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Full Length Research Paper

The moment generating function of the distribution of $X^C Y^d$

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This paper presents a proposed alternative method of expression of joint distribution of powers of two continuous random variables when both powers are necessarily not whole numbers. Hence, in finding the alternative moment generating function (AMGF) of the joint distribution of some functions of a given random variable, it is not necessary to find and use the joint distribution of these functions, it is sufficient to simply use the joint distribution of the two random variables and hence quicker to use. Unlike the regular moment generating functions, the alternative method of moment generating function is known to always exist for all continuous probability distributions.

Key words: Moment generating function, $X^c Y^d$ distribution, two random variables.

INTRODUCTION

The fundamental definitions of the moment generating function (MGF) of the distribution of random variables is $E\left(e^{tx}\right)$ for the distribution of the random variable *X* and $E\left(e^{t_1 x + t_2 y}\right)$ for the joint distribution of *x* and *y* for some non-negative real numbers $t_1 \text{ and } t_2$ (Freund, 1992).

These fundamental definitions often yield compact expressions for MGF of the distribution of random variables which when they exist uniquely identifies the probability distributions associated with them and when appropriately differentiated and evaluated at $t = 0$ yield the required moments. These MGFs do not however exist for all probability distributions. Even if they exist, they may not be used to find moments of the distributions of powers of random variables when these powers are not whole numbers (Baisnab and Manoranjan, 1993). For example, they cannot be used to find the moments of the distribution of X^c or moments of the bivariate distributions of X^c and Y^d where *c* and *d* are non-negative real numbers at least one of which is not a whole number. A more generalized expression for MGF in these cases is required. In this paper, we intend to develop expressions of the MGF for the joint distribution of X^C and Y^d, for $(c \geq 0, d \geq 0)$ both of which are not necessarily whole numbers based on the fundamental definition of MGF. The case of the MGF of the distribution of X^c where *c* is a non-negative real number not necessarily a whole number has already been discussed elsewhere (Oyeka, 1996). The cases as in which $c = d = 1$, will be treated as special cases. We assume that the random variables *x* and *y* are continuously differentiable on the real line or within their ranges of definition with joint probability density function $f(x, y)$. For lack of a better name, we here term this approach Alternative MGF (AMGF) of the joint distribution of the random variables $U = X^c \text{ and } V = Y^d$ $(c \geq 0, d \geq 0)$. Clearly, given the joint distribution of the random variables *x* and *y*, any functions of *x* and *y* such as $U = X^c \text{ and } V = Y^d$ also have their own distributions which can easily be found.

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Abbreviation: MGF, Moment generating function; **Amgf,** Alternative moment generating function.

Strictly speaking in finding the AMGF of the joint distribution of the random variables $U = X^c$ and $V = Y^d$ one would need to first find and then use the joint distribution of these random variables in the calculations.

However, as illustrated below the result obtained when using either the joint distribution of $U = X^c$ and $V = Y^d$ or simply the joint distribution of *x* and *y* is always the same.

Hence, in finding AMGF of the joint distribution of some functions such as $U = X^c$ and $V = Y^d$ $(c \geq 0, d \geq 0)$ of given random variables *x* and *y*, it is not necessary to find and use the joint distribution of these functions. It is sufficient to simply use only the joint distribution of the random variables (x and *y*) themselves in the calculations (Uche, 2003). This approach is adopted here**.**

THE PROPOSED METHOD

The MGF of the joint distribution of the continuous random variables $U = X^c$ and $V = Y^d$ with PDF $f(x, y)$ defined on the real line is:

$$M_{u,v}(t_1,t_2) = M_{X^c,Y^d}(t_1,t_2) = E\left(e^{t_1x^c + t_2y^d}\right) \qquad (t_1 \geq 0, t_2 \geq 0) \quad (1)$$

That is,

$$M_{u,v}(t_1,t_2) = M_{X^c,Y^d}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x^c + t_2y^d} f(x,.y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1 + \frac{(t_1x^c)}{1!} + \frac{(t_1x^c)^2}{2!} + \ldots + \frac{(t_1x^c)^r}{r!} + \ldots\right)\left(1 + \frac{(t_2y_d)}{1!} + \frac{(t_2y^d)^2}{2!} + \ldots + \frac{(t_2y^d)^r}{r!} + \ldots\right) f(x,y)dxc$$

$$= \sum_{r=0}^{\infty} \frac{t_1^r t_2^r}{r!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^{cr} y^{dr} f(x,y)dxdy$$

$$M_{u,v}(t_1,t_2) = M_{X^c,Y^d}(t_1,t_2) = \sum_{r=0}^{\infty} \frac{t_1^r t_2^r}{r!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^{cr} y^{dr} f(x,y)dxdy = \sum_{r=0}^{\infty} \frac{t_1^r t_2^r}{r!} \mu_r'(c,d) \quad (2)$$

Where $\mu_r'(c,d)$ is the rth moment of the joint distribution of X^C and Y^d about zero. Equation 2 is the so called "AMGF" of the joint distribution of the random variables X^C and Y^d. Equation 2 generates all conceivable moments of the joint distribution of the random variables $U = X^c$ and $V = Y^d$. The rth moment of this joint distribution is the coefficient of $\frac{t_1^r t_2^r}{r!}$ or the rth derivative of Equation 2 with respect to t_1, t_2 evaluated at $t_1 = t_2 = 0$. In the sequel, the nth moment of the joint distribution of X^C and Y^d about zero would be taken as the coefficient of $\frac{t_1^n t_2^n}{n!}$ namely $\mu_n'(c,d)$ or the nth derivative of Equation 2 with respect to t_1, t_2 evaluated at $t_1 = t_2 = 0$.

Thus,

$$\mu_n'(c,d) = M_{X^c,Y^d}(0,0)^{(n)} \quad (3)$$

Equation 2 would be used here to develop AMGFs for the joint distribution of X^C and Y^d $(c \geq 0, d \geq 0)$ given the joint distribution of the random variables X and Y. The corresponding nth moment of the joint distribution of X^C and Y^d is obtained from Equation 3. The AMGFs for cases when $c = d = 1$ that is, for joint distribution of the random variables X and Y is obtained from Equation 2 as:

$$M_{X,Y}(t_1,t_2) = \sum_{r=0}^{\infty} \frac{t_1^r t_2^r}{r!} \mu_r'(1,1) \quad (4)$$

and would be treated and discussed here as special cases. The MGF of the marginal distribution of the random variable $U = X^c$ is obtained by setting $d = 0$ and $t_2 = 1$ in Equation 2, while the corresponding nth moment is obtained by setting $d = 0$ in Equation 3. Thus,

$$M_{X^c}(t_1) = \sum_{r=0}^{\infty} \frac{t_1^r}{r!} \mu_r'(c,0) \quad (5)$$

And

$$_x\mu_n'(c) = \mu_n'(c,0) = M_{X^c}(0)^{(n)} \quad (6)$$

Similarly, the MGF of the marginal distribution of the random variable $V = Y^d$ is obtained by setting $c = 0$ and $t_1 = 1$ in Equation 2, while the corresponding nth moment is obtained by setting $c = 0$ in Equation 3. That is,

$$M_{u,v}(t_1,t_2) = M_{X^c,Y^d}(t_1,t_2) = E\left(e^{t_1x^c + t_2y^d}\right) \qquad (t_1 \geq 0, t_2 \geq 0)$$

$$M_{Y^d}(t_2) = \sum_{r=0}^{\infty} \frac{t_2^r}{r!} \mu_r'(0,d) \quad (7)$$

And

$$_y\mu_n'(d) = \mu_n'(0,d) = M_{Y^d}(0)^{(n)} \quad (8)$$

To illustrate the use of Equation 2, suppose the continuous random variables X and Y have the joint PDF,

$$f(x,y) = \frac{2}{\alpha + \beta}(\alpha x + \beta y) \qquad (0 < x < 1, 0 < y < 1) \quad (9)$$

$$g(u,v)=\frac{2\alpha}{cd(\alpha+\beta)}u^{\frac{2}{c}-1}v^{\frac{1}{d}-1}+\frac{2\beta}{cd(\alpha+\beta)}u^{\frac{1}{c}-1}v^{\frac{2}{d}-1} \qquad 0<u<1, 0<v<1 \tag{10}$$

Therefore, the AMGF for the joint distribution of $U = X^c$ and $V = Y^d$ is obtained by using Equation 10 as:

$$M_{u,v}(t_1,t_2)=E\left(e^{t_1u+t_2v}\right)=\frac{2\alpha}{cd(\alpha+\beta)}\int_0^1\int_0^1 u^{\frac{2}{c}-1}v^{\frac{1}{d}-1}e^{t_1u+t_2v}dudv+\frac{2\beta}{cd(\alpha+\beta)}\int_0^1\int_0^1 u^{\frac{1}{c}-1}v^{\frac{2}{d}-1}e^{t_1u+t_2v}dudv$$

$$=\frac{2\alpha}{cd(\alpha+\beta)}\int_0^1\int_0^1 u^{\frac{2}{c}-1}v^{\frac{1}{d}-1}\left(1+\frac{(t_1u)}{1!}+\frac{(t_1u)^2}{2!}+...+\frac{(t_1u)^r}{r!}+...\right)\left(1+\frac{(t_2v)}{1!}+\frac{(t_2v)^2}{2!}+...+\frac{(t_2v)^r}{r!}+...\right)dud+\frac{2\beta}{cd(\alpha+\beta)}\int_0^1\int_0^1 u^{\frac{1}{c}-1}v^{\frac{2}{d}-1}\left(1+\frac{(t_1u)}{1!}+\frac{(t_1u)^2}{2!}+...+\frac{(t_1u)^r}{r!}+...\right)\left(1+\frac{(t_2v)}{1!}+\frac{(t_2v)^2}{2!}+...+\frac{(t_2v)^r}{r!}+...\right)dudv$$

$$=\frac{2\alpha}{cd(\alpha+\beta)}\sum_{r=0}^{\infty}\frac{t_1^r t_2^r}{r!}\int_0^1\int_0^1 u^{\frac{2}{c}+r-1}v^{\frac{1}{d}+r-1}dudv+\sum_{r=0}^{\infty}\frac{t_1^r t_2^r}{r!}\int_0^1\int_0^1 u^{\frac{1}{c}+r-1}v^{\frac{2}{d}+r-1}dudv$$

$$\therefore M_{u,v}(t_1,t_2)=\frac{2\alpha}{(\alpha+\beta)}\sum_{r=0}^{\infty}\frac{t_1^r t_2^r}{r!(cr+2)(dr+1)}+\frac{2\alpha}{(\alpha+\beta)}\sum_{r=0}^{\infty}\frac{t_1^r t_2^r}{r!(cr+1)(dr+2)} \tag{11}$$

Equation 11 is the AMGF of the joint distribution of the random variables $U = X^c$ and $V = Y^d$ given in Equation 10. The corresponding nth moment about zero is obtained from Equation 11 as:

$$\mu_n'(c,d)=\frac{2\alpha}{(\alpha+\beta)(cn+2)(dn+1)}+\frac{2\beta}{(\alpha+\beta)(cn+1)(dn+2)} \tag{12}$$

Setting $d=0$ and $t_2=1$ in Equation 11 gives the AMGF of the marginal distribution of $U = X^c$ as:

$$\frac{2\alpha}{(\alpha+\beta)}\sum_{r=0}^{\infty}\frac{t_1^r}{r!(cr+2)}+\frac{2\beta}{(\alpha+\beta)}\sum_{r=0}^{\infty}\frac{t_1^r}{r!\,2(cr+1)} \tag{13}$$

The nth moment of this marginal distribution is obtained from Equation 12 as:

$$_x\mu_n'(c)=\mu_n'(c,0)=\frac{2\alpha}{(\alpha+\beta)(cn+2)}+\frac{2\beta}{2(\alpha+\beta)(cn+1)} \tag{14}$$

The AMGF for the marginal distribution of $V = Y^d$ and the corresponding nth moment are similarly obtained. If in Equation 11 we set $c=d=1$, we have that the AMGF of the joint distribution of X and Y given in Equation 9 is:

$$M_{X,Y}(t_1,t_2)=\frac{2\alpha}{(\alpha+\beta)}\sum_{r=0}^{\infty}\frac{t_1^r t_2^r}{r!(r+2)(r+1)}+\frac{2\beta}{(\alpha+\beta)}\sum_{r=0}^{\infty}\frac{t_1^r t_2^r}{r!(r+1)(r+2)} \tag{15}$$

So that the corresponding nth moment about zero is

$$\mu_n'(1,1)=\frac{2}{(n+1)(n+2)} \tag{16}$$

NOTE: That the usual MGF of the joint distribution of X and Y given in Equation 9 is

$$M_{X,Y}(t_1,t_2)=\frac{2}{\alpha+\beta}\left[\alpha\left(\frac{e^{t_2}-1}{t_2}\right)\left(\frac{e^{t_1}}{t_1}-\frac{(e^{t_1}-1)}{t_1^2}\right)+\beta\left(\frac{e^{t_1}-1}{t_1}\right)\left(\frac{e^{t_2}}{t_2}-\frac{(e^{t_2}-1)}{t_2^2}\right)\right]$$

which does not exist at $t_1=t_2=0$ and hence cannot be used to obtain the moments of the distribution given in Equation 15. Now, if we have in fact directly used the joint distribution of X and Y given in Equation 9 to find the AMGF of the joint distribution of $U = X^c$ and $V = Y^d$ we would have that,

$$M_{X^c,Y^d}(t_1,t_2)=E\left(e^{t_1x^c+t_2y^d}\right)$$

$$=\frac{2}{(\alpha+\beta)}\int_0^1\int_0^1 e^{t_1x^c+t_2y^d}(\alpha x+\beta y)dxdy$$

$$=\frac{2\alpha}{\alpha+\beta}\int_0^1\int_0^1 x\left(e^{t_1x^c+t_2y^d}\right)dxdy+\frac{2\beta}{\alpha+\beta}\int_0^1\int_0^1 y\left(e^{t_1x^c+t_2y^d}\right)dxdy$$

$$=\frac{2\alpha}{(\alpha+\beta)}\int_0^1\int_0^1 x\left(1+\frac{(t_1x^c)}{1!}+\frac{(t_1x^c)^2}{2!}+...+\frac{(t_1x^c)^r}{r!}+...\right)\left(1+\frac{(t_2y^d)}{1!}+\frac{(t_2y^d)^2}{2!}+...+\frac{(t_2y^d)^r}{r!}+...\right)dxd+\frac{2\beta}{(\alpha+\beta)}\int_0^1\int_0^1 y$$

$$\left(1+\frac{(t_1x^c)}{1!}+\frac{(t_1x^c)^2}{2!}+...+\frac{(t_1x^c)^r}{r!}+...\right)\left(1+\frac{(t_2y^d)}{1!}+\frac{(t_2y^d)^2}{2!}+...+\frac{(t_2y^d)^r}{r!}+...\right)dxa$$

$$=\frac{2\alpha}{\alpha+\beta}\sum_{r=0}^{\infty}\frac{t_1^r t_2^r}{r!}\int_0^1\int_0^1 x^{cr-1}y^{dr}dxdy+\frac{2\beta}{\alpha+\beta}\sum_{r=0}^{\infty}\frac{t_1^r t_2^r}{r!}\int_0^1\int_0^1 x^{cr}y^{dr+1}dxd$$

That is,

$$M_{X^c,Y^d}(t_1,t_2)=\frac{2\alpha}{(\alpha+\beta)}\sum_{r=0}^{\infty}\frac{t_1^r t_2^r}{r!(cr+2)(dr+1)}$$

$$+\frac{2\beta}{(\alpha+\beta)}\sum_{r=0}^{\infty}\frac{t_1^r t_2^r}{r!(cr+1)(dr+2)}$$

This is the same as Equation 11 obtained by using the joint distribution of X^C and Y^d. The corresponding nth moment is

$$\mu_n'(c,d)=\frac{2\alpha}{(\alpha+\beta)(cn+2)(dn+1)}+\frac{2}{(\alpha+\beta)(cn+1)(dn+1)}$$

the same as Equation 12.

The corresponding MGF of the marginal distributions of X^C and Y^d, and the corresponding nth moments are also the same. As another example, suppose

$$f(x,y)=\frac{(x+y)e^{-\left(\frac{x}{\beta_1}+\frac{y}{\beta_2}\right)}}{\beta_1\beta_2(\beta_1+\beta_2)} \qquad x>0,\ y>0 \tag{17}$$

Then the AMGF of the joint distribution of $U=X^c$ and $V=Y^d$ based on the joint distribution of X and Y given in Equation 17 is obtained using Equation 2 as:

$$M_{X^c,Y^d}(t_1,t_2)=E\left(e^{t_1x^c+t_2y^d}\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{t_1x^c+t_2y^d}f(x,y)dxdy$$

$$=\frac{1}{\beta_1\beta_2(\beta_1+\beta_2)}\int_0^{\infty}\int_0^{\infty}(x+y)e^{-\left(\frac{x}{\beta_1}+\frac{y}{\beta_2}\right)}e^{t_1x^c+t_2y^d}dxdy$$

$$=\frac{1}{\beta_1\beta_2(\beta_1+\beta_2)}\int_0^{\infty}\int_0^{\infty}xe^{-\left(\frac{x}{\beta_1}+\frac{y}{\beta_2}\right)}e^{t_1x^c+t_2y^d}dxdy$$

$$+\frac{1}{\beta_1\beta_2(\beta_1+\beta_2)}\int_0^{\infty}\int_0^{\infty}ye^{-\left(\frac{x}{\beta_1}+\frac{y}{\beta_2}\right)}e^{t_1x^c+t_2y^d}dxdy$$

$let\ z_1=\dfrac{x}{\beta_1}$ and $z_2=\dfrac{y}{\beta_2}$

Then

$$M_{X^c,Y^d}(t_1,t_2)=\frac{\beta_1}{\beta_1+\beta_2}\int_0^{\infty}\int_0^{\infty}z_1e^{-(z_1+z_2)}e^{t_1\beta_1^cz_1^c+t_2\beta_2^dz_2^d}dz_1dz_2$$

$$+\frac{\beta_1}{\beta_1+\beta_2}\int_0^{\infty}\int_0^{\infty}z_2e^{-(z_1+z_2)}e^{t_1\beta_1^cz_1^c+t_2\beta_2^dz_2^d}dz_1dz_2$$

$$=\frac{\beta_1}{\beta_1+\beta_2}\int_0^{\infty}\int_0^{\infty}z_1\left(1+\frac{(t_1\beta_1^cz_1^c)}{1!}+\frac{(t_1\beta_1^cz_1^c)^2}{2!}+\ldots+\frac{(t_1\beta_1^cz_1^c)^r}{r!}+\ldots\right)\left(1+\frac{(t_2\beta_2^dz_2^d)}{1!}+\frac{(t_2\beta_2^dz_2^d)^2}{2!}+\ldots+\frac{(t_2\beta_2^dz_2^d)^r}{r!}+\ldots\right)e^{-(z_1+z_2)}dz_1dz_2$$

$$+\frac{\beta_1}{\beta_1+\beta_2}\int_0^{-\infty}\int_0^{-\infty}z_2\left(1+\frac{(t_1\beta_1^cz_1^c)}{1!}+\frac{(t_1\beta_1^cz_1^c)^2}{2!}+\ldots+\frac{(t_1\beta_1^cz_1^c)^r}{r!}+\ldots\right)\left(1+\frac{(t_2\beta_2^dz_2^d)}{1!}+\frac{(t_2\beta_2^dz_2^d)^2}{2!}+\ldots+\frac{(t_2\beta_2^dz_2^d)^r}{r!}+\ldots\right)e^{-(z_1+z_2)}dz_1dz_2$$

$$=\frac{\beta_1}{\beta_1+\beta_2}\sum_{r=0}^{\infty}\frac{t_1^rt_2^r}{r!}\beta_1^{cr}\beta_2^{dr}\int_0^{\infty}\int_0^{\infty}z_1^{cr+1}z_2^{dr+1}e^{-(z_1+z_2)}dz_1dz_2+\frac{\beta_2}{\beta_1+\beta_2}$$

$$\sum_{r=0}^{\infty}\frac{t_1^rt_2^r}{r!}\beta_1^{cr}\beta_2^{dr}\int_0^{\infty}\int_0^{\infty}z_1^{cr}z_2^{dr+1}e^{-(z_1+z_2)}dz_1dz_2$$

$$=\frac{\beta_1}{\beta_1+\beta_2}\sum_{r=0}^{\infty}\frac{t_1^rt_2^r}{r!}\beta_1^{cr}\beta_2^{dr}\Gamma(cr+2)\Gamma(dr+1)+\frac{\beta_2}{\beta_1+\beta_2}$$

$$\sum_{r=0}^{\infty}\frac{t_1^rt_2^r}{r!}\beta_1^{cr}\beta_2^{dr}\Gamma(cr+1)\Gamma(dr+2)$$

$$i.\text{e. } M_{X^c,Y^d}(t_1,t_2)=\frac{\beta_1\sum_{r=0}^{\infty}\frac{t_1^rt_2^r}{r!}\beta_1^{cr}\beta_2^{dr}\Gamma(cr+2)\Gamma(dr+1)+\beta_2\sum_{r=0}^{\infty}\frac{t_1^rt_2^r}{r!}\beta_1^{cr}\beta_2^{dr}\Gamma(cr+1)\Gamma(dr+2)}{\beta_1+\beta_2} \tag{18}$$

Or equivalently,

$$M_{X^c,Y^d}(t_1,t_2)=\frac{\beta_1\sum_{r=0}^{\infty}\frac{t_1^rt_2^r}{r!}cdr^2\beta_1^{cr}\beta_2^{dr}\Gamma(cr)\Gamma(dr)(\beta_1(cr+1)+\beta_2(dr+1))}{\beta_1+\beta_2} \tag{19}$$

Equation 18 or equivalently Equation 19 is the AMGF of the joint distribution of X^C and Y^d where X and Y have the joint PDF given in Equation 17, and generates all conceivable moments about zero of the joint distribution of X^C and Y^d. Thus, the nth moment of this joint distribution is obtained from Equation 19 as the coefficient of $\dfrac{t_1^nt_2^n}{n!}$ namely;

$$\mu_n'(c,d)=\frac{cdn^2\beta_1^{cn}\beta_2^{dn}\Gamma(cn)\Gamma(dn)(\beta_1(cn+1)+\beta_2(dn+1))}{\beta_1+\beta_2} \tag{20}$$

The MGF of the marginal distribution of X^c is obtained by setting $d = 0$, and $t_2 = 1$ in Equation 18 yielding

$$M_{X^c}(t_1, 0) = \frac{\beta_1 \sum_{r=0}^{\infty} \frac{t_1^r}{r!} \beta_1^{cr} \Gamma(cr+2) + \beta_2 \sum_{r=0}^{\infty} \frac{t_1^r}{r!} \beta_1^{cr} \Gamma(cr+1)}{\beta_1 + \beta_2}$$

$$= \frac{\sum_{r=0}^{\infty} \frac{t_1^r}{r!} cr \beta_1^{cr} \Gamma(cr)(\beta_1(cr+1) + \beta_2)}{\beta_1 + \beta_2} \quad (21)$$

The corresponding nth moment of this marginal distribution is the coefficient of $\frac{t_1^n}{n!}$ namely

$$\mu_n'(c, 0) = \frac{cn \beta_1^{cn} \Gamma(cn)(\beta_1(cn+1) + \beta_2)}{\beta_1 + \beta_2} \quad (22)$$

The AMGF and the corresponding moments of the marginal distribution of Y^d are similarly obtained by setting $c = 0$, and $t_1 = 1$ in Equation 18. If $c = d = 1$, then we have that the AMGF of the joint distribution of X and Y given in Equation 17 is obtained by setting $c = d = 1$ in either Equation 18 or Equation 19 yielding

$$M_{X,Y}(t_1, t_2) = \frac{\sum_{r=0}^{\infty} \frac{t_1^r t_2^r}{r!} r^2 \beta_1^r \beta_2^r \Gamma(r)^2 (\beta_1(r+1) + \beta_2(r+1))}{\beta_1 + \beta_2}$$

$$i.e\ M_{X,Y}(t_1 t_2) = \sum_{r=0}^{\infty} \frac{t_1 t_2}{r!} r^2 (r+1) \beta_1^r \beta_2^r (\Gamma(r))^2 \quad (23)$$

The corresponding nth moment of the joint distribution of X and Y is

$$\mu_n'(1,1) = n^2(n+1)\beta_1^n \beta_2^n (\Gamma(n))^2 \quad (24)$$

Note that Equation 24 is much easier and faster to use in finding the nth moment of the joint distribution of X and Y given in Equation 17 than differentiating the corresponding regular MGF of this joint distribution namely $M_{X,Y}(t_1, t_2) = \frac{(\beta_1 + \beta_2 - \beta_1\beta_2(t_1 + t_2)(1-\beta_1 t_1)^{-2}(1-\beta_2 t_2)^{-2})}{\beta_1 + \beta_2}$ n times and evaluating the results at $t_1 = t_2 = 0$. Finally suppose for simplicity that X and Y are independently normally distributed with joint PDF

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right)} \qquad -\infty < x < \infty, -\infty < y < \infty \quad (25)$$

The AMGF of the joint distribution of $U = X^c$ and $V = Y^d$ is

$$M_{X^c,Y^d}(t_1, t_2) = E\left(e^{t_1 x^c + t_2 y^d}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x^c + t_2 y^d} f(x,y) dxdy$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x^c + t_2 y^d} e^{-\frac{1}{2}\left(\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right)} dxd.$$

Let $z_1 = \frac{x - \mu_1}{\sigma_1}$ and $z_2 = \frac{x - \mu_2}{\sigma_2}$

$$M_{X^c,Y^d}(t_1, t_2) = \frac{1}{2\mu} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1(\sigma_1 z_1 + \mu_1)^c + t_2(\sigma_2 z_2 + \mu_2)^d} e^{-\frac{1}{2}(z_1^2 + z_2^2)} dz_1 dz_2$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1 + \frac{t_1}{1!}(\sigma_1 z_1 + \mu_1)^c + \frac{t_1^2}{2!}(\sigma_1 z_1 + \mu_1)^{2c} + \ldots + \frac{t_1^r}{r!}(\sigma_1 z_1 + \mu_1)^{cr} + \ldots\right)\left(1 + \frac{t_2}{1!}(\sigma_2 z_2 + \mu_2)^d + \frac{t_2^2}{2!}(\sigma_2 z_2 + \mu_2)^{2d} + \ldots + \frac{t_2^r}{r!}(\sigma_2 z_2 + \mu_2)^{dr} + \ldots\right) e^{-\frac{1}{2}(z_1^2 + z_2^2)} dz_1 dz_2$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{r=0}^{\infty}\frac{t_1^r t_2^r}{r!}(\sigma_1 z_1 + \mu_1)^{cr}(\sigma_2 z_2 + \mu_2)^{dr} e^{-\frac{1}{2}(z_1^2 + z_2^2)} dz_1 dz_2$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{r=0}^{\infty}\frac{t_1^r t_2^r}{r!}\sum_{s=0}^{cr}\binom{cr}{s}\mu_1^{cr-s}(\sigma_1 z_1)^s \sum_{s=0}^{dr}\binom{dr}{s}(\sigma_2 z_2)^s e^{-\frac{1}{2}(z_1^2 + z_2^2)} dz_1 dz_2$$

Let $z_3 = \frac{z_1^2}{2}$ and $z_4 = \frac{z_2^2}{2}$ then,

$$M_{X^c,Y^d}(t_1, t_2)$$

$$= \frac{1}{\pi}\sum_{r=0}^{\infty}\frac{t_1^r t_2^r}{r!}\sum_{s=0}^{cr}\binom{cr}{s}\mu_1^{cr-s}(2\sigma_1^2)^{\frac{s}{2}}\sum_{s=0}^{dr}\binom{dr}{s}\mu_2^{dr-s}(2\sigma_2^2)^{\frac{s}{2}}\int_0^{\infty}\int_0^{\infty} z_3^{\frac{s}{2}-\frac{1}{2}} z_4^{\frac{s}{2}-\frac{1}{2}} e^{-(z_3 + z_4)} dz_3 dz_4$$

Hence, $M_{X^c,Y^d}(t_1, t_2)$

$$=\frac{1}{\pi}\sum_{r=0}^{\infty}\frac{t_1^r t_2^r}{r!}\sum_{s=0}^{cr}\binom{cr}{s}\mu_1^{cr-s}\left(2\sigma_1^2\right)^{\frac{s}{2}}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\sum_{s=0}^{dr}\binom{dr}{s}\mu_2^{dr-s}\left(2\sigma_2^2\right)^{\frac{s}{2}}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right) \quad (26)$$

Provided we set $\frac{\left(2\sigma_1^2\right)^{\frac{s}{2}}\left(2\sigma_2^2\right)^{\frac{s}{2}}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)^2}{\pi}=0$ for all odd values of s. that is, $s=1,3,5,\ldots$ since with $z_1=\frac{x-\mu_1}{\sigma_1}$ and $z_2=\frac{x-\mu_2}{\sigma_2}$,

$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_1^s z_2^s e^{-\frac{1}{2}\left(z_1^2+z_2^2\right)} dz_1 dz_2 = E\left(z_1^s z_z^s\right)=0$ for all odd values of $s=1,3,5,\ldots$

as may be easily verified. Equation 26 generates all conceivable moments of the joint distribution of $X^C \text{ and } Y^d$. Thus, the nth moment of the joint distribution of $X^C \text{ and } Y^d$ is the coefficient of $\frac{t_1^n t_2^n}{n!}$ or the nth derivative of Equation 26 with respect to $t_1 \text{ and } t_2$ evaluated at $t_1=t_2=0$. That is,

$$\mu_n'(c,d)=M_{X^c,Y^d}(0,0)=\sum_{s=0}^{cn}\binom{cn}{s}\mu_1^{cn-s}\left(2\sigma_1^2\right)^{\frac{s}{2}}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\sum_{s=0}^{dn}\binom{dn}{s}\mu_2^{dn-s}\left(2\sigma_2^2\right)^{\frac{s}{2}}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right) \quad (27)$$

subject to the provision of Equation 26. If we set $c=d=1$ in Equation 26 we obtain the AMGF for the bivariate normal distribution of Equation 26.

$$M_{X,Y}(t_1,t_2)=\sum_{r=0}^{\infty}\frac{t_1^r t_2^r}{r!}\sum_{s=0}^{r}\binom{r}{s}\mu_1^{r-s}\left(2\sigma_1^2\right)^{\frac{s}{2}}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\sum_{s=0}^{r}\binom{r}{s}\mu_2^{r-s}\left(2\sigma_2^2\right)^{\frac{s}{2}}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right) \quad (28)$$

subject to the provision of Equation 26. The corresponding nth moment of the bivariate normal distribution of Equation 25 is obtained from Equation 28 as the coefficient of $\frac{t_1^n t_2^n}{n!}$ namely

$$\mu_n'=\sum_{s=0}^{n}\binom{n}{s}\mu_1^{n-s}\left(2\sigma_1^2\right)^{\frac{s}{2}}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\sum_{s=0}^{n}\binom{n}{s}\mu_2^{n-s}\left(2\sigma_2^2\right)^{\frac{s}{2}}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right) \quad (29)$$

subject to the provision of Equation 26. Equation 29 is easier and quicker to use in finding the nth moment of the bivariate normal distribution given in Equation 25 than differentiating n times (with respect to $t_1 \text{ and } t_2$) the corresponding regular mgf of this joint distribution.

Conclusion

In this paper, we developed an alternative method of the MGF of the joint distribution of powers of two continuous random variables, where these powers need not be both whole numbers. Unlike the regular MGF, AMGF are known to always exist for all continuous probability distributions. AMGF of the joint distributions of the random variables themselves are presented as special cases. The proposed method is easier and quicker to use to obtain moments of joint distributions of continuous random variables than the usual or regular MGFs of these distributions when they exist.

REFERENCES

Baisnab AP, Manoranjan J (1993). Elements of Probability and Statistics, Tata McGraw – Hill Publishing Company Limited, New Delhi. pp. 208-232.

Freund JE (1992). Mathematical Statistics (5th Edition), Prentice – Hall International Editions, USA. pp. 161-177.

Oyeka CA (1996). An Introduction to Applied Statistical Methods, Nobern Avocation publishing Company, Enugu. pp. 113-117.

Uche PI (2003). Probability: Theory and Applications, Longman Nigeria PLC, Ikeja Lagos. pp. 149-155.