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Calculation of a class of Gaussian integrals: Derivation of payoff at expiry for European option

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This work deals with the calculation of a class of Gaussian integral of the form $\int_0^\infty d\rho \exp\left[\frac{-(x-\varepsilon\rho)^2}{4\tau}+\beta\varepsilon\rho\right]$, where $\varepsilon=\pm 1$. Completing the squares of the exponential and changing variables led to the solution $e^{\beta(x+\beta\tau)}\varphi\left(\varepsilon\frac{x+2\tau\beta}{\sqrt{2\tau}}\right)$ where φ denotes the cumulative standard normal distribution function. An equation which corresponds to pay-off at expiry for European option was derived.

Key words: Gaussian integral, Euler-Poisson integral, multivariable calculus, Gaussian function, European option.

INTRODUCTION

The Gaussian integral, also known as the Euler-Poisson integral is the integral of the Gaussian function e^{-x^2} over the entire real line. It is named after the German mathematician Carl Friedrich Gauss. The integral is: $\int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}$. Abraham de Moivre originally discovered this type of integral in 1733, while Gauss published the precise integral in 1809 (Saul, 2006). The integral has a wide range of applications. For example, with a slight change of variables it is used to compute the normalizing constant of the normal distribution. The same integral with finite limits is closely related both to the error function and the cumulative distribution function of the normal distribution. The Gaussian integral can be solved analytically through the methods of multivariable calculus. That is, there is no elementary indefinite integral for $\int e^{-x^2}dx$, but the definite integral $\int_{-\infty}^{\infty} e^{-x^2}dx$ can be evaluated. Jameson (1994) proved the Gaussian integral. This proof was rediscovered by Delgado (2003). The proof revolved the curve $z=e^{-\frac{1}{2}x^2}$ in the $xz-plane$ around the $z-axis$ to produce the "bell surface" $z=e^{-\frac{1}{2}(x^2+y^2)}$. Where the $z-axis$ is vertical and passes through the top point, the $x-axis$ lies just under the surface through the point 0 in front, and the $y-axis$ lies just under the surface through the point 0 on the left. Rozman (2016) gave another proof of Gaussian integral by differentiation under the integral sign. The method was modified on math.stackexchange (http://math.stackexchange.com/questions/390850/integrating-int-infty-o-e-x2-dx-using-feynmans-parametrization-trick), and in a slightly less elegant form it appeared much earlier in Van Yzeren (1979). The original proof of the Gaussian integral $J=\sqrt{\pi}/2$ was due to Laplace (Stigler, 1986) in 1774 {An English translation of Laplace's article was mentioned in the bibliographic citation for Stigler (1986), with preliminary comments on that article in Stigler (1986)}. He wanted to compute:

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$$\int_0^1 \frac{dx}{\sqrt{-\log x}} \quad (1)$$

Setting $y = \sqrt{-\log x}$, this integral is $2\int_0^\infty e^{-y^2} dy = 2J$, hence, Equation (1) is expected to be $\sqrt{\pi}$. Laplace's starting point for evaluating Equation (1) was a formula of Euler: $\int_0^1 \frac{x^r dx}{\sqrt{1-x^{2s}}} \int_0^1 \frac{x^{s+r} dx}{\sqrt{1-x^{2s}}} = \frac{1}{s(r+1)} \frac{\pi}{2}$ for positive r and s. (Laplace himself said this formula held 'whatever be" r or s, but if $s < 0$ then the number under the square root is negative). Besides the integral formula $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}$, another place in mathematics where $\sqrt{2\pi}$ appears is in Stirling's formula: $n! \sim \frac{n^n}{e^n} \sqrt{2\pi n}$ as $n \to \infty$. In 1730, De Moivre proved $n! \sim C(n^n/e^n)\sqrt{n}$ for some positive number C without being able to determine C. Stirling soon thereafter showed $C = \sqrt{2\pi}$ and wound up having the whole formula named after him. But determining that the constant C in Stirling's formula is $\sqrt{2\pi}$ is equivalent to showing that $J = \sqrt{\pi}/2$ in Gaussian integral.

In finance, an investor is faced with the problems of where and when to invest, ability to regularly and dynamically build his portfolio of investments, and many more. Hence, knowing the payoff at expiry will be of great help. In the process of calculating the Gaussian integrals, payoff equation for European option was derived.

THE BASIC GAUSSIAN AND ITS NORMALIZATION

The Gaussian function or the normal distribution, $exp(-\alpha x^2)$, is a widely used function in physics and mathematical physics, including in quantum mechanics. It is therefore useful to know how to calculate it. The fundamental integral is

$$\int_{-\infty}^{+\infty} \exp(-\lambda x^2) dx = \sqrt{\frac{\pi}{\lambda}} \quad (2)$$

or the related integral

$$\int_0^{+\infty} \exp(-\lambda x^2) dx = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}. \quad (3)$$

The only difference between Equations (2) and (3) is the limits of integration.

EXPECTATION VALUES WITH GAUSSIAN

In computing expectation values with Gaussian, it is vital to use normalized distributions. Using the normalized Gaussian, $\rho(x) = \sqrt{\frac{\lambda}{\pi}} \exp(-\lambda x^2)$, the expectation value of x is $\int_{-\infty}^{+\infty} x\rho(x) dx = \int_{-\infty}^{+\infty} x\sqrt{\frac{\lambda}{\pi}} \exp(-\lambda x^2) dx$.

COMPLETING THE SQUARE

Sometimes we need to evaluate an integral like

$$\int_{-\infty}^{+\infty} \exp(-ax^2 + bx) \quad (4)$$

We can solve Equation (4) by completing the square. We can always evaluate

$$\int_{-\infty}^{+\infty} \exp(-a(x - x_0)^2) \quad (5)$$

by the substitution $y = x - x_0$. This motivates us to rewrite the argument of the exponent in Equation (4) as

$$ax^2 - bx = a\left(x^2 - \frac{b}{a}x\right) = a\left(x^2 - \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) \quad (6)$$

where the same quantity was merely added and subtracted inside the parentheses. We can rewrite this as a quantity squared to have $a\left(x - \frac{b}{2a}\right)^2 - \frac{b^2}{4a}$ and the original integral (4) becomes

$$e^{\frac{b^2}{4a}} \int_{-\infty}^{+\infty} \exp\left(-a\left(x - \frac{b}{2a}\right)^2\right) = e^{\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}} \quad (7)$$

where the final substitution $y = x - \frac{b}{2a}$ was left off.

THE MODEL

$$I_\beta = \frac{1}{\sqrt{4\pi\tau}} \int_0^\infty d\rho \exp\left[\frac{-(x-\varepsilon\rho)^2}{4\tau} + \beta\varepsilon\rho\right] = e^{\beta(x+\beta\tau)} \varphi\left(\varepsilon \frac{x+2\tau\beta}{\sqrt{2\tau}}\right), \quad (8)$$

where $\varepsilon = \pm 1$ and φ denotes the cumulative standard normal distribution function

$$\varphi(\varsigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\varsigma} d\eta e^{-\eta^2/2} \quad (9)$$

Proof: By completing the squares of the exponential, we require that:

$$-\frac{(x-\varepsilon\rho)^2}{4\tau} + \beta\varepsilon\rho = -c_1(c_2 - \varepsilon\rho)^2 + c_3, \quad (10)$$

where $c_i, i = 1, \cdots, 3$ are constants by respect to the integration variable. Expanding equation (10) gives

$$-\frac{1}{4\tau}(\varepsilon\rho)^2 + (\varepsilon\rho)\left(\beta + \frac{x}{2\tau}\right) - \frac{1}{4\tau}x^2 = -c_1\rho^2 + \rho 2c_1c_2 - c_1c_2^2 + c_3. \quad (11)$$

Equating coefficients of like terms gives

$$(\varepsilon\rho)^2 : -c_1 = -\frac{1}{4\tau}$$

$$\Rightarrow c_1 = \frac{1}{4\tau} \quad (12)$$

$$\varepsilon\rho\colon\ 2c_1c_2 = \beta + \frac{x}{2\tau} = \frac{2\beta\tau + x}{2\tau}$$

$$c_2 = \frac{2\beta\tau + x}{4\tau c_1} \tag{13}$$

Substituting Equation (12) in Equation (13) gives

$$c_2 = 2\beta\tau + x. \tag{14}$$

Similarly, from Equation (11), we have

$$-c_1c_2^2 + c_3 = -\frac{x^2}{4\tau}$$

$$c_3 = -\frac{x^2}{4\tau} + c_1c_2^2 = -\frac{x^2}{4\tau} + \frac{1}{4\tau}(2\beta\tau + x)^2$$

$$c_3 = \beta^2\tau + x\beta = \beta(x + \beta\tau) \tag{15}$$

Equation (8) now becomes

$$I_\beta = e^{c_3}\frac{1}{\sqrt{4\pi\tau}}\int_0^\infty d\rho e^{-c_1(c_2 - \varepsilon\rho)^2} \tag{16}$$

Changing variables to $\eta = (c_2 - \varepsilon\rho)\sqrt{2c_1}$, $d\eta = -d\rho\varepsilon\sqrt{2c_1}$, Equation (16) becomes

$$I_\beta = \frac{e^{c_3}}{\sqrt{2c_1}}\frac{1}{\sqrt{4\pi\tau}}\left(-\varepsilon\int_{c_2\sqrt{2c_1}}^{-\varepsilon\infty} d\eta e^{-\eta^2/2}\right). \tag{17}$$

If $\varepsilon = 1$, Equation (17) becomes

$$I_\beta = e^{c_3}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c_2\sqrt{2c_1}} d\eta e^{-\eta^2/2}$$

$$= e^{c_3}\varphi\left(c_2\sqrt{2c_1}\right). \tag{18}$$

If $\varepsilon = -1$, Equation (17) becomes

$$I_\beta = e^{c_3}\frac{1}{\sqrt{2\pi}}\int_{c_2\sqrt{2c_1}}^{\infty} d\eta e^{-\eta^2/2}. \tag{19}$$

Using the change of variables $\varsigma = -\eta, d\varsigma = -d\eta$, Equation (19) becomes

$$I_\beta = e^{c_3}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-c_2\sqrt{2c_1}} d\varsigma e^{-\varsigma^2/2}$$

$$= e^{c_3}\varphi\left(-c_2\sqrt{2c_1}\right). \tag{20}$$

Combining Equations (18) and (20) gives

$$I_\beta = e^{c_3}\varphi\left(\varepsilon c_2\sqrt{2c_1}\right). \tag{21}$$

But from Equation (15)

$$e^{c_3} = e^{\beta(x + \beta\tau)}. \tag{22}$$

Similarly, from Equations (12) and (14)

$$c_2\sqrt{2c_1} = (x + 2\tau\beta)\sqrt{2\frac{1}{4\tau}}$$

$$c_2\sqrt{2c_1} = \frac{x + 2\tau\beta}{\sqrt{2\tau}}. \tag{23}$$

Replacing Equation (23) in Equation (21) gives

$$I_\beta = e^{c_3}\varphi\left(\varepsilon\frac{x + 2\tau\beta}{\sqrt{2\tau}}\right). \tag{24}$$

Substituting Equation (15) in Equation (24) gives

$$I_\beta = e^{\beta(x + \beta\tau)}\varphi\left(\varepsilon\frac{x + 2\tau\beta}{\sqrt{2\tau}}\right). \tag{25}$$

Equation (9) can be solved further. Let

$$I = \int_{-\infty}^{\varsigma} d\eta e^{-\eta^2/2} \tag{26}$$

Squaring Equation (26) gives

$$I^2 = \int_{-\infty}^{\varsigma}\exp\left(-\tfrac{\eta^2}{2}\right)d\eta\ \int_{-\infty}^{\varsigma}\exp\left(-\tfrac{y^2}{2}\right)dy \tag{27}$$

which we rewrite as

$$I^2 = \iint \exp\left(-\tfrac{1}{2}(\eta^2 + y^2)\right)d\eta dy. \tag{28}$$

Going from Cartesian coordinates (η, y) to polar coordinates (r, θ):

$$I^2 = \iint \exp\left(-\tfrac{1}{2}r^2\right)r dr d\theta. \tag{29}$$

The integral over θ is easy, leaving

$$I^2 = 2\pi\int_0^{\varsigma}\exp\left(-\tfrac{1}{2}r^2\right)r dr. \tag{30}$$

Making a substitution: $u = r^2$, with $du = 2r dr$ we have

$$I^2 = 2\pi\int_0^{\varsigma}\exp\left(-\tfrac{1}{2}u\right)\tfrac{1}{2}du.$$

This is the simple integral of an exponential and

$$I^2 = 2\pi\cdot\frac{1}{2}\left[e^{-\frac{1}{2}u}\cdot -\frac{1}{1/2}\right]_0^{\varsigma} = 2\pi\ as\ \varsigma \to \infty. \tag{31}$$

Conclusion

Gaussian integrals can be encountered in option pricing. When applied to European option, the initial payoff condition at $\tau = 0$ in Equation (25) corresponds to the payoff at expiry $t = T$. Equation (18) is for call option while Equation (19) is for put option. The derived equations and the analysis that led to it are also applicable to corporate liabilities such as common stock.

CONFLICT OF INTERESTS

The author has not declared any conflict of interests.

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