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Convergences and numerical analysis of a contact problem with normal compliance and unilateral constraint

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This paper represents a continuation of a previous study on "Analysis of a Sliding Frictional Contact Problem with Unilateral Constraint". This study considers a mathematical model which describes the equilibrium of an elastic body in frictional contact with a moving foundation. The contact is modeled with a multivalued normal compliance condition with unilateral constraints, associated to a sliding version of Coulomb's law of dry friction. After a description of the model, the variational formulation was presented. Then, the dependence of the solution was studied with respect to the data and a convergence result was proven. Regularization method was also used to study the existence and uniqueness of the contact problem for which a convergence result was presented. Finally, a semi-discrete scheme was introduced for the numerical approximation of the sliding contact problem. Under certain solution regularity assumptions, an optimal order error estimate was derived.

Key words: Elastic material, frictional contact, normal compliance, unilateral constraint, variational formulation, weak solution, regularization method, finite element, error estimate.

INTRODUCTION

The mathematical literature dedicated to the study of physical phenomena of contact is more recent. The reason for this is that, accompanied by physical phenomena and surface complexes, the contact processes are modeled by very difficult nonlinear boundary problems. One of the first mathematical publications on this subject is that of Signorini (1933), where the problem of unilateral contact between a linearly elastic body and a rigid foundation is formulated. It follows the work of Fichera (1964) where the Signorini problem has been solved, using arguments of variational inequalities of elliptic type. This being said, we can safely say that the mathematical study of contact problems begins with the monograph by Duvaut and Lions (1972), which has the merit of presenting the variational formulation of several contact problems, accompanied by1996; Kikuchi and Oden, 1988; Kinderlehrer and Stampacchia, 2000; Panagiotopoulos, 1985; Sofonea and Matei, 2012). General results on the analysis of the variational inequalities, including existence and results

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of existence and uniqueness of the solution. Considerable progress has recently been made in the fields of modeling, mathematical analysis and numerical simulation of various contact processes (Haslinger et al., 1996; Kikuchi and Oden, 1988; Kinderlehrer and Stampacchia, 2000; Panagiotopoulos, 1985; Sofonea and Matei, 2012). General results on the analysis of the variational inequalities, including existence and uniqueness results, were developed in a large number of works (Barboteu et al., 2013, 2016; Capatina, 2014; Eck et al., 2013; Han and Reddy, 1995, 1999; Rochdi et al., 1998).

Recently, a more general contact condition, called the normal compliance condition restricted by unilateral constraint introduced in Jarusek and Sofonea (2008), models the contact with an elastic-rigid foundation. The mathematical analysis of models involving the frictionless contact condition with normal compliance and unilateral constraint can be found in Eck et al. (2013; 2015), Jarusek and Sofonea (2008) and Sofonea and Matei (2012). When friction is considered, the unique solvability of the variational problems can be proven by considering a smallness assumption of the friction coefficient (Barboteu et al., 2016; Sofonea and Souleiman, 2015, 2016, Sofonea and Xiao 2016).

In this work, the frictional contact model introduced in Sofonea and Souleiman (2015) which describes the contact of deformable body with a moving foundation not perfectly rigid was considered. Therefore, the contact law with normal compliance and unilateral constraint was associated to a sliding version of Coulomb's law of dry friction. The frictional contact model are characterized condition as a multivalued normal compliance contact condition with unilateral constraints. Such kind of rigid-elastic foundation problems have been considered in Sofonea and Souleiman (2015, 2016).

PRELIMINARIES

The notation and some preliminary material which will be of use later on wer presented. In this paper, the notation $\mathbb{N}$ was used for the set of positive integer. Let $d \in \mathbb{N}$. Then, we denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by:

$$u \cdot v = u_i v_i \ , \qquad \| v \| = (v \cdot v)^{\frac{1}{2}} \qquad \forall\ u, v \in \mathbb{R}^d,$$
$$\sigma \cdot \tau = \sigma_{ij}\tau_{ij} \ , \qquad \| \tau \| = (\tau \cdot \tau)^{\frac{1}{2}} \qquad \forall\ \sigma, \tau \in \mathbb{S}^d$$

Here, the indices i, j, k, l run between 1 and d and unless stated otherwise, the summation convention over repeated indices is used.

Let Ω be a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1,2,3)$ with a Lipschitz continuous boundary Γ and let Γ_1 be a measurable part of Γ such that $\mathrm{m}eas\,(\Gamma_1) > 0$. The notation $x = (x_i)$ was used for a typical point in $\Omega \cup \Gamma$ and denoted by $\nu = (\nu_i)$ the outward unit normal at Γ.

Also, an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \partial u_i / \partial x_j$. In particular, it was recalled that the inner products on the Hilbert spaces $L^2(\Omega)^d$ and $L^2(\Gamma)^d$ are given by:

$$(u, v)_{L^2(\Omega)^d} = \int_\Omega u \cdot v \ dx, \qquad (u, v)_{L^2(\Gamma)^d} = \int_\Gamma u \cdot v \ da,$$

and the associated norms will be denoted by $\|\cdot\|_{L^2(\Omega)^d}$ and $\|\cdot\|_{L^2(\Gamma)^d}$, respectively. Moreover, the spaces are considered.

$$V = \{\ v \in H^1(\Omega)^d : v = 0 \quad on \quad \Gamma_1 \},$$
$$Q = \{\ \tau = (\tau_{ij}) \in L^2(\Omega)^d : \tau_{ij} = \tau_{ji} \}$$

These are real Hilbert spaces endowed with the inner products:

$$(u, v)_V = \int_\Omega \varepsilon(u) \cdot \varepsilon(v)\, dx, \quad (\sigma, \tau)_Q = \int_\Omega \sigma \cdot \tau \ dx,$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Here ε is the deformation operator given by:

$$\varepsilon(v) = (\varepsilon_{ij}(v)), \quad \varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i}) \quad \forall\ v \in H^1(\Omega)^d$$

Recall that the completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption $meas\,(\Gamma_1) > 0$, which allows the use of Korn's inequality.

For an element $v \in V$, v is still written for the trace of v on the boundary Γ. Let v_ν and v_τ br denoted by the normal and the tangential component of v on Γ, respectively, defined by $v_\nu = v \cdot \nu$, $v_\tau = v - v_\nu \nu$. Let Γ_3 be a measurable part of Γ. Then, by the Sobolev trace theorem, there exists a positive constant c_0 which depends on Ω, Γ_1 and Γ_3 such that:

$$\| v \|_{L^2(\Gamma_3)^d} \leq c_0 \| v \|_V \qquad \forall\ v \in V \tag{1}$$

For a regular function $\sigma : \Omega \cup \Gamma \to \mathbb{S}^d$ σ_ν and σ_τ are denoted by the normal and the tangential components of the vector $\sigma\nu$ on Γ, respectively, and recall that $\sigma_\nu = \sigma\nu \cdot \nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$. Moreover, the following Green's formula holds:

$$\int_\Omega \sigma \cdot \varepsilon(v)\, dx + \int_\Omega \mathrm{D}iv\ \sigma \cdot v \ dx = \int_\Gamma \sigma\nu \cdot v \ da \quad \forall\ v \in V \tag{2}$$

This introduction ends with the following abstract existence result.

Theorem 1

Let $(\mathrm{X}, (\cdot,\cdot)_{\mathrm{X}}, \|\cdot\|_{\mathrm{X}})$ be a real Hilbert space, K a closed convex subset of X and $A: K \to K$ a strongly monotone Lipschitz continuous operator, that is, there exists $m > 0$ and $M > 0$ such that:

$$(Au - Av, u - v)_X \geq m \parallel u - v \parallel_X^2 \qquad \forall\, u, v \in K, \tag{3}$$

$$\parallel Au - Av \parallel_X \leq M \parallel u - v \parallel_X \quad \forall\, u, v \in K \tag{4}$$

Assume that $j: K \times K \to \mathbb{R}$ is a function which satisfies the following conditions:

$$\begin{cases} (a) \ \text{For all } \eta \in K,\ j(\eta,\cdot): K \to \mathbb{R} \ \text{is convex and lower semicontinuous.} \\ (b) \ \text{There exists } \alpha \geq 0 \ \text{ such that} \\ \quad j(\eta_1, v_2) - j(\eta_1, v_1) + j(\eta_2, v_1) - j(\eta_2, v_2) \\ \qquad \leq \alpha \parallel \eta_1 - \eta_2 \parallel_X \parallel v_1 - v_2 \parallel_X \quad \forall\, \eta_1, \eta_2, v_1, v_2 \in K \end{cases} \tag{5}$$

Moreover, assume that $m > \alpha$. Then, for each $f \in X$ there exists a unique element $u \in X$ such that:

$$u \in K,\ (Au, v - u)_X + j(u, v) - j(u, u) \geq (f, v - u)_X \ \ \forall v \in K \tag{6}$$

Theorem 1 will be used to prove the existence and uniqueness of our model of contact problem regularized. Its proof can be found in Sofonea and Matei (2012) and is based on the Banach fixed point theorem.

FORMULATION OF THE PROBLEM

An elastic body that occupies the bounded domain Ω with Γ, its boundary was considered. Let ν denotes the unit outward normal, defined almost everywhere on Γ. The body is clamped on Γ_1 and, therefore, the displacement field vanishes there. A volume force of density f_0 acts in Ω, and surface tractions of density f_2 act on Γ_2. On Γ_3, the body is in frictional contact with a moving obstacle, the so-called foundation. Let v^* denotes the velocity of the foundation, that is, its velocity $v^* = v^* k^*$ is assumed to be larger than the tangential velocity u_τ on the surface contact Γ_3 (that is, $\parallel v^* \parallel \gg \parallel u_\tau \parallel$), where $k^* = \frac{v^*}{\parallel v^* \parallel}$ denotes a given unitary vector in the tangential plane and the value $v^* > 0$ is also given.

Therefore, lets consider the classical formulation of frictional contact problem that follows.

Problem P

Find a displacement field $u: \Omega \to \mathbb{R}^d$ and a stress field $\sigma: \Omega \to \mathbb{S}^d$ such that:

$$\sigma = N\varepsilon(u) \quad in \ \ \Omega, \tag{7}$$

$$Div \ \ \sigma + f_0 = 0 \ \ in \ \ \Omega, \tag{8}$$

$$u = 0 \quad on \ \ \Gamma_1, \tag{9}$$

$$\sigma \ \nu = f_2 \quad on \ \Gamma_2, \tag{10}$$

$$-\sigma_\tau = \mu T(u) \ k^* \quad on \ \ \Gamma_3, \tag{11}$$

and there exists $\pi: \Gamma_3 \to \mathbb{R}$ which satisfies:

$$\left.\begin{array}{l} u_\nu \leq g, \sigma_\nu + p(u_\nu) + \pi \leq 0, \\ (u_\nu - g)(\sigma_\nu + p(u_\nu) + \pi) = 0, \\ 0 \leq \pi \leq R, \\ \pi = 0 \quad if \quad u_\nu < 0, \\ \pi = R \quad if \ \ u_\nu > 0 \end{array}\right\} on\ \Gamma_3. \tag{12}$$

Here, for simplicity, the dependence of various functions on the spatial variable x was not indicated explicitly. Now, the physical meaning of Equations 7 to 12 were shortly described. Equation 7 represents the elastic constitutive law in which N is the elasticity operator, assumed to be nonlinear. Equation 8 represents the equation of equilibrium and was used here since the internal term in the equation of motion was neglected. Equations 9 and 10 are the displacement boundary condition and the traction boundary condition, respectively. Finally, Equations 11 and 12 represent the friction Coulomb's law and the multivalued normal compliance contact condition with unilateral constraint and crust, respectively. The friction condition of Equation 11 represents a regularized form of a version of Coulomb's law in slip status where μ represents the coefficient of friction and T is a operator which depends only on the normal displacement u_ν (Sofonea and Souleiman, 2015). Equation 12 represents the contact condition in which p is a positive Lipschitz continuous increasing function which vanishes for a negative argument, R is a positive function and $g > 0$. Note that this conditions the model's contact with a foundation made of a rigid material and covered by a layer of soft material (asperities) of thickness g with a thin crust (Sofonea and Souleiman, 2015).

Lets turn to the variational formulation of Problem P and, to this end, the assumptions on the data were listed. First, the elasticity operator N and the normal compliance function were assumed to satisfy the following conditions:

$$\begin{cases} (a) \ N: \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ (b) \ There \ exists \ L_N > 0 \ such \ that \\ \quad \parallel N(x, \varepsilon_1) - N(x, \varepsilon_2) \parallel \leq L_N \parallel \varepsilon_1 - \varepsilon_2 \parallel \\ \qquad \forall \ \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \quad \text{a.e. } x \in \Omega. \\ (c) \ There \ exists \ m_N > 0 \ such \ that \\ \quad (N(x, \varepsilon_1) - N(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_N \parallel \varepsilon_1 - \varepsilon_2 \parallel^2 \\ \qquad \forall \ \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \quad \text{a.e. } x \in \Omega. \\ (d) \ The \ mapping \ \ x \mapsto N(x, \varepsilon) \ is \ measurable \ on \Omega, \forall \ \varepsilon \in \mathbb{S}^d. \\ (e) \ The \ mapping \ \ x \mapsto N(x, 0) \ belongs \ to \ Q. \end{cases} \tag{13}$$

$$\begin{cases} (a)\ p:\Gamma_3\times\mathbb{R}\to\mathbb{R}_+.\\ (b)\ There\ exists\ L_p>0\ such\ that\\ \qquad |p(x,r_1)-p(x,r_2)|\le L_p\,|r_1-r_2|\\ \qquad \forall\, r_1,r_2\in\mathbb{R},\ \text{a.}e.\ \ x\in\Gamma_3.\\ (c)\ (p(x,r_1)-p(x,r_2))(r_1-r_2)\ge 0\\ \qquad \forall\, r_1,r_2\in\mathbb{R},\ \text{a.}e.\ \ x\in\Gamma_3.\\ (d)\ The\ mapping\ \ x\mapsto p(x,r)\ is\ measurable\ on\ \Gamma_3,\forall\, r\in\mathbb{R}\\ (e)\ p(x,r)=0\ \text{for}\ all\ r\le 0,\ \text{a.}e.x\in\Gamma_3. \end{cases} \tag{14}$$

The densities of body forces and surface tractions have the regularity

$$f_0\in L^2(\Omega)^d,\quad f_2\in L^2(\Gamma_2)^d \tag{15}$$

The surface yield and the coefficient of friction satisfy:

$$R\in L^\infty(\Gamma_3),\qquad R(x)\ge 0\quad \text{a.}e.\ \ x\in\Gamma_3 \tag{16}$$

$$\mu\in L^\infty(\Gamma_3),\qquad \mu(x)\ge 0\quad \text{a.}e.\ \ x\in\Gamma_3 \tag{17}$$

Finally, the operator T satisfies:

$$\begin{cases} (a)\ T:\Gamma_3\times\mathbb{R}^d\to\mathbb{R}_+.\\ (b)\ \text{There exists } L_T>0 \text{ such that}\\ \qquad \|T(x,u_1)-T(x,u_2)\|\le L_T\,\|u_1-u_2\|\\ \qquad \forall\, u_1,u_2\in\mathbb{R}^d,\ \text{a.}e.\ \ x\in\Gamma_3.\\ (c)\ \text{The mapping}\ \ x\mapsto T(x,u)\ \text{is measurable on}\ \Gamma_3, for\ any\ \ u\in\mathbb{R}^d.\\ (d)\ \text{The mapping}\ \ x\mapsto T(x,0_{\mathbb{R}^d})\text{belongs to } L^2(\Gamma_3). \end{cases} \tag{18}$$

Next, the set of admissible displacements fields was introduced, defined by:

$$U=\{\ v\in V\ :\ v_\nu\le g\quad a.e.\ on\,\Gamma_3\ \}. \tag{19}$$

Moreover, the operator $P:V\to V$, the function $j:V\times V\to mm\mathrm{IR}\,I\ R$ and the element $f\in V$ were defined by equalities:

$$(P\,u,v)_V=\int_{\Gamma_3} p(u_\nu)v_\nu\,da\qquad \forall\ u,\ v\in V, \tag{20}$$

$$j(u,v)=\int_{\Gamma_3} Rv_\nu^+\,da+\int_{\Gamma_3}\mu\,T(u)\,n^*\cdot\ v_\tau\,da\qquad \forall\ u,\ v\in V, \tag{21}$$

$$(f,v)_V=\int_\Omega f_0\cdot\ v\ dx,+\int_{\Gamma_2} f_2\cdot\ v\ da\qquad \forall\ v\in V.$$

Here, r^+ denotes the positive part of r, that is, $r^+=max\,\{r,0\}$.

Assume in what follows that (u,σ) are sufficiently regular functions which satisfy Equations 7 to 12 and let $v\in U$. Green's formula of Equations 2, 8 to 10 and Definition 22 were used to see that:

$$(\sigma,\varepsilon(v)-\varepsilon(u))_Q\ge(f,v-u)_V-\int_{\Gamma_3}p(u_\nu)(v_\nu-u_\nu)\,da-\int_{\Gamma_3}R\,(v_\nu^+-u_\nu^+)\,da-\int_{\Gamma_3}\mu T(u)\,n^*\cdot(v_\tau-u_\tau)\,da \tag{23}$$

Finally, the constitutive law of Equation 7, the variational inequality (Equation 23) and Definitions 19 to 21 were gathered to obtain the following variational formulation of the contact problem P .

Problem P_V

Find a displacement field u such that:

$$\mathrm{u}\in\mathrm{U},\ (\mathrm{N}\,\varepsilon(\mathrm{u}),\varepsilon(\mathrm{v})-\varepsilon(\mathrm{u}))_{\mathrm{Q}}+(\mathrm{P}\,\mathrm{u},\mathrm{v}-\mathrm{u})_{\mathrm{V}}+\mathrm{j}(\mathrm{u},\mathrm{v})-\mathrm{j}(\mathrm{u},\mathrm{u})\ge(\mathrm{f},\mathrm{v}-\mathrm{u})_{\mathrm{V}}\ \forall\,\mathrm{v}\in\mathrm{U} \tag{24}$$

A result of existence and uniqueness for the problem P_V was provided in Sofonea and Souleiman (2015).

A CONTINUOUS DEPENDENCE RESULT

The dependence of the solution Problem P^V was studied with respect to perturbations of the data. To this end, it was assumed in what follows that Equations 13 to 18 hold, and denoted by u the solution of Problem P^V. For each $\eta>0$, let $f_{0\eta}, f_{2\eta}, p_\eta, R_\eta$ and μ_η, represent perturbations of f_0, f_2, p, R and μ, respectively, which satisfy conditions of Equations 14 to 17, respectively. With these data, the operator $P_\eta:V\to V$ and the functions $j_\eta:V\times V\to\mathbb{R}$ the element $f_\eta\in V$ were defined by equalities:

$$(P_\eta\,u,v)_V=\int_{\Gamma_3}p_\eta(u_\nu)\,v_\nu\,da\qquad\forall\ u,\ v\in V, \tag{25}$$

$$j_\eta(u,v)=\int_{\Gamma_3}R_\eta v_\nu^+\,da+\int_{\Gamma_3}\mu_\eta\,T(u)\,n^*\cdot\ v_\tau\,da\qquad\forall\ u,\ v\in V, \tag{26}$$

$$(f_\eta,v)_V=\int_\Omega f_{0\eta}\cdot\ v\ dx,+\int_{\Gamma_2}f_{2\eta}\cdot\ v\ da\qquad\forall\ v\in V. \tag{27}$$

Then, the following perturbation of Problem P^V was considered

Problem $\mathrm{P}_{\mathrm{V}}^{\eta}$

Find a displacement field u_η such that:

$$u_\eta\in U,(N\,\varepsilon(u_\eta),\varepsilon(v)-\varepsilon(u_\eta))_Q+(P_\eta u_\eta,v-u_\eta)_V+j_\eta(u_\eta,v)-j_\eta(u_\eta,u_\eta)\ge(f,v-u_\eta)_V\ \forall v\in U \tag{28}$$

It follows from Sofonea and Souleiman (2015) that, for $\eta > 0$, Problem P_V^η has a unique solution u_η. Consider now the following assumptions:

$$\begin{cases} \text{There exists } G: \mathbb{R}_+ \to \mathbb{R}_+ \ such\ that \\ (a)\ |p_\eta(x,r) - p(x,r)| \leq G(\eta)(|r|+1) \\ \forall\, r \in \mathbb{R},\ \text{a.}e.\ \ x \in \Gamma_3 \quad for\ each\ \eta > 0. \\ (b)\ \lim_{\eta\to 0} G(\eta) = 0. \end{cases} \quad (29)$$

$$R_\eta \to R \quad in \quad L^\infty(\Gamma_3) \quad as \quad \eta \to 0 \quad (30)$$

$$\mu_\eta \to \mu \quad in \quad L^\infty(\Gamma_3) \quad as \quad \eta \to 0 \quad (31)$$

$$f_{0\eta} \to f_0 \quad in \quad L^2(\Omega)^d \quad as \quad \eta \to 0 \quad (32)$$

$$f_{2\eta} \to f_2 \quad in \quad L^2(\Gamma_2)^d \quad as \quad \eta \to 0 \quad (33)$$

The following convergence result represents the main result here.

Theorem 2

Assume that Equations 29 to 33 hold, then the solution u_η of Problem $\mathrm{P}_\mathrm{V}^\eta$ converges to the solution u of Problem P_V, that is:

$$u_\eta \to u \quad in \quad U \quad as \quad \eta \to 0 \ (34)$$

Proof

Let $\eta > 0$, then $v = u$ in Equation 28 and $v = u_\eta$ in Equation 24 and add the resulting inequalities to obtain:

$$\begin{aligned}&(N\,\varepsilon(u_\eta) - N\,\varepsilon(u), \varepsilon(u_\eta) - \varepsilon(u))_Q \leq \\ &(P_\eta u_\eta - P u, u - u_\eta)_V + (f_\eta - f, u_\eta - u)_V + \\ &j(u, u_\eta) - j(u,u) + j_\eta(u_\eta, u) - j_\eta(u_\eta, u_\eta)\end{aligned} \quad (35)$$

Estimating each term in previous inequality using Assumption (Equation 13), to deduce that

$$(N\,\varepsilon(u_\eta) - N\,\varepsilon(u), \varepsilon(u_\eta) - \varepsilon(u))_Q \geq m_N \|\, u_\eta - u\,\|_V^2 \quad (36)$$

To proceed, the Definitions (Equations 20 and 25), the monotonicity of the function p_η and Assumption (Equation 29) were used to see that:

$$\begin{aligned}&(P_\eta u_\eta - P u, u_\eta - u)_V = \\ &\int_{\Gamma_3} \big(p_\eta(u_{\eta\nu}) - p(u_\nu)\big)(u_\nu - u_{\eta\nu})da \leq \\ &\int_{\Gamma_3} \big(p_\eta(u_\nu) - p(u_\nu)\big)(u_\nu - u_{\eta\nu})da \leq \\ &\int_{\Gamma_3} \big|p_\eta(u_\nu) - p(u_\mu)\big|\,|u_\nu - u_{\eta\nu}|da \leq \\ &\int_{\Gamma_3} G(\eta)(|u_\nu| + 1)|u_\nu - u_{\eta\nu}|\,da\end{aligned}$$

Therefore, using the trace Inequality (Equation 1), after some elementary calculus, it was found that:

$$(P_\eta u_\eta - P u, u_\eta - u)_V \leq G(\eta)(c_0^2 \|\, u\,\|_V + c_0\, meas\,(\Gamma_3)^{1/2}) \|\, u_\eta - u\,\|_V \quad (37)$$

Next, using Definitions (Equations 21 and 26), thus:

$$\begin{aligned}&j(u, u_\eta) - j(u,u) + j_\eta(u_\eta, u) - j_\eta(u_\eta, u_\eta) = \\ &\int_{\Gamma_3} (R - R_\eta)(u_{\eta\nu}^+ - u_\nu^+)da + \\ &\int_{\Gamma_3} (\mu T(u) - \mu_\eta T(u_\eta))\, n^*(u_{\eta\tau} - u_\tau)\,da\end{aligned}$$

Therefore, writing:

$$\mu T(u) - \mu_\eta T(u_\eta) = \mu(T(u) - T(u_\eta)) + (\mu - \mu_\eta)T(u_\eta)$$

Assumptions 17 and 18b combined with Equation 1 were used to get:

$$\begin{aligned}&j(u, u_\eta) - j(u,u) + j_\eta(u_\eta, u) - j_\eta(u_\eta, u_\eta) \\ &\leq c_0 \|\, R - R_\eta\,\|_{L^\infty(\Gamma_3)}\, meas\,(\Gamma_3)^{\frac{1}{2}} \|\, u_\eta - u\,\|_V + \\ &c_0^2 L_T \|\,\mu\,\|_{L^\infty(\Gamma_3)} \|\, u_\eta - u\,\|_V^2 + \\ &c_0 \|\,\mu - \mu_\eta\,\|_{L^\infty(\Gamma_3)} \|\, T(u_\eta)\,\|_{L^2(\Gamma_3)} \|\, u_\eta - u\,\|_V \leq \\ &c_0 \|\, R - R_\eta\,\|_{L^\infty(\Gamma_3)}\, meas\,(\Gamma_3)^{\frac{1}{2}} \|\, u_\eta - u\,\|_V + \\ &c_0^2 L_T \|\,\mu\,\|_{L^\infty(\Gamma_3)} \|\, u_\eta - u\,\|_V^2 + \\ &c_0 \|\,\mu - \mu_\eta\,\|_{L^\infty(\Gamma_3)} (c_0 L_T \|\, u_\eta\,\|_V + \\ &\|\, T(0_V)\,\|_{L^2(\Gamma_3)}) \|\, u_\eta - u\,\|_V\end{aligned} \quad (38)$$

Finally, using the Cauchy-Schwartz inequality, we obtain that:

$$(f_\eta - f, u_\eta - u)_V \leq \| f_\eta - f \|_V \| u_\eta - u \|_V \qquad (39)$$

Inequalities 35 to 39 were combined to deduce that:

$$(m_N - c_0^2 L_T \| \mu \|_{L^\infty(\Gamma_3)}) \| u_\eta - u \|_V \leq G(\eta)(c_0^2 \| u \|_V + c_0\, meas\, (\Gamma_3)^{\frac{1}{2}}) + c_0 \| R - R_\eta \|_{L^\infty(\Gamma_3)}\, meas\, (\Gamma_3)^{\frac{1}{2}} + c_0 \| \mu - \mu_\eta \|_{L^\infty(\Gamma_3)} (c_0 L_T \| u_\eta \|_V + \| T(0_V) \|_{L^2(\Gamma_3)}) + \| f_\eta - f \|_V \qquad (40)$$

Take $v = 0$ in inequality (Equation 28) and using inequality (Equation 13c and 13e) to see that:

$$m_N \| u_\eta \|_V^2 \leq (f_\eta, u_\eta)_V,$$

which implies:

$$\| u_\eta \|_V \leq \frac{1}{m_N} \| f_\eta \|_V$$

On the other hand, using Definitions (Equations 32 and 33), there exists a constant α which does not depend on η such that:

$$\| f_\eta \|_V \leq \alpha$$

and since $m_N > c_0^2 L_T \| \mu \|_{L^\infty(\Gamma_3)}$, it was deduced that:

$$\| u_\eta - u \|_V \leq c\, G(\eta)(\| u \|_V + 1) + c \| R - R_\eta \|_{L^\infty(\Gamma_3)} + c \| \mu - \mu_\eta \|_{L^\infty(\Gamma_3)} (\frac{\alpha}{m_N} + \| T(0_V) \|_{L^2(\Gamma_3)}) + c \| f_\eta - f \|_V \qquad (41)$$

where c is a positive constant which does not depend on η.

The convergence in Definition 34 is now a consequence of the Inequality 41 combined with Assumptions 29 to 33.

In addition to the mathematical interest in the convergence results in Definition 34, it is of importance from mechanical point of view, since it states that the weak solution of problem in Equations 7 to 12 depends continuously on the normal compliance function, the surface yield, the coefficient of friction and the densities of body forces and surface tractions, as well.

REGULARIZATION

In what follows, Problem P using the regularization method was studied. To this end, for each $\rho > 0$, the difference arises from the fact that here the function j define by Equation 21 with its regularization the function $j_\rho : V \times V \to \mathbb{R}$ defined by equalities were replaced:

$$j_\rho(u, v) = \int_{\Gamma_3} R\psi_\rho(v_\nu)\, da + \int_{\Gamma_3} \mu\phi_\rho(T(u))n^*.v_\tau\, da \quad \forall\, u \in V,\ v \in V \qquad (42)$$

Where $\psi_\rho : \mathbb{R} \to \mathbb{R}_+$ and $\phi_\rho : \mathbb{R} \to \mathbb{R}_+$ are the functions differentiable defined by the equalities:

$$\psi_\rho(x) = \begin{cases} \sqrt{x^2 + \rho^2} - \rho, & if \quad x > 0 \\ 0, & if \quad x \leq 0 \end{cases} \qquad (43)$$

and,

$$\phi_\rho(x) = \sqrt{x^2 + \rho^2} - \rho, \quad \forall x \in \mathbb{R} \qquad (44)$$

Let $\rho > 0$, considering the following lemma.

Lemma 3

Let $\rho > 0$. The functions ψ_ρ and ϕ_ρ defined by Equations 43 and 44, the following satisfies conditions:

$$\begin{cases} \text{Consider the function } h: \mathbb{R} \to \mathbb{R}_+ \text{ such that:} \\ (a)\ |h(x) - h(y)| \leq |x - y| \quad \forall x, y \in \mathbb{R}, \\ (b) |h(x) - |x|| \leq \rho \quad \forall x \in \mathbb{R}, \\ (c) |h(x)| \leq |x| \quad \forall x \in \mathbb{R}. \end{cases} \qquad (45)$$

Proof

(a) Let $\rho > 0$. Using Equation 44 of the function ϕ, to see that:

$$\phi_\rho(x) - \phi_\rho(y) = \sqrt{x^2 + \rho^2} - \sqrt{y^2 + \rho^2} = \frac{x^2 - y^2}{\sqrt{x^2+\rho^2} + \sqrt{y^2+\rho^2}} = (x - y).\frac{x+y}{\sqrt{x^2+\rho^2} + \sqrt{y^2+\rho^2}} \qquad (46)$$

and, since

$$\frac{x+y}{\sqrt{x^2+\rho^2} + \sqrt{y^2+\rho^2}} \leq 1 \qquad (47)$$

Combining Equation 46 and inequality (Equation 47) to obtain

$$|\phi_\rho(x) - \phi_\rho(y)| \leq |x - y| \qquad 48)$$

Consider $x > 0$, it follows from the Definitions (Equations 43 and 44) that:

$$\phi_\rho(x) = \psi_\rho(x) \quad if \quad x > 0 \tag{49}$$

Moreover, from Equations 48 and 49, it is deduced that:

$$|\psi_\rho(x) - \psi_\rho(y)| = |\phi_\rho(x) - \phi_\rho(y)| \le |x - y| \tag{50}$$

which conclude the first part of the proof.

(b) Let $\rho > 0$. Noting that for all $x \in \mathbb{R}$ it is obtained that:

$$\left|\phi_\rho(x) - |x|\right| = \left|\sqrt{x^2+\rho^2} - \rho - |x|\right| = \rho + |x| - \sqrt{x^2+\rho^2} \le \rho \tag{51}$$

Moreover, using Equation 49, it is deduced that:

$$\left|\psi_\rho(x) - |x|\right| = \left|\phi_\rho(x) - |x|\right| \le \rho \tag{52}$$

which concludes the second part of the proof.

(c) Next, using Equations 43 and 44, it is easy to see that:

$$\phi_\rho(0) = \psi_\rho(0) = 0 \tag{53}$$

Therefore, using Equation 50 with $y = 0$ and Equation 53, it is found that:

$$|\psi_\rho(x)| = |\phi_\rho(x))| \le |x| \tag{54}$$

Which completes the proof.

For all $u, v \in V$ and note that, again, the integral in Equation 42 is well-defined.
Using argument similar to those used in the previous section, using the previous equality, the following variational formulation of the sliding friction contact problem regularized was obtained.

Problem P_V^ρ

Find u_ρ such that:

$$u_\rho \in U, (N\,\varepsilon\,(u_\rho), \varepsilon(v) - \varepsilon\,(u_\rho))_Q + (Pu_\rho, v - u_\rho)_V + j_\rho(u_\rho, v) - j_\rho(u_\rho, u_\rho) \ge (f, v - u_\rho)_V \forall v \in U \tag{55}$$

The following are the existence, uniqueness and convergence results.

Theorem 4

Under the Assumptions (Equations13 to 18), there exist a constant μ_0, which depends only on Ω, Γ_1, Γ_3, N and T, such that:

1) The Problem P_V^ρ admits a unique solution if:

$$\| \mu \|_{L^\infty(\Gamma_3)} < \mu_0 \tag{56}$$

2) The solution u_ρ of Problem P_V^ρ converges strongly to the solution u of the Problem P_V, that is:

$$u_\rho \to u \quad in \ \ V \quad as \ \ \rho \to 0 \tag{57}$$

Proof 1

To solve the variational inequality (Equation 55), Theorem 1 with $X = V$ and $K = U$ was used. To this end, it is noted that U is a nonempty closed convex subset of V. Considering the operators $A: V \to V$ defined by:

$$(A\,u_\rho, v)_V = (N\,\varepsilon\,(u_\rho), \varepsilon\,(v))_Q + (P\,u_\rho, v)_V \quad \forall\ u,\ v \in V \tag{58}$$

Moreover, using definitions (Equations 13c and 20c) to see that:

$$(A\,u - A\,v, u - v)_V \ge m_N \| u - v \|_V^2 \qquad \forall\ u,\ v \in V \tag{59}$$

On the other hand, using definitions (Equations 13b, 14b) and the trace inequality (Equation 1) yield:

$$||A\,u - A\,v||_V \le (L_N + c_0^2 L_p) \| u - v \|_V \quad \forall\ u,\ v \in V \tag{60}$$

To see that A is a strongly monotone Lipschitz continuous operator on the space V.
Next, using the functional defined by Equation 42.

$$j_\rho(u, v) = \int_{\Gamma_3} R\psi_\rho(v_\nu)\,da + \int_{\Gamma_3} \mu\phi_\rho(T(u))n^*.v_\tau\,da$$

Moreover, using Equation 44 to obtain:

$$j_\rho(u, v) \le \int_{\Gamma_3} R|v_\nu|\,da + \int_{\Gamma_3} \mu|S(u)|n^*.v_\tau\,da.$$

Conditions (Equations 13 to 18), inequality (Equation 1) and the previous inequality were combined to see that:

$$j_\rho(u, v) \le c_0\left(\| R \|_{L^\infty(\Gamma_3)}\ meas\ (\Gamma_3)^{1/2} + \| \mu \|_{L^\infty(\Gamma_3)} \| T(u) \|_{L^2(\Gamma_3)}\right) \| v \|_V \tag{61}$$

Therefore, it is easy to see that the functional j_ρ satisfies Equation 5a.
Let u_1, u_2, v_1, $v_2 \in V$ using Equations 42 and 48, we find that:

$$j_\rho(u_1, v_2) - j_\rho(u_1, v_1) + j_\rho(u_2, v_1) - j_\rho(u_2, v_2) =$$
$$\int_{\Gamma_3} \mu\Big(\phi_\rho\big(T(u_1)\big) - \phi_\rho\big(T(u_2)\big)\Big) n^* \cdot (v_{2\tau} - v_{1\tau}) da \le$$
$$\int_{\Gamma_3} |\mu| \, \| T(u_1) - T(u_2) \| \| \, v_{2\tau} - v_{1\tau} \| \, da$$

Then we Definitions (Equations 17 and 18b) and Inequality (Equation 1) to see that:

$$j_\rho(u_1, v_2) - j_\rho(u_1, v_1) + j_\rho(u_2, v_1) - j_\rho(u_2, v_2) \le$$
$$c_0^2 L_T \| \mu \|_{L^\infty(\Gamma_3)} \| u_1 - u_2 \|_V \| v_1 - v_2 \|_V \tag{62}$$

Let

$$\mu_0 = \frac{m_N}{c_0^2 L_T} \tag{63}$$

and note that, clearly, μ_0 depends only on $\Omega, \Gamma_1, \Gamma_3, N$ and T. It follows from Inequality (Equation 62) that j_ρ satisfies condition (Equation 5b) with $\alpha = c_0^2 L_T \| \mu \|_{L^\infty(\Gamma_3)}$ and $m = m_F$. Assume Inequality (Equation 56), $c_0^2 L_T \| \mu \|_{L^\infty(\Gamma_3)} < m_N$ which implies that $m > \alpha$ was obtained, which concludes the first part the proof.

Let $\rho > 0, v = u$ in inequality (Equation 55) and $v = u_\rho$ in inequality (Equation 24). Then, adding the resulting inequalities to obtain:

$$(A u_\rho - A u, u_\rho - u)_V \le j_\rho(u_\rho, u) - j_\rho(u_\rho, u_\rho) + j(u, u_\rho) - j(u, u). \tag{64}$$

Then by Equations 13c, 21 and 42,

$$m_N \| u_\rho - u \|_V^2 \le \Delta_1 + \Delta_2 \tag{65}$$

where

$$\Delta_1 = \int_{\Gamma_3} R(u_{\rho\nu}^+ - u_\nu^+ + \psi_\rho(u_\nu) - \psi_\rho(u_{\rho\nu}))\, da$$
$$\Delta_2 = \int_{\Gamma_3} \mu(\phi_\rho(T(u_\rho)) - T(u))n^*.(u_\tau - u_{\rho\tau})\, da.$$

Next, using the definitions (Equations 43 and 52) to obtain:

$$\Delta_1 \le \int_{\Gamma_3} R(|\psi_\rho(u_{\rho\nu}) - u_{\rho\nu}^+| + |\psi_\rho(u_\nu) - u_\nu^+|)\, da$$
$$\le \int_{\Gamma_3} 2\rho R \, da$$

and, by Equation 13:

$$\Delta_1 \le 2\rho \| R \|_{L^\infty(\Gamma_3)} \, meas\,(\Gamma_3) \tag{66}$$

On the other hand, using the triangle inequality, it follows that:

$$\phi_\rho\Big(T(u_\rho)\Big) - T(u) = \phi_\rho\Big(T(u_\rho)\Big) - \phi_\rho(T(u)) + \phi_\rho(T(u)) - T(u) \le \Big|\phi_\rho\Big(T(u_\rho)\Big) - \phi_\rho(T(u))\Big| + |\phi_\rho(T(u)) - T(u)|$$

Combining definitions (Equations 18a, 18b, 50 and 52) and the previous inequality, it is seen that:

$$\phi_\rho(T(u_\rho)) - T(u) \le L_T \| u_\rho - u \| + \rho \tag{67}$$

Finally, using inequalities (Equations 66, 67, 17 and 1) to obtain:

$$\Delta_1 + \Delta_2 \le 2\rho \| R \|_{L^\infty(\Gamma_3)} \, meas\,(\Gamma_3) +$$
$$c_0^2 L_T \| \mu \|_{L^\infty(\Gamma_3)} \| u_\rho - u \|_V^2 +$$
$$c_0 \, \rho \, meas\,(\Gamma_3)^{1/2} \| u_\rho - u \|_V \tag{68}$$

Assume condition (Equation 56), it follows from Inequalities (Equations 65 and 68) that:

$$(m_N - \alpha) \| u_\rho - u \|_V^2 \le 2\rho \| R \|_{L^\infty(\Gamma_3)} \, meas\,(\Gamma_3) + c_0 \, \rho \, meas\,(\Gamma_3)^{1/2} \| u_\rho - u \|_V \tag{69}$$

where $\alpha = c_0^2 L_T \| \mu \|_{L^\infty(\Gamma_3)}$

(64)

Using inequality (Equation 69), the elementary inequality:

$$x, a, b \ge 0 \quad and \quad x^2 \le ax + b \quad \Longrightarrow \quad x^2 \le a^2 + 2b$$

As a result it is deduced that:

$$\| u_\rho - u \|_V^2 \to 0 \quad as \quad \rho \to 0 \tag{70}$$

Therefore, Equation 70 implies the convergence in Equation 57, which concludes the proof.

The interest in Theorem 4 is twofold: first, it provides the existence and uniqueness of the solution to the variational inequality (Equation 55); second, it shows that the solution of inequality (Equation 24) represents the strong limit of the sequence of the solution u_ρ of the problem of inequality (Equation 55), as $\rho \to 0$.

The convergence result in Equation 57 is important from the mechanical point of view, since it shows that the weak solution of the elastic the sliding frictional contact problem with normal compliance, and unilateral constraint may be approached as closely as one wishes by the solution of the sliding friction contact problem with normal compliance, unilateral constraint and regularized friction, with a sufficiently small regularization parameter.

NUMERICAL APPROXIMATION

Here, is devoted to the numerical discretization of the Problem P_V. Let $V^h \subset V$ be a linear finite element space on the domain, which is vanishing on the boundary Γ_1. We define the space:

$$U^h = \{ v^h \in V^h : v_\nu^h \leq g \quad a.e.\ on \quad \Gamma_3\} \tag{71}$$

where $h > 0$ denotes the spatial discretization parameter. It is easy to see that the finite dimensional space $U^h \subset U$ for the polygonal domain. The constraint condition $v^h \leq g$ on the boundary Γ_3 is satisfied at nodes, that is, $v^h \leq g^I$, where g^I is the linear interpolation of function g.

The following approximated solution for the Problem P_V are discussed.

Problem P_V^h

Find a displacement field u^h such that:

$$u^h \in U^h, (N\varepsilon(u^h), \varepsilon(v^h) - \varepsilon(u^h))_Q + (Pu^h, v^h - u^h)_V + j(u^h, v^h) - j(u^h, u^h) \geq (f, v^h - u^h)_V \forall v^h \in U^h \tag{72}$$

Under the assumptions (by Sofonea and Souleiman, 2015) and inequality (Equation 56), the discrete system of inequality (Equation 72) has a unique solution. Focusing on the error analysis between the solutions to problems P_V and P_V^h.

Theorem 5

Assume that conditions (Equations 13 to 18) and inequality (Equation 56) hold, then there exists a constant c independent of h such that:

$$\| u - u^h \|_V \leq c \inf_{v^h \in U^h} \left(\| u - v^h \|_V + \| u - v^h \|_V^{1/2} + \| u - v^h \|_{L^2(\Gamma_3)}^{1/2} \right) \tag{73}$$

Proof

By the assumptions (Equations 13c and 14c), and Equations 20 and 58 for any $v_h \in U^h$:

$$m_N \| \varepsilon(u) - \varepsilon(u^h) \|_Q^2 \leq \left(N\varepsilon(u) - N\varepsilon(u^h), \quad \varepsilon(u) - \varepsilon(u^h)\right) \leq \left(N\varepsilon(u) - N\varepsilon(u^h), \quad \varepsilon(u) - \varepsilon(u^h)\right) + \int_{\Gamma_3} \left(p(u_\nu) - p(u_\nu^h)\right)(u_\nu - u_\nu^h) da = (Au - Au^h, u - u^h) = \left(Au - Au^h, u - v^h\right) + (Au, v^h - u^h) - (Au^h, v^h - u^h) \tag{74}$$

Using the inequality (Equation 24) with $v = u^h$ and inequality (Equation 72) to find that:

$$(Au, v^h - u^h) - (Au^h, v^h - u^h) \leq (Au, v^h - u) + j(u^h, v^h) - j(u^h, u^h) + j(u, u^h) - j(u, u) + (f, u - v^h)_V \tag{75}$$

Then by inequalities (Equations 74 and 75):

$$m_F \| \varepsilon(u) - \varepsilon(u^h) \|_Q^2 \leq K_1 + K_2 + K_3 + K_4 \tag{76}$$

Where:

$$K_1 = (Au - Au^h, u - v^h)$$
$$K_2 = (Au, v^h - u) + j(u, v^h) - j(u, u) - (f, v^h - u)_V$$
$$K_3 = j(u, u^h) - j(u^h, u^h) + j(u^h, u) - j(u, u)$$
$$K_4 = j(u^h, v^h) - j(u, v^h) + j(u, u) - j(u^h, u)$$

Estimating term by term; for the first term, the same inequality as inequality (Equation 13b) is obtained:

$$|K_1| \leq | \int_\Omega (N\varepsilon(u) - N\varepsilon(u^h)) \cdot \varepsilon(u - v^h)\, dx + \int_{\Gamma_3} (p(u_\nu) - p(u_\nu^h))(u_\nu - v_\nu^h)\, da| \leq L_N \| u - u^h \|_V \| u - v^h \|_V + L_p \| u - u^h \|_{L^2(\Gamma_3)} \| u - v^h \|_{L^2(\Gamma_3)} \leq \left(L_N + c_0^2 L_p\right) \| u - u^h \|_V \| u - v^h \|_V \tag{77}$$

For the second term K_2, it can be viewed as a residual. Hence:

$$K_2 = \int_{\Omega} N\,\varepsilon(u)\cdot\varepsilon(v^h - u)dx +$$

$$\int_{\Gamma_3} p(u_\nu)(v_\nu^h - u_\nu)da +$$

$$\int_{\Gamma_3} R(v_\nu^{+h} - u_\nu)da +$$

$$\int_{\Gamma_3} \mu T(u)n^*.(v_\tau^h - u_\tau)da - (f, v^h - u)_V$$

After some elementary calculus based on the Assumptions (Equations 13 to 18):

$$|K_2| \le (\| N\,\varepsilon(u)\|_V + \| f \|_V)\, \| u - v^h \|_V +$$

$$(\| p(u_\nu)\|_{L^2(\Gamma_3)} + \| R \|_{L^\infty(\Gamma_3)}\, meas(\Gamma_3)^{\frac{1}{2}} +$$

$$L_T \| \mu \|_{L^\infty(\Gamma_3)} \| u \|_{L^2(\Gamma_3)} +$$

$$\| \mu \|_{L^\infty(\Gamma_3)} \| T(0_V)\|_{L^2(\Gamma_3)})\ \| u - v^h \|_{L^2(\Gamma_3)} \tag{78}$$

For the term K_3:

$$K_3 = \int_{\Gamma_3} \mu(T(u) - T(u^h))n^*.(u_\tau^h - u_\tau)\, da$$

Therefore, using Equations 1, 17 and 18b to obtain:

$$|K_3| \le c_0^2 L_S \| \mu \|_{L^\infty(\Gamma_3)} \| u - u^h \|_V^2 \tag{79}$$

For the last term K_4, using Equation 18b to obtain:

$$|K_4| = |j(u^h, v^h) - j(u^h, u) + j(u, u) - j(u, v^h)| =$$

$$\left|\int_{\Gamma_3} \mu\big(T(u^h) - T(u)\big)n^*.(v_\tau^h - u_\tau)da\right| \le$$

$$c_0^2 L_T \| \mu \|_{L^\infty(\Gamma_3)} \| u - u^h \|_V \| u - v^h \|_V \tag{80}$$

Under the hypothesis of Inequality (Equation 56), absorbing the third term K_3 of Inequality (Equation 79), using the elementary inequality:

$$ab \le \delta a^2 + \frac{1}{4\delta} b^2, \qquad \forall \delta > 0$$

The result of inequality 73 can be proved easily.

Note we obtain the error estimate by the trace inequality on boundary Γ_3:

$$\| u - u^h \|_V = O\left(\sqrt{\inf_{v^h \in U^h} \| u - v^h \|_V}\right)$$

It is the same order error estimates as presented in Inequality (Equation 1) and for the other mathematical model and not the optimal order. The optimal error estimates under extra regularity for the solution was derived.

Theorem 6

Under the assumptions of Theorem 5 and $\sigma \in (L^2(\Gamma_3))^d$, there exists a constant c independent of h such that:

$$\| u - u^h \|_V \le c \inf_{v^h \in U^h} (\| u - v^h \|_V + \| u - v^h \|_{L^2(\Gamma_3)}^{1/2}). \tag{81}$$

Proof

The estimate K_2 was done under extra regularity of the solution $\sigma \in L^2(\Gamma_3)^d$. Green's formula (Equation 2), Equations 7 to 10 and Definition (Equation 22) were used to show that:

$$(\sigma, \varepsilon(v) - \varepsilon(u))_Q = (f, v - u)_V + \int_{\Gamma_3} \sigma_\nu(v_\nu - u_\nu)da + \int_{\Gamma_3} \sigma_\tau\cdot(v_\tau - u_\tau)da \tag{82}$$

By Equation 82:

$$\begin{aligned}|K_2| &\le | \int_{\Gamma_3} \sigma_\nu(v_\nu^h - u_\nu)\, da + \int_{\Gamma_3} \sigma_\tau\cdot(v_\tau^h - u_\tau)\, da| \\ &\quad + \int_{\Gamma_3} \mu T(u)n^*.(v_\tau^h - u_\tau)\, da| \\ &\le (\| \sigma_\nu \|_{L^2(\Gamma_3)} + \| \sigma_\tau \|_{L^2(\Gamma_3)} + \| p(u_\nu)\|_{L^2(\Gamma_3)} + \| R \|_{L^\infty(\Gamma_3)}\, meas(\Gamma_3)^{1/2} \\ &\quad + L_T \| \mu \|_{L^\infty(\Gamma_3)} \| u \|_{L^2(\Gamma_3)} + \| \mu \|_{L^\infty(\Gamma_3)} \| S(0_V)\|_{L^2(\Gamma_3)})\ \| u - v^h \|_{L^2(\Gamma_3)} \\ &\le c\ \| u - v^h \|_{L^2(\Gamma_3)}\end{aligned}$$

Thus, the result of inequality (Equation 81) following the proof of Theorem 6 can be obtained.

To derive an order error estimate, similar theory (*cf.* Han and Sofonea, 2002) was used. Assume:

$$u \in H^2(\Omega)^d, \quad u_{|\Gamma_3} \in H^2(\Gamma_3)^d \tag{83}$$

Let $\Pi_h u \in V_h$ be the linear finite element interpolant of the solution u. As the solution $u \in U$, that is, $u_\nu \le g$, then $\Pi_h u \in U^h$. The standard finite element interpolation theory yields (*cf.* Ciarlet, 1978):

$$\| u - \Pi_h u \|_V \le c\, h\, | u |_{H^2(\Omega)^d},$$

$$\| u - \Pi_h u \|_{L^2(\Gamma_3)^N} \le c\, h^2 | u |_{H^2(\Gamma_3)^d}.$$

Therefore, under the regularity Assumption (Equation 83), the optimal error estimate is obtained:

$$\| u - u^h \|_V \le c\, h$$

where the constant c is independent of h.

CONFLICT OF INTERESTS

The author has not declared any conflict of interests.

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