \alpha, \beta \in \mathbb{Z}$

If $C \subseteq C_{K1}$ (this corresponds

$$C \subseteq C_{K1}, C = 1.C_{K1} + 0.C_{12}) \text{ clear.}$$

If $C \subseteq C_{12}$ (corresponds $C = C_{12}$)

If $C \subseteq C_{K1}$ and $C \subseteq C_{12}$ that is, $(C.C_{K1}) \geq 0$ and

$$(C.C_{12}) \geq 0 \text{ but on } H_1 \text{ we have}$$

$$C_{K1}^2 = -1, C_{K1} \cdot C_{12} = 1 \text{ and } C_{12}^2 = 0$$

Alexiou and Alwadi (2003), therefore:

$$\alpha, \beta \geq 0 \Leftrightarrow \begin{cases} (C.C_{K1}) = -\alpha + \beta \geq 0 \\ (C.C_{12}) = \alpha \geq 0 \end{cases}$$

(iii) $C \subseteq H_2$ (symmetric to(2)).

(iv) $C \subseteq K$, $K \cong \mathbb{P}^1 \times \mathbb{P}^1$ Lazarsfeld (2005) \Rightarrow

$$C = \alpha_1.C_{K1} + \beta_1.C_{K2} \text{ in } (\mathbb{P}^1 \times \mathbb{P}^1) = Z.C_{K1} \oplus Z.C_{K2}$$

$$\text{If } C \subseteq C_{K1} \text{ (} C = 1.C_{K1} + 0.C_{K2}\text{)}.$$

If $C \subseteq C_{K2}$ and $C \subseteq C_{K1}$ i.e. $(C.C_{K1}) \geq 0$ and

$$(C.C_{K2}) \geq 0 \text{ but on } C_{K1}.C_{K2} = 1, C_{K1}^2 = C_{K2}^2 = 0$$

$$(C.C_{K1}) = \beta_1 \geq 0, \quad (C.C_{K2}) = \alpha_1 \geq 0$$

Finally, we deduce the following:

Theorem

$$L = O_R(a_1H_1 + a_2H_2 + bK) \text{ is ample over } R \Leftrightarrow a_1 - b, a_2 - b, b > 0$$

Proof:

(\Rightarrow)

Let L be ample, then by the Kleiman's criterion we have:

$$(L.C_{K1}) > 0, \quad (L.C_{K2}) > 0, \quad \text{and } (L.C_{12}) > 0,$$

But:

$$L.C_{K1} = L.K.H_1 = a_2 - b > 0$$

$$L.C_{K2} = L.K.H_2 = a_1 - b > 0$$

$$L.C_{12} = L.H_1.H_2 = b > 0$$

(\Leftarrow)

Can be obtained from the above proposition.

DISCUSSION AND CONCLUSION

This paper put conditions for ampleness on the irreducible components of the canonical resolution of the simple 4-dimensional singularity A_n . It will be fruitful to be done as a similar study in higher dimensions, because for 3-dimensional it was done Roczen (1984), also it will be fruitful to be done as a similar study for another simple singularities (D_n, E_n) . Our studies enable us to study the so called "negative embedded" of the exceptional locus. The conditions that we found are important cohomological properties of the irreducible components of the canonical resolution of the exceptional locus.

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