A common fixed point theorem and some fixed point theorems in \( D^* \)-Metric spaces

T. Veerapandi\(^1\) and Aji .M. Pillai\(^2\)

\(^1\)P.M.T College, Melaneelithanallur - 627 953, India.
\(^2\)Manomaniam Sundaranar University, Tirunelveli, India.

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In this paper we establish some common fixed point theorems for contraction and some generalized contraction mappings in \( D^* \)- metric space which is introduced by Shaban et al. (2007). In what follows, \((X, D^*)\) will denote \( D^* \)- metric space, \( N \) is the set of all natural number and \( R^+ \) is the set of all positive real number.

Key words: \( D^* \)- metric, contraction mapping, complete \( D^* \)- metric space, common fixed point theorem.

INTRODUCTION

There have been a number of generalization in generalized metric space (or \( D \)-Metric space) initiated by Dhage (1992). He proved the existence of unique fixed point theorems of a self map satisfying contractive conditions in complete and bounded \( D \)-Metric space. Dealing with \( D \)-metric space, (Ahmad et al., 2001; Dhage, 1992, 1999; Dhang et al., 2000; Rhoades, 1996; Singh and Sharma, 2002) and others made a significant contribution in fixed point theory of \( D \)-metric space. Unfortunately almost all theorems in \( D \)-metric space are not valid (Naidu et al., 2004, 2005a, b). Here our aim is to prove some common fixed point theorems in \( D \)-metric space as a probable modification of the definition of \( D \)-metric spaces introduced by Dhage (1992).

Definition 1

Let \( X \) be a non empty set. A generalized metric (or \( D^* \)-metric) on \( X \) is a function
\[
D^*: X^3 \rightarrow [0, \infty) \text{ that satisfies the following conditions for each } x, y, z, a \in X.
\]
(1) \( D^* (x, y, z) \geq 0 \)
(2) \( D^* (x, y, z) = 0 \) if and only if \( x = y = z \)
(3) \( D^* (x, y, z) = D^* (\rho(x, y, z)) \) where \( \rho \) is permutation (function).
(4) \( D^* (x, y, z) \leq D^* (x, y, a) + D^* (a, z, z) \).

The pair \((X, D^*)\) is called generalized metric (or \( D^* \)-metric) space.

Examples 1

(a) \( D^*(x, y, z) = \max \{d(x, y), d(y, z), d(z, x)\} \),
(b) \( D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x) \).

Here, \( d \) is the ordinary metric on \( X \).

(c) If \( X = R^+ \) then we define
\[
D^*(x, y, z) = (||x - y||^p + ||y - z||^p + ||z - x||^p)^{1/p} \text{ for every } p \in R^+.
\]

(d) If \( X = R \) then we define
\[
D^*(x, y, z) = \begin{cases} 
0 & \text{if } x = y = z, \\
\max \{x, y, z\} & \text{otherwise},
\end{cases}
\]
Remark 1
In $D^*$ - metric space $D^* (x, y, y) = D^* (x, x, y)$

Definition 2
A open ball in a $D^*$ - metric space $X$ with centre $x$ and radius $r$ is denoted by $B_{D^*} (x, r)$ and is defined by $B_{D^*} (x, r) = \{ y \in X: D^* (x, y, y) < r \}$

Example 2
Let $X = R$ Denote $D^* (x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in R$.

Thus, $B_{D^*} (0, 1) = \{ y \in R / D^* (0, y, y) < 1 \}
= \{ y \in R / |0 - y| + |y - y| + |y - 0| < 1 \}
= \{ y \in R / |y| < 1/2 \}
= \{ y \in R / - 1/2 < y < 1/2 \}
= (-1/2, 1/2).

Definition 3
Let $(X, D^*)$ be a $D^*$ - metric space and $A \subseteq X$

(1) If for every $x \in A$, there exist $r > 0$ such that $B_{D^*} (x, r) \subseteq A$, then subset $A$ is called open subset of $X$.

(2) Subset $A$ of $X$ is said to be $D^*$ - bounded if there exist $r > 0$ such that $D^* (x, y, y) < r$ for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in $X$ converges to $x$ if and only if $D^* (x_n, x, x) \rightarrow 0$ as $n \rightarrow \infty$.

That is, for each $\varepsilon > 0$ there exist $n_0 \in N$ such that for all $n \geq n_0$ implies $D^* (x_n, x, x) < \varepsilon$. This is equivalent for each $\varepsilon > 0$, there exist $n_0 \in N$ such that for all $n, m \geq n_0$ implies $D^* (x_n, x_m, x_m) < \varepsilon$.

(4) A sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if for each $\varepsilon > 0$, there exist $n_0 \in N$ such that for all $n, m \geq n_0$ implies $D^* (x_n, x_m, x_m) < \varepsilon$ for each $n, m \geq n_0$ The $D^*$ - metric space $(X, D^*)$ is said to be complete if every Cauchy sequence is convergent.

Remark 2
(1) $D^*$ is continuous function on $X^3$

(2) If sequence $\{x_n\}$ in $X$ converges to $x$, then $x$ is unique.

(3) Any convergent sequence in $(X, D^*)$ is a Cauchy sequence.

Definition 4
A point $x$ in $X$ is a common fixed point of two maps $T_1, T_2$: $X \rightarrow X$ if $T_1 (x) = T_2 (x) = x$.

MAIN RESULTS
Common fixed point theorems for Banach contraction mappings type in $D^*$ - metric space.

Theorem 1
Let $(X, D^*)$ be a complete $D^*$ - metric space and $T_1, T_2, T_3$: $X \rightarrow X$ be three maps such that $D^* (T_1 x, T_2 y, T_3 z) \leq a D^* (x, y, z)$ for all $x, y, z \in X$ and $0 \leq a < 1$. Then $T_1, T_2, T_3$ have a unique fixed point in $X$.

Proof
Let $x_0 \in X$ be a fixed arbitrary element. Define a sequence $\{x_n\}$ in $X$ as
$x_{3n+1} = T_1 x_{3n}$ for $n = 0, 1, 2 \ldots$
$x_{3n+2} = T_2 x_{3n+1}$ for $n = 0, 1, 2 \ldots$
$x_{3n+3} = T_3 x_{3n+2}$ for $n = 0, 1, 2 \ldots$

For all $n > 0$ we define
$d_n = D^* (x_n, x_{n+1}, x_{n+2})$
$d_{n+1} = D^* (x_{3n+1}, x_{3n+2}, x_{3n+3})$
$= D^* (T_1 x_{3n}, T_2 x_{3n+1}, T_3 x_{3n+2})$
$\leq a D^* (x_{3n}, x_{3n+1}, x_{3n+2})$
$< d_n$

Similarly,
$d_{3n+2} = D^* (x_{3n+2}, x_{3n+3}, x_{3n+4})$
$= D^* (T_2 x_{3n+1}, T_3 x_{3n+2}, T_1 x_{3n+3})$
$\leq a D^* (x_{3n+1}, x_{3n+2}, x_{3n+3})$
$< d_{3n+1}$

Hence
$d_{3n+2} < d_{3n+1} < d_{3n}$ for all $n$.

Thus $(d_n)$ is strictly monotonically decreasing sequence of positive real numbers and bounded below, it is convergent to a positive real number. Let it be $l$. suppose that $l \neq 0$. Then $l > 0$.

$l = \lim_{n \to \infty} d_n$
$= \lim_{n \to \infty} d_{3n+1}$
$= \lim_{n \to \infty} D^* (x_{3n+1}, x_{3n+2}, x_{3n+3})$
$= \lim_{n \to \infty} D^* (T_1 x_{3n}, T_2 x_{3n+1}, T_3 x_{3n+2})$
$\leq \lim_{n \to \infty} a D^* (x_{3n}, x_{3n+1}, x_{3n+2})$
\[
\lim_{n \to \infty} a d_{3n} = 0.
\]

Thus \( I = 0 \).

Hence \( \lim_{n \to \infty} d_n = 0 \).

we shall prove that \( \{x_n\} \) is a \( D^* \)-cauchy sequence in \( X \).

For \( m > n \geq n_0 \), we have

\[
D^* (x_n, x_m) \leq D^* (x_n, x_{n+1}) + D^* (x_{n+1}, x_m).
\]

\[
\leq a \lim_{n \to \infty} D^* (x, x_{n+1}) = 0.
\]

Thus \( D^* (x_n, x_m) < a \) for all \( m, n \geq n_0 \) for some \( n_0 \in \mathbb{N} \).

Thus \( \{x_n\} \) is a \( D^* \)-Cauchy sequence in \( X \).

Since \( X \) is a \( D^* \)-complete space, \( x_n \to x \) in \( X \) as \( n \to \infty \).

Now we shall prove that \( T_1 x = x \).

We have

\[
D^* (x, x) = \lim_{n \to \infty} D^* (T_n x, T_{n+1} x).
\]

Thus \( D^* (x, x, x) = 0 \). Hence \( T_1 x = x \).

Similarly, we prove that \( T_2 x = x \) and \( T_3 x = x \).

Uniqueness:

Suppose \( x \neq y \) such that \( T_1 y = y, T_2 y = y \) and \( T_3 y = y \).

Now \( D^* (x, y, y) = D^* (T_1 y, T_2 y, T_3 y) \).

\[
\leq a \lim_{n \to \infty} D^* (x, x_{n+1}) = 0.
\]

This implies \( 1-a D^* (x, y, y) \leq 0 \).

Since \( x \neq y \), \( D^* (x, y, y) > 0 \) we have \( 1-a < 0 \).

This implies \( a > 1 \) which is contradiction to \( a < 1 \).

Hence \( T_1, T_2 \) and \( T_3 \) have a unique common fixed point.

\[\text{Theorem 2}\]

Let \( (X, D^*) \) be a \( D^* \)-complete metric space and \( T: X \to X \) be a map such that

\[
D^* (T_1 x, x, x) = \lim_{n \to \infty} D^* (T_n x, T_{n+1} x, T_{n+2} x) \leq \lim_{n \to \infty} D^* (x, x_{n+1}, x_{n+2}) = 0.
\]

Thus \( x_n \to x \) in \( X \).

Now we prove that \( T x = x \).

Suppose \( x \neq y \).

\[D^* (x, y, y) = D^* (T_1 y, y, y) \leq a \lim_{n \to \infty} D^* (x, x_{n+1}, x_{n+2}) = 0.\]

Thus \( x = y \).

Now we prove the uniqueness. Suppose \( x \neq y \) such that \( T y = y \).

\[D^* (x, y, y) = D^* (T_1 y, y, y) \leq a \lim_{n \to \infty} D^* (x, x_{n+1}, x_{n+2}) = 0.\]

Hence \( T \) has a unique fixed point.

\[\text{Theorem 3}\]

Let \( (X, D^*) \) be a \( D^* \)-complete metric space and \( T: X \to X \) be a map such that

\[
D^* (T_1 y, y, y) \leq a \{ D^* (y, y, z) + D^* (y, T_1 y, T_2 y) \} \]

\[\text{for all } x, y, z \in X\]

and \( 0 \leq a < 1/4 \). Then \( T \) has a unique fixed point.

\[\text{Proof}\]

Let \( x_0 \in X \) a fixed arbitrary element.

Define the sequence \( \{x_n\} \) in \( X \) as \( x_{n+1} = T x_n \) for \( n = 0, 1, 2 \).

\[\text{For } n \geq 0 \text{ we have,}\]

\[D^* (x_n, x_{n+1}) = D^* (T x_n, T x_{n+1}, T x_{n+2}) \leq a \{ D^* (x_n, x_{n+1}) + D^* (x_{n+1}, x_{n+2}) \} + \frac{a}{2} D^* (x_{n+2}, x_{n+3}).\]
and $0 \leq a_1 + \frac{3}{2} a_2 + \frac{3}{2} a_3 < 1$. Then $T$ has a unique fixed point.

**Proof**

Let $x_0 \in X$ be a fixed arbitrary element. Define the sequence $\{x_n\}$ in $X$ as $x_{n+1} = Tx_n$ for $n = 0, 1, 2$.

For $n \geq 0$ we have

$$D^*(x_n, x_{n+1}) = D^*(Tx_n, Tx_{n+1}, Tx_n) \leq \left\{ a_1 D^*(x_n, x_{n+1}, x_0) + \frac{a_2}{2} (D^*(x_n, x_0, x_0) + D^*(x_n, x_{n+1}, x_0)) + \frac{a_3}{2} (D^*(x_n, x_0, x_{n+1}) + D^*(x_n, x_{n+1}, x_n)) \right\}$$

$$= \left\{ a_1 D^*(x_n, x_0, x_0) + \frac{a_2}{2} (D^*(x_n, x_0, x_0) + D^*(x_n, x_{n+1}, x_0)) + \frac{a_3}{2} (D^*(x_n, x_0, x_{n+1}) + D^*(x_n, x_{n+1}, x_n)) \right\}$$

$$\leq \left[ (a_1 + a_2 + a_3) D^*(x_n, x_0, x_{n+1}) \right]$$

$$D^*(x_n, x_{n+1}) \leq \left\{ a_1 + a_2 + a_3 \right\} D^*(x_{n+1}, x_n, x_0)$$

$$a = \frac{a_1 + a_2 + a_3}{1 - \frac{a_2}{2} - \frac{a_3}{2}} < 1,$$

Thus $\{x_n\}$ is a Cauchy sequence in $X$.

Now we prove that $\{x_n\}$ is a Cauchy sequence in $X$.

For $n > m$, we have

$$D^*(x_n, x_m) \leq D^*(x_n, x_{n+1}) + D^*(x_{n+1}, x_{n+2}) + \ldots + D^*(x_m, x_{m+1}) \rightarrow 0 \text{ as } n \rightarrow \infty.$$  

Thus $\{x_n\}$ is a Cauchy sequence in $X$.

Now we prove that $Tx = x$. Suppose $x \neq Tx$. Then $D^*(Tx, x, x) = \lim_{n \to \infty} D^*(Tx_n, x_n, x_n)$

$$\leq \lim_{n \to \infty} \left\{ a_1 D^*(x_n, x_0, x_0) + \frac{a_2}{2} (D^*(x_n, x_0, x_0) + D^*(x_n, x_{n+1}, x_0)) + \frac{a_3}{2} (D^*(x_n, x_0, x_{n+1}) + D^*(x_n, x_{n+1}, x_n)) \right\}$$

$$= \frac{a_2}{2} D^*(x, x, x) \rightarrow 0 \text{ as } n \to \infty.$$  

Hence $\{x_n\}$ is a Cauchy sequence in $X$.

**Theorem 4**

Let $(X, D^*)$ be a complete $D^*$-metric space and $T: X \to X$ be a map such that

$$D^*(x, y, y) \leq a_1 D^*(x, y, y) + \frac{a_2}{2} \{D^*(x, x, y) + D^*(y, y, y)\} + \frac{a_3}{2} (D^*(x, x, x) + D^*(x, x, x))$$

$$\leq \lim_{n \to \infty} \left\{ a_1 D^*(x_n, x_0, x_0) + \frac{a_2}{2} (D^*(x_n, x_0, x_0) + D^*(x_n, x_{n+1}, x_0)) + \frac{a_3}{2} (D^*(x_n, x_0, x_{n+1}) + D^*(x_n, x_{n+1}, x_n)) \right\}$$

$$= \frac{a_2}{2} D^*(x, x, x) \rightarrow 0 \text{ as } n \to \infty.$$  

Thus $x = y$. $T$ has a unique fixed point.
Thus, $x_n \to x$ in $X$.

Now we prove $Tx = x$. Supper $x \neq Tx$

$$D^*(x, x, Tx) = \lim_{n\to\infty} D^*(x_{n+2}, x_{n+1}, Tx)$$

$$\leq \lim_{n\to\infty} D^*(T^2x_n, x)$$

$$= \lim_{n\to\infty} D^*(x_{n+1}, y)$$

Thus, $x = Tx$

Uniqueness:

Supper $x \neq y$ such that $Ty = y$.

Then $D^*(x, y, y) = D^*(T^2x, T^2y, Ty)$

$$\leq \lim_{n\to\infty} D^*(x_{n+1}, y)$$

This implies

$$D^*(x, y, y) \leq 0$$

Hence $1 - a < 0$ (since $D^*(x, y, y) > 0$)

Therefore $a > 1$. This is contradiction to $a < 1$.

Thus $T$ has a unique fixed point.

**Theorem 5**

Let $(X, D^*)$ be a complete $D^*$-metric space and $T : X \to X$ be a map such that

$$d(Tx, Ty, Tz) \leq a \max \{D^*(x, y, z), D^*(x, Tx, Ty), D^*(y, Ty, Tz), D(x, y, Tx), D(y, z, Tz)\}$$

for all $x, y, z \in X$ and $0 \leq a < \frac{1}{2}$. Then $T$ has a unique fixed point.

**Proof**

Let $x_0 \in X$ be a fixed arbitrary element.

Define the sequence $(x_n)$ in $X$ as $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$

For $n \geq 0$, we have

$$D^*(x_{n+1}, x_n, x_{n+1}) = D^*(Tx_n, x_n, x_{n+1})$$

$$\leq a \max \{D^*(x_{n+1}, x_n, x_{n+1}), D^*(x_{n+1}, x_n, x_{n+1}), D^*(x_{n+1}, x_n, x_{n+1}), D^*(x_{n+1}, x_n, x_{n+1})\}$$

$$\leq a \max \{D^*(x_{n+1}, x_n, x_{n+1}), D^*(x_{n+1}, x_n, x_{n+1}), D^*(x_{n+1}, x_n, x_{n+1}), D^*(x_{n+1}, x_n, x_{n+1})\}$$

$$\leq a D^*(x_{n+1}, x_n, x_{n+1}) + a D^*(x_n, x_{n+1}) + D^*(x_n, x_{n+1})$$

$$\leq b D^*(x_{n+1}, x_{n+1}, x_{n+1})$$

where $b = \frac{a}{1-a} < 1$ for all $n$

Thus $(x_n)$ is a $D^*$-cauchy sequence in $X$ and is $D^*$-complete $x_n \to x$ in $X$.

Now we prove that $Tx = x$.

$$D^*(Tx, x, x) = \lim_{n\to\infty} D^*(Tx_n, x, x)$$

$$\leq a \lim_{n\to\infty} D^*(x_{n+1}, x, x)$$

$$= a D^*(x, y, y)$$

Thus $x = Tx$.

Uniqueness:

Suppose $x \neq y$ such that $Ty = y$.

Then $D^*(x, y, y) = D^*(T^2x, T^2y, Ty)$

$$\leq a \lim_{n\to\infty} D^*(x_{n+1}, y, y)$$

Therefore $a > 1$. This is contradiction. Thus $x = Ty$.

**Theorem 6**

Let $(X, D^*)$ be a complete $D^*$-metric space and $T : X \to X$ be a map such that

$$D^*(Tx, Ty, Tz) \leq a_1 D^*(x, y, z) + a_2 \max \{D^*(x, Tx, Ty), D^*(y, Ty, Tz)\}$$

for all $x, y, z \in X$ and $0 \leq a_1 + 2a_2 < 1$.

Then $T$ has a unique fixed point.

**Proof**

Let $x_0 \in X$ be any arbitrary fixed element and define a sequence element and define a sequence $(x_n)$ in $X$ as $x_{n+1} = x_n$ for $n = 0, 1, 2, \ldots$
For \( n \geq 0 \), we have.

\[
D^*(x_n, x_{n+1}) = D^*(Tx_n, Tx_{n+1}, Tx_n) \\
\leq a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max \{D^*(x_{n-1}, Tx_{n-1}, Tx_n), D^*(x_{n-1}, Tx_n, x_n)\} \\
= a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max \{D^*(x_{n-1}, x_{n-1}, x_{n+1}), D^*(x_{n-1}, x_{n+1}, x_n)\}
\]

\[
\leq a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max \{D^*(x_{n-1}, x_{n-1}, x_{n+1}) + D^*(x_{n-1}, x_{n+1}, x_n)\}
\]

\[
(1 - a_2) D^*(x_{n-1}, x_{n-1}, x_n) \leq (a_1 + a_2) D^*(x_{n-1}, x_{n-1}, x_{n+1})
\]

\[
D^*(x_n, x_{n+1}) \leq \frac{a_1 + a_2}{1 - a_2} D^*(x_{n-1}, x_{n-1}, x_n)
\]

Thus \( D^*(x_n, x_{n+1}) \leq a D^*(x_{n-1}, x_{n-1}, x_n) \), where

\[
a = \frac{a_1 + a_2}{1 - a_2} < 1.
\]

Now for \( m > n \geq 0 \) we have

\[
D^*(x_n, x_m) \leq \sum_{k=n}^{m-1} D^*(x_k, x_{k+1}) \to 0 \text{ as } m, n \to \infty.
\]

Thus \( \{x_n\} \) is a Cauchy sequence in complete \( D^* \)-metric space. Hence there exist a point \( x \in X \) such that \( x_n \to x \) in \( X \).

Now we shall prove that \( x \) is a fixed point of \( T \).

\[
D^*(x, x, Tx) = \lim_{n \to \infty} D^*(x_{n+1}, x_{n+1}, Tx)
\]

\[
= \lim_{n \to \infty} D^*(Tx_n, Tx_n, Tx)
\]

\[
\leq \lim_{n \to \infty} \{a_1 D^*(x_n, x_n, x) + a_2 \max \{D^*(x_n, Tx_n, Tx_n), D^*(x_n, Tx_n, Tx_n)\}\}
\]

\[
\leq \lim_{n \to \infty} \{a_1 D^*(x_n, x_n, x) + a_2 \max \{D^*(x_n, x_{n+1}, x_{n+1}), D^*(x_n, x_{n+1}, x_{n+1})\}\}
\]

\[
\to a_1 (0) + a_2 D^*(x_n, x, Tx) \text{ as } n \to \infty.
\]

Thus, \( D^*(x, x, Tx) < D^*(x, x, Tx) \), which is contradiction. Thus implies \( x = Tx \).

Now we shall prove uniqueness. Suppose \( x \neq y \) such that \( Ty = y \). Then \( D^*(x, y, y) = D^*(Tx, Ty, Ty) \)

\[
\leq a_1 D^*(x, y, y) + a_2 \max \{D^*(x, Tx, Ty), D^*(y, y, Ty)\}
\]

\[
= a_1 D^*(x, y, y) + a_2 \max \{D^*(x, x, y), D^*(y, y, y)\}
\]

\[
= (a_1 + a_2) D^*(x, y, y)
\]

\[
< D^*(x, y, y), \text{ which is contradiction. Therefore } T \text{ has a unique fixed point.}
\]

Remark 3

If we put \( a_2 = 0 \) and \( a_1 = a \) in the above theorem we get the following Theorem as corollary.

Corollary 1

Let \( (X, D^*) \) be a complete \( D^* \)-metric space and \( T: X \to X \) be a map such that

\[
D^*(Tx, Ty, Tz) \leq a \max \{D^*(x, y, z), D^*(y, Ty, Tz)\}
\]

for all \( x, y, z \in X \) and \( 0 \leq a < 1 \). Then \( T \) has a unique fixed point.

The above theorem is known as Banach contraction type theorem in \( D^* \)-metric space.

Remark 4

If we put \( a_1 = 0 \) and \( a_2 = a \) in the previous theorem 6. We get the following theorem as corollary.

Corollary 2

Let \( (X, D^*) \) be a complete \( D^* \)-metric space and \( T: X \to X \) be a map such that

\[
D^*(Tx, Ty, Tz) \leq \frac{a}{2} \max \{D^*(x, x, Ty), D^*(y, Ty, Tz)\}
\]

for all \( x, y, z \in X \) and \( 0 \leq a < \frac{1}{2} \). Then \( T \) has a unique fixed point.

Theorem 7

Let \( (X, D^*) \) be a complete \( D^* \)-metric space and \( T: X \to X \) be a map such that

\[
D^*(Tx, Ty, Tz) \leq \left\{ a_1 D^*(x, y, z) + a_2 \max \{D^*(x, x, Ty), D^*(y, y, Tz)\} + a_3 \max \{D^*(x, y, Ty), D^*(y, z, Tz)\} \right\}
\]

for all \( x, y, z \in X \) and \( 0 \leq a_1 + 2a_2 + 2a_3 < 1 \). Then \( T \) has a unique fixed point.

Proof

Let \( x_0 \in X \) be a fixed arbitrary element. Define a sequence \( \{x_n\} \) in \( X \) as \( x_{n+1} = Tx_n \) for \( n = 0, 1, 2 \ldots \) Now for \( n \geq 0 \) we have

\[
D^*(x_n, x_{n+1}) = D^*(Tx_n, Tx_{n+1}, Tx_n)
\]

\[
= \left\{ a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max \{D^*(x_{n-1}, Tx_{n-1}, Tx_{n+1}), D^*(x_{n-1}, Tx_{n+1}, Tx_n)\} + a_3 \max \{D^*(x_{n-1}, x_{n+1}, Tx_{n+1}), D^*(x_{n-1}, x_{n}, Tx_n)\} \right\}
\]

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Theorem 8

Let \((X, D^*)\) be a complete \(D^*\) - metric space and \(T: X \to X\) be a map such that

\[
D^*(Tx, Ty, Tz) \leq a_1 D^*(x, y, z) + a_2 \max \left\{ \frac{D^*(x, Ty, Tz) + D^*(y, Tx, Tz)}{2}, \frac{D^*(x, y, Ty) + D^*(y, z, Tz)}{2} \right\}
\]

for all \(x, y, z \in X\) and

\[
0 \leq a_1 + 3a_2 \leq 1.
\]

Then \(T\) has a unique fixed point.

**Proof**

Let \(x_0 \in X\) be any fixed arbitrary element.

Define a sequence \(\{x_n\}\) in \(X\) as \(x_{n+1} = Tx_n\) for \(n = 0, 1, 2 \ldots\).

For \(n \geq 0\), we have

\[
D^*(x_n, x_{n+1}) = D^*(Tx_{n-1},Tx_n) \leq a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max \left\{ \frac{D^*(x_{n-1}, Tx_{n-1}) + D^*(x_{n-1}, Tx_n)}{2}, \frac{D^*(x_{n-1}, x_{n-1}, x_{n+1})}{2} \right\}
\]

\[
\leq a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max \{ D^*(x_{n-1}, x_{n-1}, x_{n+1}), D^*(x_{n-1}, x_{n-1}, x_n) \}
\]

\[
\leq \lim_{n \to \infty} \left( a_1 D^*(x_{n-1}, x_{n-1}, x_n) + a_2 \max \{ D^*(x_{n-1}, x_{n-1}, x_{n+1}), D^*(x_{n-1}, x_{n-1}, x_n) \} \right)
\]

Thus \(\{x_n\}\) is a Cauchy sequence in \(D^*\) - complete metric space \(X\).

Hence there is a point \(x\) in \(X\) such that \(x_n \to x\) in \(X\).

Now we shall prove that \(x\) is a fixed point of \(T\).

Now \(D^*(x, x, Tx) = \lim_{n \to \infty} D^*(x_{n+1}, x_{n+1}, Tx)
\]

\[
= \lim_{n \to \infty} D^*(Tx_n, x, Tx)
\]

\[
\leq \lim_{n \to \infty} \left\{ a_1 D^*(x_n, x, x) + a_2 \max \{ D^*(x_n, Tx_n), D^*(x_n, Tx, x) \} + a_3 \max \{ D^*(x_n, x, x), D^*(x_n, x, Tx) \} \right\}
\]

\[
= \lim_{n \to \infty} \left\{ a_1 D^*(x_n, x, x) + a_2 \max \{ D^*(x_n, x, x), D^*(x_n, x, x) \} + a_3 \max \{ D^*(x_n, x, x), D^*(x_n, x, x) \} \right\}
\]

\[
= a_1 (0) + a_2 D^*(x, x, Tx) + a_3 D^*(x, x, Tx)
\]

Thus \(Tx = x\).

Uniqueness:

Suppose \(y \neq x\) such that \(Ty = y\).

Now \(D^*(x, y, y) = D^*(Tx, Ty, Ty) \leq \left\{ a_1 D^*(x, Ty, Ty) + a_2 \max \{ D^*(x, Ty, Ty), D^*(y, Ty, Ty) \} + a_3 \max \{ D^*(x, y, Ty), D^*(y, y, Ty) \} \right\}
\]

\[
= \left\{ a_1 D^*(x, y, y) + a_2 \max \{ D^*(x, y, y), D^*(y, y, y) \} + a_3 \max \{ D^*(x, y, y), D^*(y, y, y) \} \right\}
\]

\[
= a_1 D^*(x, y, y) + a_2 D^*(x, y, y) + a_3 D^*(x, y, y)
\]

\[
< D^*(x, y, y), \text{ which is contradiction. Hence} \ T \text{ has a unique fixed point.}
\]
Now we shall prove that \( \{x_n\} \) is a cauchy sequence in \( X \). For \( m > n \geq 0 \), we have

\[
D^*(x_n, x_n, x_m) \leq \sum_{k=n}^{m-1} D^*(x_k, x_{k+1}) \to 0 \text{ as } m, n \to \infty.
\]

Thus \( \{x_n\} \) is a \( D^* \)-Cauchy sequence in \( X \). Since \( X \) is complete \( D^* \)-metric space and \( x_n \to x \) in \( X \).

Now we prove that \( x = Tx \) suppose \( x \neq Tx \).

\[
D^*(x, x, Tx) = \lim_{n \to \infty} D^*(x_{n+1}, x_{n+1}, Tx) \leq \lim_{n \to \infty} \left\{ a_1 D^*(x_n, x_n, x) \right\} + \frac{a_2}{2} \max \left\{ D^*(x_n, Tx_n, Tx_n) + D^*(x, Tx_n, Tx) \right\}
\]

\[
= \lim_{n \to \infty} \left\{ a_1 D^*(x_n, x_n, x) \right\} + \frac{a_2}{2} \max \left\{ D^*(x_n, x_{n+1}, x_{n+1}) + D^*(x, x, Tx) \right\}
\]

\[
< D^*(x, x, Tx), \text{ which is contradiction. Thus } Tx = x.
\]

Uniqueness:
Suppose \( y \neq x \) such that \( Ty = y \).
Now \( D^*(x, y, y) = D^*(Tx, Ty, Ty) \)

\[
\leq \left\{ a_1 D^*(x, y, y) + \frac{a_2}{2} \max \left\{ D^*(x, y, Ty) + D^*(y, y, Ty) \right\} \right\} 
\]

\[
= a_1 D^*(x, y, y) + \frac{a_2}{2} \max \left\{ D^*(x, y, y) + D^*(y, y, y) \right\} 
\]

\[
\leq \left\{ a_1 D^*(x, y, y) + \frac{a_2}{2} D^*(x, y, y) \right\} 
\]

\[
= \left\{ a_1 + \frac{a_2}{2} \right\} D^*(x, y, y) 
\]

\[
< D^*(x, y, y), \text{ which is contradiction. Hence } T \text{ has a unique fixed point.}
\]

REFERENCES


