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Full Length Research Paper

An empirical optimal portfolio selection model

A. C. Okoroafor and B. O. Osu*

Department of Mathematics, Abia State University, Uturu, Nigeria.

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We consider risk arising from changes in the prices of financial assets. We propose a risk measure based on asymptotic power law behaviour for optimal portfolio selection in a single period. We apply this measure to compute explicitly the optimal portfolio when the underlying security prices follow a Weibull distribution. An illustrative example is given.

Key words: Asymptotic power law, asset price change, Weibull distribution, portfolio selection, mathematics subject classification: 91B60, 91B26.

INTRODUCTION

The purpose of portfolio selection is to find an optimal strategy for allocating wealth among a number of secureties. The mean-variance approach initiated in Markowitz (1952) and Markowitz (1959), as basis for portfolio selection in single period has the goal of minimizing risk using the variance as a criterion.

The literature has mostly implicitly assumed that investors are primarily affected in their decision by the expected returns and its variance, and therefore it was acceptable to focus on a distribution characterized by its first two moments. Thus diffusion has been the standard model of uncertainty, despite empirical evidence that asset returns are not normally distributed.

Instead, in this paper we assume returns follow Weibull distribution (this is because Weibull distribution enables us to model asset returns in a natural way, make inferences about the parameters of the reduction of the process and predict the growth rate of the selected portfolio), and show that this distribution follows asymptotic power- law behaviour.

We propose a risk measure based on the power-law behaviour. We analyze the probability distribution of returns of 36 securities (Table 1) in Nigeria for a period of ten months with aim to quantify the incurred risk, as the variance of portfolio returns provides only limited quantification of incurred risk, as the distributions of returns have “fat tails” (Anderson et al., 1999).

The advantage of our approach is that it is a much simplified model and could be used as guide to obtain portfolio selection policies that are nearly as good as the optimal ones from practical concern.

*Corresponding author. E-mail: megaobrait@yahoo.com, dracokoroafor@yahoo.com

The model

The investment opportunities are represented by n ‘long live’ securities with price process S_n and price return $\frac{S_n(t+\Delta t)}{S_n}-1$ distributed according to Weibull distribution. We consider risk arising from changes in the prices of the financial assets on a single time period T.

Now consider the problem of an investor, who at the beginning of an investment period is faced with a series of decision on the optimal choice of investment that minimizes risk incurred. His goal will be to find an optimal strategy for allocating wealth among a number of securities. Mataz (2000) showed that it is possible to find an optimal investment strategy in terms of the probability density function describing the prices returns of a security. This strategy optimizes some appropriate measure of risk.

Let $S_n(t)$ be the price of the nth asset at time t (time is counted for trading days in multiples of a fundamental units, say days).

We define the continuous returns as:

$$Z_n(t) = InS_n(t+\Delta t) - InS_n(t) = In\left(\frac{S_n(t+\Delta t)}{S_n(t)}\right), \quad (1)$$

Table 1. Returns of 36 securities (in thousands of naira), for ten months (January – October, 2007). Source of data: Aba exchange market, Abia State, Nigeria.

January	February	March	April	May	June	July	August	September	October
1.685	1.642	1.570	1.537	1.657	1.980	1.702	1.897	1.506	1.787
15.26	15.84	15.47	15.02	15.62	18.53	15.22	17.21	13.65	16.89
6.909	6.613	6.315	6.322	5.983	7.078	6.042	6.543	5.347	6.592
168.7	166.5	161.1	157.8	165.7	193.7	168.9	188.3	152.1	175.2
1.963	1.943	1.874	1.828	1.893	2.210	1.769	2.089	1.685	2.033
14501	10840	6947	6741	2345	8937	3181	4861	1102	5213
1.778	1.754	1.689	1.644	1.701	1.014	1.562	1.767	1.401	1.751
1.907	1.863	1.815	1.772	1.800	2.014	1.674	1.926	1.588	1.989
1.921	1.902	1.837	1.808	1.725	2.134	1.784	1.996	1.583	1.899
66.34	62.90	59.54	60.07	71.71	84.23	14.21	74.47	61.3	74.62
13.952	16.367	13.271	12.948	13.156	15.602	13.544	15.492	12.452	14.448
1.928	1.878	1.848	1.799	1.772	2.098	1.833	2.030	1.63	1.921
13.8	13.50	13.17	12.78	13.09	15.54	13.49	15.36	12.33	14.29
15.1	14.96	14.46	14.24	14.54	17.25	14.66	17.20	13.77	15.69
14.75	14.38	13.99	13.87	13.57	16.05	11.93	15.29	12.57	15.52
2018	1962	1896	186.3	185.2	2208	191.2	2143	171.5	2031
1602	15750	1517	15864	15171	1850	1503	18297	14442	1804
190.1	1898	184.8	176.1	1855	2210	177.9	208.5	1641	198.2
260.0	257.3	252.4	246.7	255.5	289.8	253.9	283.7	229.6	275.3
2069	2030	19762	19355	19291	2268	1896	22063	17856	2148
159.1	154.7	149.8	94.87	94.67	97.51	84.25	86.71	132.3	164.6
164.4	160.6	156.5	15.23	15.13	17.97	136.5	163.3	128.4	167.8
142.6	139.4	135.5	13.22	13.73	16.31	132.4	150.4	118.6	149.1
12.182	12.08	12.207	9.954	9.116	15.487	8.980	15.425	8.069	9.440
139.7	136.5	133.2	12.96	13.11	15.90	119.9	136.6	106.6	138.1
183.3	179.1	174.2	175.1	176.6	209.6	170.6	200.2	160.1	195.7
164.0	169.2	159.5	153.0	1584	1861	1598	1813	1469	1691
2.204	2.109	2.010	2.002	2.612	3.071	2.703	3.341	2.726	3.257
1.587	1.553	1.514	1.477	1.505	1.772	1.542	1.759	1.429	1.645
20104	19960	19250	19145	19560	22880	20120	22645	18426	20747
3.290	3.171	3.080	2.889	3.142	4.162	2.704	3.300	2.280	3.483
912.4	862.8	815.8	819.9	937.5	11090	973.7	10487	856.8	10371
17765	17406	16967	16403	17304	20271	16743	18680	14983	18288
144.2	145.1	140.9	137.2	137.2	165.2	123.7	147.5	117.7	155.5
4.554	4.397	14.225	4.063	4.259	6.141	4.051	4.913	3.304	5.302
171.1	166.7	162.6	159.4	164.0	192.12	159.4	179.8	151.6	176.7

and the discrete returns as ;

$$r(t) = \frac{S_n(t+\Delta t) - S_n(t)}{S_n(t)}. \quad (2)$$

The basic quantity of our study is the relative return rate of assets given by;

$$G_{\Delta t} = \exp\{Z_n(t)\} - 1, \text{ for each n} \quad (3)$$

and the normalized price return;

$$g_{\Delta t} = \frac{G_{\Delta t} - \mu_{\Delta t}}{\beta_{\Delta t}}, \quad (4)$$

where $\mu_{\Delta t}$ and $\beta_{\Delta t}$ are the mean and the standard deviation respectively of $G_{\Delta t}$ and Δt is the time lag. Following Anderson et al. (1999), we assume that the normalized price return (4) is distributed according to the following probability distribution function;

$$P(g_{\Delta t}) = \begin{cases} \frac{\beta}{\alpha}\left(\frac{|g_{\Delta t}(t)|}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{|g_{\Delta t}(t)|}{\alpha}\right)^{\beta}\right\} & \text{if } g_{\Delta t}(t) \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

Where $\beta > 0, \alpha > 0,$ the scale parameter α is directly proportional to the mean of $g_{\Delta t}(t)$, while the shape parameter β (or slope) provides more information about the properties of incurred risk mode. Given our assumption of Weibull distribution of asset returns, we define the strategy that optimizes the variance of the return distribution as:

$$H(g_{\Delta t}(t)) = \int_0^{\infty} P(g_{\Delta t}(t))dg_{\Delta t}(t) \quad (6)$$

It is well known (Gopikrishman et al., 1998) that the distribution of large asset price changes shows characteristic power-law behaviour. We shall show that the optimal investment strategy $H(g_{\Delta t}(t))$ of (6) reduces to a power-law.

In the sequel we shall need the following lemma.

Lemma 1: If $g_{\Delta t}(t)$ has the Weibull distribution of (5) and is given as in (4), then

$$Y = \left(\frac{G_{\Delta t}(t) - \mu_{\Delta t}}{\beta_{\Delta t}}\right)^{\gamma},$$

has an exponential distribution with $\alpha = 1$.

Proof

To verify this assertion, we find the probability density function of *Y*:

$$f_Y(y) = \left|\frac{d}{dy}\left(\alpha y^{1/\beta} + \mu_{\Delta t}\right)\right| f_{G_{\Delta t}}\left(\alpha y^{1/\beta} + \mu_{\Delta t}\right)$$

$$= \begin{cases} \exp\left\{-\left(\frac{G_{\Delta t}(t) - \mu_{\Delta t}}{\beta_{\Delta t}}\right)^{\gamma}\right\} & \text{if } g_{\Delta t} \geq 0 \\ 0 & \text{if } g_{\Delta t} < 0 \end{cases}, \quad (7)$$

(Olkin et al., 1980).

Theorem 1: If $g_{\Delta t}(t)$ has the Weibull distribution, $\beta_{\Delta t}$ given as in (13) and γ (the parameter representing risk) given as in (14). Then the optimal strategy has the power law distribution given as:

$$H(g_{\Delta t}(t) = \frac{\beta_{\Delta t}}{\gamma}(g_{\Delta t}(t))^{\alpha(\gamma)}$$

Proof: Using lemma 1, we can now show the power-law characteristics of (6) as follows:

$$H(g_{\Delta t}(t)) = \int_0^{\infty} \exp\left\{-\left(\frac{G_{\Delta t} - \mu_{\Delta t}}{\beta_{\Delta t}}\right)^{\gamma}\right\} dG_{\Delta t}(t). \quad (8)$$

But from (4),

$$dG_{\Delta t}(t) = \beta_{\Delta t} dg_{\Delta t}(t) \quad (9)$$

so that

$$H(g_{\Delta t}(t)) = \beta_{\Delta t} \int \exp\left\{-(g_{\Delta t}(t))^{\gamma}\right\} dg_{\Delta t}(t) \quad (10)$$

Let

$$x = g_{\Delta t}^{\gamma},$$

then

$$dg_{\Delta t}(t) = \frac{1}{\gamma}(g_{\Delta t}(t))^{1-\gamma} dx,$$

and

$$H(g_{\Delta t}(t) = \frac{\beta_{\Delta t}}{\gamma}(g_{\Delta t}(t))^{\alpha(\gamma)} \quad (11)$$

Where

$$\alpha(\gamma) = 1 - \gamma, \quad (12)$$

is the characteristic exponent of (11) -the power-law distribution- which we shall use as the risk measure for the incurred risk of the investor in an investment decision. This risk measure of the portfolio can be explicitly computed as follows. In term of the normalized price returns, we estimate $\beta_{\Delta t}$ thus:

$$\beta_{\Delta t} = \frac{\sum_{j=1}^{N} Z_j}{N} \quad (13)$$

where N is the number of securities and $Z_j(t)$ the continuous return of each security, and

Table 2. Empirical result.

N	$\beta_{\Delta t}$	γ	$\alpha(\gamma)$	H_{γ}
1	0.525	0.031	0.969	0.907×10^{1}
2	2.761	0.186	0.814	3.393×10^{1}
3	1.847	0.025	0.975	1.347×10^{2}
4	5.130	0.022	0.978	1.154×10^{3}
5	0.654	0.025	0.975	1.729×10^{1}
6	8.560	2.343	-1.343	2.04×10^{-1}
7	0.462	0.169	0.831	0.144×10^{1}
8	0.605	0.059	0.941	0.638×10^{1}
9	0.617	0.026	0.974	1.483×10^{1}
10	4.068	1.011	-0.011	0.396×10^{1}
11	2.546	0.072	0.928	8.417×10^{1}
12	2.929	0.015	0.985	5.628×10^{2}
13	2.618	0.010	0.990	6.788×10^{2}
14	2.718	0.021	0.979	3.445×10^{2}
15	2.649	0.033	0.967	2.046×10^{2}
16	9.766	0.025	0.975	4.135×10^{2}
17	8.540	5.591	-4.591	8.09×10^{-5}
18	6.393	5.791	-4.791	1.52×10^{-4}
19	5.560	0.0185	0.9815	1.619×10^{3}
20	8.762	5.486	-4.486	9.44×10^{-5}
21	4.314	1.20	-0.20	0.271×10^{1}
22	4.355	4.675	-3.675	4.18×10^{-3}
23	4.248	4.336	-3.336	7.86×10^{-3}
24	2.401	0.203	0.797	2.377×10^{1}
25	4.195	4.573	-3.573	5.46×10^{-3}
26	5.203	0.026	0.974	9.975×10^{2}
27	6.484	6.058	-5.058	8.38×10^{-5}
28	0.939	0.151	0.849	0.594×10^{1}
29	0.454	0.020	0.980	1.047×10^{1}
3O	9.915	0.019	0.981	4.953×10^{3}
31	1.136	0.098	0.902	1.301×10^{1}
32	7.529	5.677	-4.677	1.05×10^{-4}
33	9.766	0.024	0.976	3.762×10^{3}
34	4.967	0.041	0.959	5.635×10^{2}
35	1.496	0.110	0.890	1.946×10^{1}
36	5.123	0.019	0.981	1.339×10^{3}

$$\gamma = \left(\frac{Z_j(t) - \beta_{\Delta t}}{\lambda} \right)^{0.5} \tag{14}$$

Where

$\lambda = In\,\Delta t$.

We further define for any γ ,

$$H_{\gamma}\left(g_{\Delta t}(t)\right) = \frac{\beta_{\Delta t}\left(g_{\Delta t}(t)\right)^{1-\gamma}}{\gamma} \tag{15}$$

Where H_{γ} is the optimal portfolio. Let γ be random variable representing a risk. It follows immediately from (15) that given a portfolio *H* the associated risk measure is H_{γ}. Then the investor evaluates a risk γ by calculating its incurred risk as defined in (14) and hence in (12). Thus, given given a choice among N securities, an investor would then pick the one having the optimal risk.

It follows that for $0<\gamma<1$ the investment T_1 is preferred to T_2 if and only if $H_{\gamma_1} \geq H_{\gamma_2}$, and for $\gamma>1$ the investment T_2 is preferred to T_1 if and only if

$H_{\gamma_1} \leq H_{\gamma_2}$.

One expects in this model that the behaviour of H_γ depends on the parameter γ in the following manna; If γ is small, say $\gamma < 1, \alpha(\gamma) > 0$ and H_γ is strictly increasing and the interval between the successive events *Z(t)* is stochastically decreasing. For a large value of γ, say $\gamma > 1, \alpha(\gamma) < 0$, then H_γ is strictly decreasing and the interval between the successive events *Z(t)* is stochastically increasing. For $\gamma = 1, \alpha(\gamma) = 0$ and H_γ is constant.

DISCUSSION AND CONCLUSION

Risk is an important factor in determining how to efficiently manage a portfolio of investment because it determines the variation in returns on the asset and portfolio and gives investors a mathematical basis for investment decisions. A representation of the risk associated with a security (stocks, bonds, property, etc), or the risk of a portfolio of securities is the standard deviation. The overall concept of risk is that investors should expect a higher return on an investment when a said investment carries a higher level of risk.

However, this concept is not generally true. Our findings (Table 2) show that some investments have high level of returns with a lower level of risk while some have low level of returns with a higher level of risk, when we measure the risk of a portfolio security using γ (which is equivalent to the standard deviation of the investment tool in question). But this concept seems to agree with our findings if we are using (12) as our risk measure (Table 2). We therefore leave open, the investment decision to the investors depending whether he is risk-averse, risk-loving, or risk-neutral.

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