Full Length Research Paper

An empirical optimal portfolio selection model

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We consider risk arising from changes in the prices of financial assets. We propose a risk measure based on asymptotic power law behaviour for optimal portfolio selection in a single period. We apply this measure to compute explicitly the optimal portfolio when the underlying security prices follow a Weibull distribution. An illustrative example is given.

Key words: Asymptotic power law, asset price change, Weibull distribution, portfolio selection, mathematics

INTRODUCTION

The purpose of portfolio selection is to find an optimal strategy for allocating wealth among a number of securities. The mean-variance approach initiated in Markowitz (1952) and Markowitz (1959), as basis for portfolio selection in single period has the goal of minimizing risk using the variance as a criterion.

The literature has mostly implicitly assumed that investors are primarily affected in their decision by the expected returns and its variance, and therefore it was acceptable to focus on a distribution characterized by its first two moments. Thus diffusion has been the standard model of uncertainty, despite empirical evidence that asset returns are not normally distributed.

Instead, in this paper we assume returns follow Weibull distribution (this is because Weibull distribution enables us to model asset returns in a natural way, make inferences about the parameters of the reduction of the process and predict the growth rate of the selected portfolio), and show that this distribution follows asymptotic power-law behaviour.

We propose a risk measure based on the power-law behaviour. We analyze the probability distribution of returns of 36 securities (Table 1) in Nigeria for a period of ten months with aim to quantify the incurred risk, as the variance of portfolio returns provides only limited quantification of incurred risk, as the distributions of returns have “fat tails” (Anderson et al., 1999).

The advantage of our approach is that it is a much simplified model and could be used as guide to obtain portfolio selection policies that are nearly as good as the optimal ones from practical concern.

The model

The investment opportunities are represented by n ‘long live’ securities with price process $S_n$ and price return $\frac{S_n(t + \Delta t) - 1}{S_n}$. We consider risk arising from changes in the prices of the financial assets on a single time period $T$.

Now consider the problem of an investor, who at the beginning of an investment period is faced with a series of decision on the optimal choice of investment that minimizes risk incurred. His goal will be to find an optimal strategy for allocating wealth among a number of securities. Mataz (2000) showed that it is possible to find an optimal investment strategy in terms of the probability density function describing the prices returns of a security. This strategy optimizes some appropriate measure of risk.

Let $S_n(t)$ be the price of the nth asset at time $t$ (time is counted for trading days in multiples of a fundamental units, say days).

We define the continuous returns as:

$$Z_n(t) = \ln S_n(t + \Delta t) - \ln S_n(t)$$

$$= \ln \left( \frac{S_n(t + \Delta t)}{S_n(t)} \right),$$

(1)

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Table 1. Returns of 36 securities (in thousands of naira), for ten months (January – October, 2007). Source of data: Aba exchange market, Abia State, Nigeria.

<table>
<thead>
<tr>
<th>January</th>
<th>February</th>
<th>March</th>
<th>April</th>
<th>May</th>
<th>June</th>
<th>July</th>
<th>August</th>
<th>September</th>
<th>October</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.685</td>
<td>1.642</td>
<td>1.570</td>
<td>1.537</td>
<td>1.587</td>
<td>1.702</td>
<td>1.897</td>
<td>1.506</td>
<td>1.787</td>
<td>1.787</td>
</tr>
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<td>16.5</td>
<td>16.02</td>
<td>16.62</td>
<td>18.53</td>
<td>15.22</td>
<td>17.21</td>
<td>13.65</td>
<td>16.89</td>
<td>16.89</td>
</tr>
<tr>
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<td>166.5</td>
<td>161.1</td>
<td>157.8</td>
<td>165.7</td>
<td>193.7</td>
<td>168.9</td>
<td>183.3</td>
<td>152.1</td>
<td>175.2</td>
</tr>
<tr>
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<td>1.828</td>
<td>1.893</td>
<td>2.210</td>
<td>1.769</td>
<td>2.089</td>
<td>1.685</td>
<td>2.033</td>
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<td>14501</td>
<td>10840</td>
<td>6947</td>
<td>6741</td>
<td>2345</td>
<td>8937</td>
<td>3181</td>
<td>4861</td>
<td>1102</td>
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</tr>
<tr>
<td>1.778</td>
<td>1.754</td>
<td>1.689</td>
<td>1.644</td>
<td>1.701</td>
<td>1.014</td>
<td>1.562</td>
<td>1.767</td>
<td>1.401</td>
<td>1.751</td>
</tr>
<tr>
<td>1.907</td>
<td>1.863</td>
<td>1.815</td>
<td>1.772</td>
<td>1.800</td>
<td>2.014</td>
<td>1.674</td>
<td>1.926</td>
<td>1.588</td>
<td>1.989</td>
</tr>
<tr>
<td>1.921</td>
<td>1.902</td>
<td>1.837</td>
<td>1.808</td>
<td>1.725</td>
<td>2.134</td>
<td>1.784</td>
<td>1.996</td>
<td>1.583</td>
<td>1.899</td>
</tr>
</tbody>
</table>

and the discrete returns as;

\[ r(t) = \frac{S_n(t + \Delta t) - S_n(t)}{S_n(t)}. \]  

(2)

The basic quantity of our study is the relative return rate of assets given by;

\[ G_n = \exp \{ Z_n(t) \} - 1, \text{ for each } n \]  

(3)

and the normalized price return;

\[ g_\Delta = \frac{G_\Delta - \mu_\Delta}{\beta_\Delta} , \]  

(4)

where \( \mu_\Delta \) and \( \beta_\Delta \) are the mean and the standard deviation respectively of \( G_\Delta \) and \( \Delta t \) is the time lag. Following Anderson et al. (1999), we assume that the normalized price return (4) is distributed according to the following probability distribution function;
\[
P(g_M) = \begin{cases} 
\frac{\beta}{\alpha} \left( \frac{g_M}{\alpha} \right)^{\beta-1} \exp \left( - \left( \frac{g_M}{\alpha} \right)^\beta \right) & \text{if } g_M > 0 \\
0 & \text{otherwise}
\end{cases}
\]

(5)

Where \( \beta > 0, \alpha > 0 \), the scale parameter \( \alpha \) is directly proportional to the mean of \( g_M(t) \), while the shape parameter \( \beta \) (or slope) provides more information about the properties of incurred risk mode. Given our assumption of Weibull distribution of asset returns, we define the strategy that optimizes the variance of the return distribution as:

\[
H(g_M(t)) = \int_0^\infty P(g_M(t)) dg_M(t)
\]

(6)

It is well known (Gopikrishman et al., 1998) that the distribution of large asset price changes shows characteristic power-law behaviour. We shall show that the optimal investment strategy \( H(g_M(t)) \) of (6) reduces to a power-law.

In the sequel we shall need the following lemma.

**Lemma 1:** If \( g_M(t) \) has the Weibull distribution of (5) and is given as in (4), then

\[
Y = \left( \frac{g_M(t) - \mu_M}{\beta_M} \right) ^\gamma
\]

has an exponential distribution with \( \alpha = 1 \).

**Proof**

To verify this assertion, we find the probability density function of \( Y \):

\[
f_Y(y) = \left| \frac{d}{dy} \left( \alpha y^{\beta_M} + \mu_M \right) \right| f_{g_M} \left( \alpha y^{\beta_M} + \mu_M \right)
\]

\[
= \begin{cases} 
\exp \left( - \left( \frac{g_M(t) - \mu_M}{\beta_M} \right)^\gamma \right) & \text{if } g_M > 0 \\
0 & \text{if } g_M < 0
\end{cases}
\]

(7)

(Olkin et al., 1980).

**Theorem 1:** If \( g_M(t) \) has the Weibull distribution, \( \beta_M \) given as in (13) and \( \gamma \) (the parameter representing risk) given as in (14). Then the optimal strategy has the power law distribution given as:

\[
H(g_M(t)) = \int_0^\infty \left[ - \left( \frac{G_M - \mu_M}{\beta_M} \right)^\gamma \right] dG_M(t).
\]

(8)

But from (4),

\[
dG_M(t) = \beta_M dg_M(t)
\]

so that

\[
H(g_M(t)) = \beta_M \int \exp \left[ - \left( g_M(t) \right)^\gamma \right] dg_M(t)
\]

(10)

Let

\[
x = g_M^\gamma,
\]

then

\[
dg_M(t) = \frac{1}{\gamma} (g_M(t))^{-\gamma} dx,
\]

and

\[
H(g_M(t)) = \frac{\beta_M}{\gamma} (g_M(t))^{\alpha(\gamma)}
\]

(11)

Where

\[
\alpha(\gamma) = 1 - \gamma
\]

(12)

is the characteristic exponent of (11) - the power-law distribution - which we shall use as the risk measure for the incurred risk of the investor in an investment decision. This risk measure of the portfolio can be explicitly computed as follows. In term of the normalized price returns, we estimate \( \beta_M \) thus:

\[
\beta_M = \frac{\sum_{j=1}^N Z_j}{N}
\]

(13)

where \( N \) is the number of securities and \( Z_j(t) \) the continuous return of each security, and
Table 2. Empirical result.

<table>
<thead>
<tr>
<th>N</th>
<th>$\beta_{\Delta t}$</th>
<th>$\gamma$</th>
<th>$\alpha(\gamma)$</th>
<th>$H_{\gamma}$</th>
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<tbody>
<tr>
<td>1</td>
<td>0.525</td>
<td>0.031</td>
<td>0.969</td>
<td>$0.907 \times 10^1$</td>
</tr>
<tr>
<td>2</td>
<td>2.761</td>
<td>0.186</td>
<td>0.814</td>
<td>$3.393 \times 10^1$</td>
</tr>
<tr>
<td>3</td>
<td>1.847</td>
<td>0.025</td>
<td>0.975</td>
<td>$1.347 \times 10^2$</td>
</tr>
<tr>
<td>4</td>
<td>5.130</td>
<td>0.022</td>
<td>0.978</td>
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</tr>
<tr>
<td>5</td>
<td>0.654</td>
<td>0.025</td>
<td>0.975</td>
<td>$1.729 \times 10^1$</td>
</tr>
<tr>
<td>6</td>
<td>8.560</td>
<td>2.343</td>
<td>-1.343</td>
<td>$2.04 \times 10^1$</td>
</tr>
<tr>
<td>7</td>
<td>0.462</td>
<td>0.169</td>
<td>0.831</td>
<td>$0.144 \times 10^1$</td>
</tr>
<tr>
<td>8</td>
<td>0.605</td>
<td>0.059</td>
<td>0.941</td>
<td>$0.638 \times 10^1$</td>
</tr>
<tr>
<td>9</td>
<td>0.617</td>
<td>0.026</td>
<td>0.974</td>
<td>$1.483 \times 10^1$</td>
</tr>
<tr>
<td>10</td>
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<td>$0.396 \times 10^1$</td>
</tr>
<tr>
<td>11</td>
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<td>0.072</td>
<td>0.928</td>
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</tr>
<tr>
<td>12</td>
<td>2.929</td>
<td>0.015</td>
<td>0.985</td>
<td>$5.628 \times 10^2$</td>
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<tr>
<td>13</td>
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<td>0.010</td>
<td>0.990</td>
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</tr>
<tr>
<td>14</td>
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<td>0.021</td>
<td>0.979</td>
<td>$3.445 \times 10^2$</td>
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<td>0.967</td>
<td>$2.046 \times 10^2$</td>
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<tr>
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<td>0.975</td>
<td>$4.135 \times 10^2$</td>
</tr>
<tr>
<td>17</td>
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<td>5.591</td>
<td>-4.591</td>
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</tr>
<tr>
<td>18</td>
<td>6.393</td>
<td>5.791</td>
<td>-4.791</td>
<td>$1.52 \times 10^4$</td>
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<tr>
<td>19</td>
<td>5.560</td>
<td>0.0185</td>
<td>0.9815</td>
<td>$1.619 \times 10^5$</td>
</tr>
<tr>
<td>20</td>
<td>8.762</td>
<td>5.486</td>
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<td>$9.44 \times 10^5$</td>
</tr>
<tr>
<td>21</td>
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<tr>
<td>22</td>
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<td>4.675</td>
<td>-3.675</td>
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<tr>
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<td>-3.336</td>
<td>$7.86 \times 10^3$</td>
</tr>
<tr>
<td>24</td>
<td>2.401</td>
<td>0.203</td>
<td>0.797</td>
<td>$2.377 \times 10^1$</td>
</tr>
<tr>
<td>25</td>
<td>4.195</td>
<td>4.573</td>
<td>-3.573</td>
<td>$5.46 \times 10^3$</td>
</tr>
<tr>
<td>26</td>
<td>5.203</td>
<td>0.026</td>
<td>0.974</td>
<td>$9.975 \times 10^2$</td>
</tr>
<tr>
<td>27</td>
<td>6.484</td>
<td>6.058</td>
<td>-5.058</td>
<td>$8.38 \times 10^5$</td>
</tr>
<tr>
<td>28</td>
<td>0.939</td>
<td>0.151</td>
<td>0.849</td>
<td>$0.594 \times 10^1$</td>
</tr>
<tr>
<td>29</td>
<td>0.454</td>
<td>0.020</td>
<td>0.980</td>
<td>$1.047 \times 10^1$</td>
</tr>
<tr>
<td>30</td>
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<td>0.019</td>
<td>0.981</td>
<td>$4.953 \times 10^5$</td>
</tr>
<tr>
<td>31</td>
<td>1.136</td>
<td>0.098</td>
<td>0.902</td>
<td>$1.301 \times 10^1$</td>
</tr>
<tr>
<td>32</td>
<td>7.529</td>
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<td>-4.677</td>
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<td>33</td>
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<td>0.976</td>
<td>$3.762 \times 10^5$</td>
</tr>
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<td>4.967</td>
<td>0.041</td>
<td>0.959</td>
<td>$5.635 \times 10^2$</td>
</tr>
<tr>
<td>35</td>
<td>1.496</td>
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<td>$1.946 \times 10^1$</td>
</tr>
<tr>
<td>36</td>
<td>5.123</td>
<td>0.019</td>
<td>0.981</td>
<td>$1.339 \times 10^3$</td>
</tr>
</tbody>
</table>

\[ \gamma = \left( \frac{Z_{\Delta t}(t) - \beta_{\Delta t}}{\lambda} \right)^{0.5} \quad (14) \]

Where

\[ \lambda = \ln(\Delta t) . \]

We further define for any $\gamma$ :

\[ H_{\gamma}(g_{\Delta t}(t)) = \frac{\beta_{\Delta t}(g_{\Delta t}(t))^{\gamma}}{\gamma} \quad (15) \]

Where $H_{\gamma}$ is the optimal portfolio. Let $\gamma$ be random variable representing a risk. It follows immediately from (15) that given a portfolio $H$ the associated risk measure is $H_{\gamma}$. Then the investor evaluates a risk $\gamma$ by calculating its incurred risk as defined in (14) and hence in (12). Thus, given a choice among $N$ securities, an investor would then pick the one having the optimal risk.

It follows that for $0 < \gamma < 1$ the investment $T_1$ is preferred to $T_2$ if and only if $H_{\gamma_1} \geq H_{\gamma_2}$, and for $\gamma > 1$ the investment $T_2$ is preferred to $T_1$ if and only if
\( \gamma_1 \leq \gamma_2 \).

One expects in this model that the behaviour of \( H_\gamma \) depends on the parameter \( \gamma \) in the following manner: If \( \gamma \) is small, say \( \gamma < 1 \), \( \alpha(\gamma) > 0 \) and \( H_\gamma \) is strictly increasing and the interval between the successive events \( Z(t) \) is stochastically decreasing. For a large value of \( \gamma \), say \( \gamma > 1 \), \( \alpha(\gamma) < 0 \), then \( H_\gamma \) is strictly decreasing and the interval between the successive events \( Z(t) \) is stochastically increasing. For \( \gamma = 1 \), \( \alpha(\gamma) = 0 \) and \( H_\gamma \) is constant.

**DISCUSSION AND CONCLUSION**

Risk is an important factor in determining how to efficiently manage a portfolio of investment because it determines the variation in returns on the asset and portfolio and gives investors a mathematical basis for investment decisions. A representation of the risk associated with a security (stocks, bonds, property, etc), or the risk of a portfolio of securities is the standard deviation. The overall concept of risk is that investors should expect a higher return on an investment when a said investment carries a higher level of risk.

However, this concept is not generally true. Our findings (Table 2) show that some investments have high level of returns with a lower level of risk while some have low level of returns with a higher level of risk, when we measure the risk of a portfolio security using \( \gamma \) (which is equivalent to the standard deviation of the investment tool in question). But this concept seems to agree with our findings if we are using (12) as our risk measure (Table 2). We therefore leave open, the investment decision to the investors depending whether he is risk-averse, risk-loving, or risk-neutral.

**REFERENCES**


