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# Multivalent harmonic uniformly convex functions

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**In this paper, several properties of the multivalent harmonic uniformly convex classes $K_H(m,\alpha)$ and $\overline{K_H}(m,\alpha)$ were investigated. Coefficient bounds, distortion theorem, extreme points, convolution condition, convex combinations and integral operator for these classes were obtained.**

**Key words:** Harmonic, multivalent functions, convex, convolution.

## INTRODUCTION

A continuous complex valued function $f=u+iv$ which is defined in a simply connected complex domain *D* is said to be harmonic in $D$ if both $u$ and $v$ are real harmonic in *D*. In any simply connected domain we can write:

$$f(z)=h(z)+\overline{g(z)}, \tag{1}$$

where *h* and *g* are analytic in *D*. We call *h* the analytic part and *g* the co-analytic part of *f*. A necessary and sufficient condition for *f* to be locally univalent and sense-preserving in $D$ is that $|h'(z)|>|g'(z)|$ in *D* (Clunie and Sheil-Small, 1984).

Denote by $S_H$, the class of functions *f* of the form (2) that are harmonic univalent and sense preserving in the unit disc $U=\{z:|z|<1\}$ for which $f(0)=f_z(0)-1=0.$ For $f=h+\overline{g}\in S_H$, we may express:

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k+\sum_{k=1}^{\infty}\overline{b_k z^k},\ |b_1|<1, \tag{2}$$

where the analytic functions $h$ and $g$ are of the form:

$$h(z)=z+\sum_{k=2}^{\infty}a_k z^k,\quad g(z)=\sum_{k=2}^{\infty}b_k z^k,\ |b_1|<1. \tag{3}$$

Clunie and Sheil-Small (1984) investigated the class $S_H$ as well as its geometric subclasses and some coefficient bounds for functions in $S_H$ were obtained. Since then, various subclasses of $S_H$ were investigated by several authors (Al-Shaqsi and Darus, 2008; Chandrashekar et al., 2009; Jahangiri, 1999; Murugusundaramoorthy, 2003; Murugusundaramoorthy et al., 2009; Rosy et al., 2001).

Recently, Kanas and Wisniowska (1999), Kanas and Srivastava (2000) studied the class of $k$ -uniformly convex analytic functions. For $m\geq 1$ and $0\leq\alpha<1,$ we let

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$H(m)$ denote the class of multivalent harmonic functions $f(z)=h(z)+\overline{g(z)},$ where

$$h(z)=z^m+\sum_{k=m+1}^{\infty}a_k z^k,\ g(z)=\sum_{k=m}^{\infty}b_k z^k,\ |b_m|<1. \quad (4)$$

We consider the class $K_H(m,\alpha)$ of functions of the form (1) where $h$ and $g$ are given by Equation (4) satisfying the inequality

$$\operatorname{Re}\left(\frac{z^2h''(z)+zh'(z)+\overline{z^2g''(z)}+\overline{zg'(z)}}{zh'(z)-\overline{zg'(z)}}\right) \geq \left|\frac{z^2h''(z)+zh'(z)+\overline{z^2g''(z)}+\overline{zg'(z)}}{zh'(z)-\overline{zg'(z)}}-m\right|+m\alpha, \quad (5)$$

where $m\geq 1$ and $0\leq\alpha<1.$

Using the fact that $\operatorname{Re}(w)>|w-m|+m\alpha \Leftrightarrow \operatorname{Re}[(1+e^{i\varphi})w-me^{i\varphi}]\geq m\alpha,$ it follows from the condition (5) that *f* is in the class $K_H(m,\alpha)$ if and only if

$$\operatorname{Re}\left\{m+(1+e^{i\varphi})\left(\frac{z^2h''(z)+zh'(z)+\overline{z^2g''(z)}+\overline{zg'(z)}}{zh'(z)-\overline{zg'(z)}}-m\right)\right\}\geq m\alpha, \quad (6)$$

where $m\geq 1$ and $0\leq\alpha<1.$

We note that:

Putting $m=1,$ $K_H(1,\alpha)=HCV(1,\alpha)$ (Kim et al., 2002).

Further, let $\overline{K}_H(m,\alpha)$ be the subclass of $K_H(m,\alpha)$ consisting of functions of the form:

$$f(z)=z^m-\sum_{k=m+1}^{\infty}|a_k|z^k-\sum_{k=m}^{\infty}|b_k|\overline{z}^k\ \ (|b_m|<1). \quad (7)$$

Recent interest in the study of multivalent harmonic functions in the plan prompted the publication of several articles, such as Ahuja and Jahangiri (2001, 2002, 2003), Bshouty et al. (1999), Guney and Ahuja (2006).

In this paper, the coefficient bounds for the classes $K_H(m,\alpha)$ and $\overline{K}_H(m,\alpha)$. as well as distortion theorem, extreme points, convolution, convex combinations and integral operator for functions in the class $\overline{K}_H(m,\alpha)$ were obtained.

## COEFFICIENTS BOUNDS AND DISTORTION THEOREM

Unless otherwise mentioned, it was assumed in the course of this study that $0\leq\alpha<1,\ m\geq 1$ and $z\in U$. We began with a sufficient condition for functions in the classes $K_H(m,\alpha)$ and $\overline{K}_H(m,\alpha)$ and obtained distortion theorem for functions in the class $\overline{K}_H(m,\alpha)$.

### Theorem 1

*Let* $f=h+\overline{g},$ where $h$ and $g$ are given by Equation (4), and satisfy the condition

$$\sum_{k=m+1}^{\infty}\frac{k[2k-m(1+\alpha)]}{m(1-\alpha)+1-|m(1-\alpha)-1|}|a_k| + \sum_{k=m}^{\infty}\frac{k[2k+m(1+\alpha)]}{m(1-\alpha)+1-|m(1-\alpha)-1|}|b_k|\leq\frac{1}{2}. \quad (8)$$

Then $f(z)\in K_H(m,\alpha).$

### Proof

Assume that Equation (8) holds. It suffices to prove that

$$\operatorname{Re}\left\{m+(1+e^{i\varphi})\left(\frac{z^2h''(z)+zh'(z)+\overline{z^2g''(z)}+\overline{zg'(z)}}{zh'(z)-\overline{zg'(z)}}-m\right)-m\alpha\right\} =\operatorname{Re}\frac{A(z)}{B(z)}\geq 0. \quad (9)$$

Using the fact that $\operatorname{Re}\{w\}\geq 0$ if and only if $|1+w|\geq|1-w|$, it suffices to show that

$$|A(z)+B(z)|-|A(z)-B(z)|\geq 0, \quad (10)$$

where

$$A(z)=[(1-m)\left(1+e^{i\varphi}\right)+m(1-\alpha)]zh'(z)+(1+e^{i\varphi})z^2h''(z) +[(1+m)\left(1+e^{i\varphi}\right)-m(1-\alpha)]\overline{zg'(z)}+(1+e^{i\varphi})\overline{z^2g''(z)}$$

and

$$B(z)=zh'(z)-\overline{zg'(z)}.$$

Substituting for *A(z)* and *B(z)* in the left side of Equation (10) we obtain:

$$|A(z)+B(z)|-|A(z)-B(z)| =\left|m[1+m(1-\alpha)]z^m+\sum_{k=m+1}^{\infty}k[k-m\alpha+(k-m)e^{i\varphi}+1]a_k z^k\right.$$

$$+\sum_{k=m}^{\infty} k[k+m\alpha+(k+m)e^{i\varphi}-1]\overline{b_k z^k}\Big|$$

$$-\Big|m[-1+m(1-\alpha)]z^m+\sum_{k=m+1}^{\infty} k[k-m\alpha+(k-m)e^{i\varphi}-1]a_k z^k$$

$$+\sum_{k=m}^{\infty} k[k+m\alpha+(k+m)e^{i\varphi}+1]\overline{b_k z^k}\Big|$$

$$\geq m[m(1-\alpha)+1)]|z|^m-\sum_{k=m+1}^{\infty} k[2k-m(1+\alpha)+1]|a_k||z|^k$$

$$-\sum_{k=m+1}^{\infty} k[2k+m(1+\alpha)-1]|b_k||z|^k$$

$$-|[m(m(1-\alpha)-1)]||z|^m-\sum_{k=m+1}^{\infty} k[2k-m(1+\alpha)-1]|a_k||z|^k$$

$$-\sum_{k=m}^{\infty} k[2k+m(1+\alpha)+1]|b_k||z|^k$$

$$\geq m[(m(1-\alpha)+1)-|(m(1-\alpha)-1)|]|z|^m\cdot$$

$$\cdot\left\{1-\sum_{k=m+1}^{\infty}\frac{2k[2k-m(1+\alpha)]}{m[(m(1-\alpha)+1)-|(m(1-\alpha)-1)|]}|a_k||z|^{k-m}\right.$$

$$\left.-\sum_{k=m}^{\infty}\frac{2k[2k+m(1+\alpha)]}{m[(m(1-\alpha)+1)-|(m(1-\alpha)-1)|]}|a_k||z|^{k-m}\right\}$$

$$\geq m[m(1-\alpha)+1)-|(m(1-\alpha)-1)|]\cdot$$

$$\cdot\left\{1-\sum_{k=m+1}^{\infty}\frac{2k[2k-m(1+\alpha)]}{m[(m(1-\alpha)+1)-|(m(1-\alpha)-1)|]}|a_k|\right.$$

$$\left.-\sum_{k=m}^{\infty}\frac{2k[2k+m(1+\alpha)]}{m[(m(1-\alpha)+1)-|(m(1-\alpha)-1)|]}|b_k|\right\} \geq 0,$$

using Equation (8).

The functions

$$f(z)=z^m+\sum_{k=m+1}^{\infty}\frac{m[(m(1-\alpha)+1)-|(m(1-\alpha)-1)|]}{k[2k-m(1+\alpha)]}x_k z^k$$

$$+\sum_{k=m}^{\infty}\frac{m[(m(1-\alpha)+1)-|(m(1-\alpha)-1)|]}{k[2k+m(1+\alpha)]}\overline{y_k z^k}, \tag{11}$$

where $\sum_{k=m+1}^{\infty}|x_k|+\sum_{k=m}^{\infty}|y_k|=1,$ shows that the coefficient bound given by Equation (8) is sharp. This completes the proof of Theorem 1.

**Corollary 1**

Let $f=h+\overline{g},$ where $h$ and $g$ are given by Equation (4). Also, let $1\leq m\leq 1/(1-\alpha)$ and if the condition:

$$\sum_{k=m+1}^{\infty}\frac{k[2k-m(1+\alpha)]}{m^2(1-\alpha)}|a_k|+\sum_{k=m}^{\infty}\frac{k[2k+m(1+\alpha)]}{m^2(1-\alpha)}|b_k|\leq 1, \tag{12}$$

holds, then $f(z)\in K_H(m,\alpha).$

**Corollary 2**

Let $f=h+\overline{g},$ where $h$ and $g$ are given by Equation (4). Also, let $m\geq 1/(1-\alpha)$ and if the condition

$$\sum_{k=m+1}^{\infty} k[2k-m(1+\alpha)]|a_k|+\sum_{k=m}^{\infty} k[2k+m(1+\alpha)]|b_k|\leq 1, \tag{13}$$

holds, then $f(z)\in K_H(m,\alpha).$

In the following theorem, it is shown that the condition (12) is also necessary for function $f=h+g$, where $f$ is of the form (7).

**Theorem 2**

Let $f=h+\overline{g},$ be given by the form (7). Then $f(z)\in \overline{K}_H(m,\alpha),$ if and only if the coefficient bound (12) holds.

**Proof**

Since $\overline{K}_H(m,\alpha)\subseteq K_H(m,\alpha),$ we only need to prove this part of the theorem. To this end, for functions $\overline{K}_H(m,\alpha),$ it was noticed that the necessary and sufficient condition to be in the class $\overline{K}_H(m,\alpha)$ is that:

$$\operatorname{Re}\left\{m+(1+e^{i\varphi})\left(\frac{z^2h''(z)+zh'(z)+\overline{z^2g''(z)}+\overline{zg'(z)}}{zh'(z)-\overline{zg'(z)}}-m\right)-m\alpha\right\}\geq 0. \tag{14}$$

This is equivalent to

$$\operatorname{Re}\left\{\frac{m^2(1-\alpha)z^m-\sum_{k=m+1}^{\infty}k[2k-m(1+\alpha)]|a_k|z^k-\sum_{k=m}^{\infty}k[2k+m(1+\alpha)]|b_k|\overline{z}^k}{z^m-\sum_{k=m+1}^{\infty}k|a_k|z^k+\sum_{k=m}^{\infty}k|b_k|\overline{z}^k}\right\}\geq 0.$$

This condition must hold for all values of $z\in U$ and for real α so that on taking $z=r<1$ , the above inequality reduces to:

$$\frac{1-\sum_{k=m+1}^{\infty}\frac{k[2k-m(1+\alpha)]}{m^2(1-\alpha)}|a_k|r^{k-m}-\sum_{k=m}^{\infty}\frac{k[2k+m(1+\alpha)]}{m^2(1-\alpha)}|b_k|r^{k-m}}{1-\sum_{k=m+1}^{\infty}k|a_k|r^{k-m}+\sum_{k=m}^{\infty}k|b_k|r^{k-m}}\geq 0. \tag{15}$$

Letting $r\to 1^-$ through real values, we obtain the condition (12). This completes the proof of Theorem 2.

**Theorem 3**

Let the function $f(z)$ given by Equation (7) be in the class $\overline{K}_H(m,\alpha)$, then for $z=r<1$

$$|f(z)|\le\begin{cases}(1+|b_m|)r^m+\frac{1}{m+1}\left(\frac{m^2(1-\alpha)}{2+m(1-\alpha)}-\frac{m^2(3+\alpha)}{2+m(1-\alpha)}|b_m|\right)r^{m+1} & ,\ m(1-\alpha)\le 1\\ (1+|b_m|)r^m+\frac{1}{m+1}\left(\frac{1}{2+m(1-\alpha)}-\frac{m^2(3+\alpha)}{2+m(1-\alpha)}|b_m|\right)r^{m+1} & ,\ m(1-\alpha)\ge 1,\end{cases} \tag{16}$$

and

$$|f(z)|\ge\begin{cases}(1-|b_m|)r^m-\frac{1}{m+1}\left(\frac{m^2(1-\alpha)}{2+m(1-\alpha)}-\frac{m^2(3+\alpha)}{2+m(1-\alpha)}|b_m|\right)r^{m+1} & ,\ m(1-\alpha)\le 1\\ (1-|b_m|)r^m-\frac{1}{m+1}\left(\frac{m^2(1-\alpha)}{2+m(1-\alpha)}-\frac{m^2(3+\alpha)}{2+m(1-\alpha)}|b_m|\right)r^{m+1} & ,\ m(1-\alpha)\ge 1,\end{cases} \tag{17}$$

where $|b_m|\le\dfrac{(1-\alpha)}{(3+\alpha)}.$

**Proof**

If $m(1-\alpha)\le 1,$ we have,

$$|f(z)|\le(1+|b_m|)r^m+\sum_{k=m+1}^{\infty}(|a_k|+|b_k|)r^k$$

$$\le(1+|b_m|)r^m+\sum_{k=m+1}^{\infty}(|a_k|+|b_k|)r^{m+1}$$

$$\le(1+|b_m|)r^m+\frac{m^2(1-\alpha)}{(m+1)[2+m(1-\alpha)]}\sum_{k=m+1}^{\infty}(m+1)\left(\frac{2+m(1-\alpha)}{m^2(1-\alpha)}\right)(|a_k|+|b_k|)r^{m+1}$$

$$\le(1+|b_m|)r^m+\frac{m^2(1-\alpha)}{(m+1)[2+m(1-\alpha)]}\left[\sum_{k=m+1}^{\infty}\left(\frac{k[2k-m(1+\alpha)]}{m^2(1-\alpha)}\right)|a_k|+\left(\frac{k[2k+m(1+\alpha)]}{m^2(1-\alpha)}\right)|b_k|\right]r^{m+1}$$

$$\le(1+|b_m|)r^m+\frac{m^2(1-\alpha)}{(m+1)[2+m(1-\alpha)]}\left[1-\frac{(3+\alpha)}{(1-\alpha)}\right]r^{m+1}$$

$$=(1+|b_m|)r^m+\frac{1}{m+1}\left(\frac{m^2(1-\alpha)}{2+m(1-\alpha)}-\frac{m^2(3+\alpha)}{2+m(1-\alpha)}|b_m|\right)r^{m+1},$$

which proves the assertion (16) of Theorem 3. The proof of the assertion (17) is similar, thus, it was omitted.

**Remark 1**

Putting $m=1$ in Theorem 3, we improve the result obtained by Kim et al. (2002) by adding the condition $|b_1|\le\frac{(1-\alpha)}{(3+\alpha)}.$

The following covering result follows the left hand inequality Theorem 3.

**Corollary 3**

Let the function $f(z)$ given by (7) be in the class $\overline{K}_H(m,\alpha)$, then for $|z|=r<1,$ we have:

$$\left\{w:|w|<\begin{bmatrix}\frac{2+m(3-\alpha)}{(m+1)(2+m(1-\alpha)}-\frac{2+m(3-2m-\alpha(2m+1)}{(m+1)(2+m(1-\alpha)}|b_m| & ,\ m(1-\alpha)\le 1\\ \frac{1+m[3+m(1-\alpha)-\alpha]}{(m+1)(2+m(1-\alpha)}-\frac{2+m(3-2m-\alpha(2m+1)}{(m+1)(2+m(1-\alpha)}|b_m| & ,\ m(1-\alpha)\ge 1\end{bmatrix}\right\}\subset f(U),$$

where $|b_m|<\dfrac{2+m(3-\alpha)}{2+m(3-2m-\alpha(2m+1)},$ or $|b_m|<\dfrac{1+m[3+m(1-\alpha)-\alpha]}{2+m(3-2m-\alpha(2m+1)}.$

**EXTREME POINTS**

Here, the extreme points of the closed convex hull of the class $\overline{K}_H(m,\alpha)$ denoted by $clco\,\overline{K}_H(m,\alpha).$ was determined

**Theorem 4**

Let *f(z)* be given by (7), then $f(z)\in clco\,\overline{K}_H(m,\alpha)$ if and only if

$$f(z)=\sum_{k=m}^{\infty}\left[x_k h_k(z)+y_k g_k(z)\right], \tag{18}$$

where

$$h_m(z)=z^m,$$

$$h_k(z)=\begin{cases}z^m-\frac{m^2(1-\alpha)}{k[2k-m(1+\alpha)]}z^k\ (k\ge m+1) & ,\ m(1-\alpha)\le 1\\ z^m-\frac{1}{k[2k-m(1+\alpha)]}z^k\ (k\ge m+1) & ,\ m(1-\alpha)\ge 1,\end{cases}$$

And

$$g_k(z)=\begin{cases}z^m-\frac{m^2(1-\alpha)}{k[2k+m(1+\alpha)]}\overline{z}^k\ (k\ge m) & ,\ m(1-\alpha)\le 1\\ z^m-\frac{1}{k[2k+m(1+\alpha)]}\overline{z}^k\ (k\ge m) & ,\ m(1-\alpha)\ge 1,\end{cases}$$

Where $\sum_{k=m}^{\infty}(x_k+y_k)=1,\ x_k\ge 0$ and $y_k\ge 0.$

In particular, the extreme points of the class $\overline{K}_H(m,\alpha)$ are $\{h_k\}(k\ge m+1)$ and $\{g_k\}(k\ge m),$ respectively.

**Proof**

For a function *f(z)* of the form (18), we have:

$$f(z)=\sum_{k=m}^{\infty}\left[x_k h_k(z)+y_k g_k(z)\right]$$
$$=\sum_{k=m}^{\infty}x_k\left(z^m-\frac{m^2(1-\alpha)}{k[2k-m(1+\alpha)]}z^k\right)+y_k\left(z^m-\frac{m^2(1-\alpha)}{k[2k+m(1+\alpha)]}\overline{z}^k\right)$$

$$=z^m-\sum_{k=m+1}^{\infty}\frac{m^2(1-\alpha)}{k[2k-m(1+\alpha)]}x_k z^k-\sum_{k=m}^{\infty}\frac{m^2(1-\alpha)}{k[2k+m(1+\alpha)]}y_k\overline{z}^k$$

but,

$$\sum_{k=m+1}^{\infty}\frac{k[2k-m(1+\alpha)]}{m^2(1-\alpha)}|a_k|+\sum_{k=m}^{\infty}\frac{k[2k+m(1+\alpha)]}{m^2(1-\alpha)}|b_k|$$
$$=\sum_{k=m+1}^{\infty}x_k+\sum_{k=m}^{\infty}y_k=1-x_k\le 1,$$

and $f(z)\in clco\,\overline{K}_H(m,\alpha).$

Conversely, assume that $f(z)\in clco\,\overline{K}_H(m,\alpha).$ Then

$$a_k=\frac{m^2(1-\alpha)}{k[2k-m(1+\alpha)]},$$

and

$$b_k=\frac{m^2(1-\alpha)}{k[2k+m(1+\alpha)]}$$

set

$$x_k=\frac{k[2k-m(1+\alpha)]}{m^2(1-\alpha)}|a_k|,$$

and

$$y_k=\frac{k[2k+m(1+\alpha)]}{m^2(1-\alpha)}|b_k|$$

Then by using Equation (12), we have $0\le x_k\le 1(k=m+1,m+2,...)$ and $0\le y_k\le 1(k=m,m+1,...).$

$x_m=1-\sum_{k=m+1}^{\infty}x_k-\sum_{k=m}^{\infty}y_k.$ is defined and the equation: $f(z)=\sum_{k=m}^{\infty}(x_k h_k+y_k g_k).$ is obtained. This completes the proof of Theorem 4.

## CONVOLUTION AND CONVEX COMBINATION

In this section, the convolution properties and convex combination were determined.

Let the functions $f_j(z)$ be defined by:

$$f_j(z)=z^m-\sum_{k=m+1}^{\infty}|a_{k,j}|z^k-\sum_{k=m}^{\infty}|b_{k,j}|\overline{z}^k\ (j=1,2), \tag{19}$$

be in the class $\overline{K}_H(m,\alpha)$, we denote by $(f_1*f_2)(z)$ the convolution or (Hadamard Product) of the function $f_1(z)$ and $f_2(z)$, that is,

$$(f_1*f_2)(z)=z^m-\sum_{k=m+1}^{\infty}|a_{k,1}||a_{k,2}|z^k-\sum_{k=m}^{\infty}|b_{k,1}||b_{k,2}|\overline{z}^k. \tag{20}$$

while the integral convolution is defined by

$$(f_1\Diamond f_2)(z)=z^m-\sum_{k=m+1}^{\infty}\frac{m|a_{k,1}||a_{k,2}|}{k}z^k-\sum_{k=m}^{\infty}\frac{m|b_{k,1}||b_{k,2}|}{k}\overline{z}^k. \tag{21}$$

We first show that the class $\overline{K}_H(m,\alpha)$ is closed under convolution.

**Theorem 5**

For $0\le\delta\le\alpha<1$, let the functions $f_1(z)\in\overline{K}_H(m,\alpha)$ and $f_2(z)\in\overline{K}_H(m,\delta).$

Then

$$(f_1*f_2)(z)\in\overline{K}_H(m,\alpha)\subset\overline{K}_H(m,\delta), \tag{22}$$

$$(f_1\Diamond f_2)(z)\in\overline{K}_H(m,\alpha)\subset\overline{K}_H(m,\delta). \tag{23}$$

**Proof**

Let $f_j(z)(j=1,2)$ given by Equation (19), where $f_1(z)$ is in the class $\overline{K}_H(m,\alpha)$ and $f_2(z)$ be in the class $\overline{K}_H(m,\delta).$ It therefore shows that the coefficients of $(f_1*f_2)(z)$ satisfy the required condition given in Equation (12).

For $f_2(z)\in\overline{K}_H(m,\delta)$, we note that $|a_{k,2}|<1$ and $|b_{k,2}|<1$. Now for the convolution functions $(f_1*f_2)(z),$ we obtain

$$\sum_{k=m+1}^{\infty}\frac{k[2k-m(1+\delta)]}{m^2(1-\delta)}|a_{k,1}||a_{k,2}|+\sum_{k=m}^{\infty}\frac{k[2k+m(1+\delta)]}{m^2(1-\delta)}|b_{k,1}||b_{k,2}|$$

$$\le\sum_{k=m+1}^{\infty}\frac{k[2k-m(1+\delta)]}{m^2(1-\delta)}|a_{k,1}|+\sum_{k=m}^{\infty}\frac{k[2k+m(1+\delta)]}{m^2(1-\delta)}|b_{k,1}|$$
$$\le\sum_{k=m+1}^{\infty}\frac{k[2k-m(1+\alpha)]}{m^2(1-\alpha)}|a_{k,1}|+\sum_{k=m}^{\infty}\frac{k[2k+m(1+\alpha)]}{m^2(1-\alpha)}|b_{k,1}|\le 1,$$

since $0 \le \delta \le \alpha < 1$ and $f_1(z) \in \overline{K}_H(m,\alpha)$.

Thus $(f_1 * f_2)(z) \in \overline{K}_H(m,\alpha) \subset \overline{K}_H(m,\delta)$. The proof of the assertion (23) is similar, thus, it was omitted. This completes the proof of Theorem 5.

Next we show that $\overline{K}_H(m,\alpha)$ is closed under convex combinations of its members.

**Theorem 6**

The class $\overline{K}_H(m,\alpha)$ is closed under convex combination.

**Proof**

For $i = 1, 2, \ldots$, let $f_i(z) \in \overline{K}_H(m,\alpha)$, where

$$f_i(z) = z^m - \sum_{k=m+1}^{\infty} |a_{k,i}| z^k - \sum_{k=m}^{\infty} |b_{k,i}| \overline{z}^k \quad (z \in U; i = 1, 2, \ldots), \tag{24}$$

then from (12), for $\sum_{i=1}^{\infty} t_i = 1,\ 0 \le t_i < 1$, the convex combination of $f_i(z)$ may be written as:

$$\sum_{i=1}^{\infty} t_i f_i(z) = z^m - \sum_{k=m+1}^{\infty} \left( \sum_{i=1}^{\infty} t_i |a_{k,i}| \right) z^k - \sum_{k=m}^{\infty} \left( \sum_{i=1}^{\infty} t_i |b_{k,i}| \right) \overline{z}^k. \tag{25}$$

Then by using Equation (12), we have

$$\sum_{k=m+1}^{\infty} \frac{k[2k - m(1+\alpha)]}{m^2(1-\alpha)} \left( \sum_{i=1}^{\infty} t_i |a_{k,i}| \right) + \sum_{k=m}^{\infty} \frac{k[2k - m(1+\alpha)]}{m^2(1-\alpha)} \left( \sum_{i=1}^{\infty} t_i |b_{k,i}| \right)$$

$$= \sum_{i=1}^{\infty} t_i \left[ \sum_{k=m+1}^{\infty} \frac{k[2k - m(1+\alpha)]}{m^2(1-\alpha)} |a_{k,i}| + \sum_{k=m}^{\infty} \frac{k[2k - m(1+\alpha)]}{m^2(1-\alpha)} |b_{k,i}| \right]$$

$$\le \sum_{i=1}^{\infty} t_i \le 1.$$

This completes the proof of Theorem 6.

**Integral operator**

Here, a closure property of the class $\overline{K}_H(m,\alpha)$ was examined under the generalized Bernardi-Libera-Livingston integral operator (Saitoh et al., 1992), $L_{c,m}(f(z))$ which is defined by:

$$L_{c,m}(f(z)) = \left( \frac{c+m}{z^c} \right) \int_0^z t^{c-1} f(t)\,dt, \ c > -m. \tag{26}$$

**Theorem 7**

Let $\overline{K}_H(m,\alpha)$.

Then

$$L_{c,m}(f(z)) \in \overline{K}_H(m,\alpha).$$

**Proof**

From Equation (26), it follows that

$$L_{c,m}(f(z)) = \left( \frac{c+m}{z^c} \right) \int_0^z t^{c-1} \left[ h(t) + \overline{g(t)} \right] dt$$

$$= \left( \frac{c+m}{z^c} \right) \left[ \int_0^z t^{c-1} \left( t^m - \sum_{k=m+1}^{\infty} a_k t^k \right) dt - \int_0^z \overline{\left( t^{c-1} \sum_{k=m}^{\infty} b_k t^k \right)} dt \right]$$

$$= z^m - \sum_{k=m+1}^{\infty} A_k z^k - \sum_{k=m}^{\infty} B_k z^k,$$

Where

$$A_k = \left( \frac{c+m}{c+k} \right) a_k, \ B_k = \left( \frac{c+m}{c+k} \right) b_k.$$

Therefore,

$$\sum_{k=m+1}^{\infty} \frac{k[2k - m(1+\alpha)]}{m^2(1-\alpha)} \left( \frac{c+m}{c+k} \right) |a_k| + \sum_{k=m}^{\infty} \frac{k[2k + m(1+\alpha)]}{m^2(1-\alpha)} \left( \frac{c+m}{c+k} \right) |b_k|$$

$$\le \sum_{k=m+1}^{\infty} \frac{k[2k - m(1+\alpha)]}{m^2(1-\alpha)} |a_k| + \sum_{k=m}^{\infty} \frac{k[2k + m(1+\alpha)]}{m^2(1-\alpha)} |b_k| \le 1.$$

Since $f(z) \in \overline{K}_H(m,\alpha)$, by using Corollary 1, then $L_{c,m}(f(z)) \in \overline{K}_H(m,\alpha)$.

This completes the proof of Theorem 7.

**Remark 2**

Putting $m = 1$ in the above results, the corresponding results by Kim et al. (2002), with $k = 1$ is obtained.

**Conflict of Interest**

The authors have not declared any conflict of interest.

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