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# Euler-Maruyama method for solving first order uncertain stochastic differential equations

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**Two forms of uncertainty are identified to be associated with dynamical systems, which are randomness and belief degree. The uncertain stochastic differential equation (USDE) is used to describe dynamical systems driven simultaneously by randomness and human uncertainty (belief degree). In this paper, the Euler-Maruyama method for solving USDEs is examined. The method is used to solve a stock pricing problem and the results are compared with those of Runge Kutta of order 4. The Euler-Maruyama method yields lower stock prices, while the stock prices from the Runge Kutta method proved to converge faster to those from the analytical method. At α = 0.5 where α ∈ (0, 1), the USDE reverts to the stochastic differential equation, with the uncertain component eliminated, showing that the USDE is indeed a hybrid of the uncertain differential equation and stochastic differential equation.**

**Key words:** Euler-Maruyama, uncertain stochastic differential equation, stock price, stochastic contour process.

## INTRODUCTION

The significance of differential equations in modeling dynamical systems cuts across biology, engineering, physics, and finance, etc. In simple term, dynamical systems are time-dependent systems. Two forms of uncertainty are identified to be associated with dynamical systems, which are randomness and belief degree. The class of differential equations that model dynamical systems associated with randomness only is known as the stochastic differential equation (SDE). While the class of differential equation that models dynamical systems associated with belief degrees only is known as uncertain differential equations (UDE). The foundation for SDEs is probability theory which started from the work of Kolmogorov (1933), and has since evolved, and studied by different scholars including Bachelier (1900), Black and Scholes (1973), Merton (1973), Hull and White (1990) and Dmouj (2006). Since the ground breaking work of the Japanese mathematician, Ito Kiyoshi in 1949 on stochastic processes' differentiation and integration, SDEs have been successfully applied by different researchers when it comes to discussing the modeling of dynamical systems. As an example, in financial processes, Black and Scholes (1973) won the Nobel Prize in economics for successfully using this concept to price options in their model. On the other hand, the UDE was first mentioned by Liu (2009) sequel to the ground breaking work of Liu (2007) which birthed uncertainty theory. Since then, researchers have followed in expanding the boundaries of knowledge in this new branch of mathematics. UDEs have been successfully

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applied in describing the evolution of dynamical systems. For example, Liu (2009) proposed an uncertain stock model for option pricing. Overtime as researchers continue in study, it was observed that the two forms of uncertainty (randomness and belief degree) can simultaneously drive a dynamical system. As a result, a new class of differential equations called as uncertain stochastic differential equations was created. As a result, a new class of differential equations called uncertain stochastic differential equations (USDE) was created. Just as the SDEs are modeled by probability theory and the UDEs by uncertainty theory, the USDEs are modeled by chance theory. The USDEs are driven by uncertain random processes, which are a collection of uncertain random variables. Chance theory came into being through the work of Liu (2013a), where he introduced the concept of chance measure by combining uncertain and probability measures. Researchers have studied this new concept of chance theory, and have equally contributed to its development over these years. Using probability and uncertainty distribution, Liu (2013b) provided a method for determining an uncertain random variable's expected value as well as a definition of their law of operation. Fei (2014) combined the Itˆo integral and Liu integral into the Itˆo-Liu integral to integrate an uncertain random process. He also provided the Itˆo-Liu formula for finding the derivative of the hybrid process.

Like the SDEs and UDEs, the USDEs have also been successfully applied in describing dynamical systems. Matenda and Chikodza (2018) suggested an uncertain stochastic stock model with jump using an USDE. They also outlined the analytical solutions to some USDE. The exact and analytical solutions to USDEs are not easily solved and do not exist for some. However, it is possible to approximate the solution numerically, just as with SDEs and UDEs. Hence to address this challenge, it is necessary to create algorithms to numerically solve USDEs. Chirima et al. (2020) used the Runge-Kutta of order four (RK4) to numerically solve USDEs. In order to use numerical schemes of solving ODEs for USDEs, they converted the USDEs into ODEs with α-path, whose solution is a contour process. They also gave the theorems and proofs for existence, uniqueness and stability of USDEs. Alternatively, the Euler-Maruyama (EM) method is used to numerically solve USDEs in this paper. The EM method is applied to calculate the stock price of an uncertain stochastic market, and the results are compared to that of RK4 proposed by Chirima et al. (2020).

One idea this paper contributes is using the EM method to solve USDEs. In preparation for the main work of this paper, the next section presents some basic definitions of useful terms. In the following section, we present the uncertain stochastic differential equation (USDE) and the theorems for their existence, uniqueness and stability. In the numerical example section we use the EM technique with α-path to compute the price of the stock from a considered stock model of an uncertain random market. We also show the results in table and figures. In the final section, conclusions are given.

## PRELIMINARIES

We present some preliminary material which will be of use later on.

### Stochastic analysis

#### *Stochastic process*

**Definition 2.1:** A stochastic process is a function:

$$\mathrm{Y}_t(\psi): T \times (\Psi, \mathcal{A}, \mathcal{P}) \to \Re$$

such that $\{\mathrm{Y}_t \in \mathrm{B}\}$ is an event for any Borel set B at each time t. Where $(\Psi, \mathcal{A}, \mathcal{P})$ is a probability space and T is a totally ordered set.

**Definition 2.2 (Brownian motion/Wiener process):** A stochastic process $\mathrm{B}_t$ is called a standard Brownian motion or Wiener process if it satisfies the following properties

i. $\mathrm{B}_0 = 0$ and almost all sample paths are continuous,
ii. $\mathrm{B}_t$ has stationary and independent increments,
iii. every increment $\mathrm{B}_{s+t} - \mathrm{B}_t$ is a normal random variable with expected value 0 and variance t.

#### *Stochastic differential equation*

**Definition 2.3.** The differential equation: $\mathrm{d}Y_t = m(t, Y_t)\mathrm{dt} + n(t, Y_t)\mathrm{d}B_t$ is called a stochastic differential equation. Where, $B_t$ is a standard Brownian motion and m and n are two given functions. An Ito process $Y_t$ that satisfies the equation about identically in *t* is called the solution of the stochastic differential equation.

### Uncertain analysis

#### *Uncertain process*

**Definition 2.4.** An uncertain process is a function:

$$\mathrm{Y}_t(\pi): T \times (\Pi, \mathcal{B}, \mathcal{U}) \to \Re$$

such that $\{\mathrm{Y}_t \in \mathrm{B}\}$ is an event for any Borel set B at each time *t*. Where $(\Pi, \mathcal{B}, \mathcal{U})$ is an uncertainty space, and T is a totally ordered set.

#### *Contour process*

**Definition 2.5:** Let $Y_t$ be an uncertain process. If for each

$\alpha \in (0,1)$, there exist a real function $Y_t^\alpha$ such that

$$\mathcal{M}\{Y_t \le Y_t^\alpha, \forall t\} = \alpha$$
$$\mathcal{M}\{Y_t > Y_t^\alpha, \forall t\} = 1 - \alpha$$

then $Y_t$ is called a contour process. In this case, $Y_t^\alpha$ is called an α-path of the contour process $Y_t$.

**Theorem 2.6:** (Yao and Chen, 2013) Let $Y_t$ be an uncertain process with an α-path $Y_t^\alpha$. Then $Y_t$ has an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = Y_t^\alpha, \forall \alpha \in (0,1)$$

This theorem is also known as the inverse uncertainty distribution theorem of a contour process.

**Definition 2.7:** An uncertain process $L_t$ is said to be a canonical Liu process if:

i. $L_0 = 0$ and almost all sample paths are Lipschitz continuous,
ii. $L_t$ has stationary and independent increments,
iii. every increment $L_{s+t} - L_t$ is a normal random variable with expected value 0 and variance $t^2$.

**Definition 2.8:** The inverse uncertainty distribution of the canonical Liu process is given by

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}$$

***Uncertain differential equation***

**Definition 2.8:** The differential equation

$$\mathrm{d}Y_t = m(t, Y_t)\mathrm{d}t + n(t, Y_t)\mathrm{d}L_t \quad (1)$$

is called an uncertain differential equation. Where, $L_t$ is a canonical Liu process and *m* and *n* are two given functions. A Liu process $Y_t$ that satisfies Equation 1 identically in *t* is known as the solution of the uncertain differential equation. The uncertain differential equation defined above is equivalent to the uncertain integral equation

$$Y_s = Y_0 + \int_0^s m(t, Y_t)\mathrm{d}t + \int_0^s n(t, Y_t)\mathrm{d}L_t$$

**Chance theory**

We shall be presenting previous literatures developed on chance theory that are important to this study.

**Definition 2.9:** A chance space is defined as the product space $(\Pi, \mathcal{B}, \mathcal{U}) \times (\Psi, \mathcal{A}, \mathcal{P})$, where $(\Pi, \mathcal{B}, \mathcal{U})$ is an uncertainty space and $(\Psi, \mathcal{A}, \mathcal{P})$ is a probability space.

**Definition 2.10:** Given a chance space $(\Pi, \mathcal{B}, \mathcal{U}) \times (\Psi, \mathcal{A}, \mathcal{P})$ and an uncertain random event $\Phi \in \mathcal{B} \times \mathcal{A}$, the chance measure C of Φ is defined by

$$\mathcal{C}\{\Phi\} = \int_0^1 \mathcal{P}\{\psi \in \Psi \mid \mathcal{U}\{\pi \in \Pi \mid (\pi, \psi) \in \Phi\} \ge x\}dx$$

A chance measure satisfies the following properties:

1. (Liu, 2013b) Normality: $\mathcal{C}\{\Pi \times \Psi\} = 1, \mathcal{C}\{0\} = 0$
2. (Liu, 2013b) Monotonicity: $\mathcal{C}\{\Phi_1\} \le \mathcal{C}\{\Phi_2\}$ for any event $\Phi_1 \subset \Phi_2$
3. (Liu, 2013b) Self-duality: $\mathcal{C}\{\Phi\} + \mathcal{C}\{\Phi^c\} = 1$ for any event Φ
4. (Hou, 2014) Sub-additivity: For any countable sequence of events $\Phi_1, \Phi_2, \ldots$,

$$\mathcal{C}\left\{\bigcup_{i=1}^{\infty} \Phi_i\right\} \le \sum_{i=1}^{\infty} \mathcal{C}\{\Phi_i\}$$

5. (Hou, 2014) Null-additivity: Suppose $\Phi_1, \Phi_2, \ldots$, are a sequence of events with $\mathcal{C}\{\Phi_i\} \to 0$ as $i \to \infty$. Then for any event $\lim_{i\to\infty} \mathcal{C}\{\Phi \cup \Phi_i\} = \lim_{i\to\infty} \mathcal{C}\{\Phi \setminus \Phi_i\} = \mathcal{C}\{\Phi\}$ That means $\mathcal{C}\{\Phi_1 \cup \Phi_2\} = \mathcal{C}\{\Phi_1\} + \mathcal{C}\{\Phi_2\}$ if either $\mathcal{C}\{\Phi_1\} = 0$ or $\mathcal{C}\{\Theta_2\} = 0$
6. (Hou, 2014) Asymptotic: For any sequence of events $\Phi_1, \Phi_2, \ldots$,
$\lim_{i\to\infty} \mathcal{C}\{\Phi_i\} > 0, \text{if } \Phi_i \uparrow \Pi \times \Psi$
$\lim_{i\to\infty} \mathcal{C}\{\Phi_i\} < 0, \text{if } \Phi_i \downarrow \emptyset$

**Definition 2.11:** Given a chance space$(\Pi, \mathcal{B}, \mathcal{U}) \times (\Psi, \mathcal{A}, \mathcal{P})$, an uncertain random variable is defined as a function $\zeta : (\Pi, \mathcal{B}, \mathcal{U}) \times (\Psi, \mathcal{A}, \mathcal{P}) \to B \in \Re$such that $\zeta \in B \in (\mathcal{B} \times \mathcal{A})$.

**Definition 2.12:** Given a chance space $(\Pi, \mathcal{B}, \mathcal{U}) \times (\Psi, \mathcal{A}, \mathcal{P})$ and a totally ordered set T, an uncertain random process is a function:

$$Y_t(\pi, \psi): T \times (\Pi, \mathcal{B}, \mathcal{U}) \times (\Psi, \mathcal{A}, \mathcal{P}) \to \Re$$

such that $Y_t \in B \in (\mathcal{B} \times \mathcal{A})$ for any Borel set $B(\Re)$ at each time t.

**Remark 1:** Suppose $Y_t$ and $Z_t$ are uncertain and stochastic processes, respectively, then $X_t = m(Y_t, Z_t)$ is an uncertain random process given a measurable function f.

**Definition 2.13:** Given fixed $\pi^* \in \Pi$ and $\psi^* \in \Psi$, the sample path of an uncertain random process $Y_t$ on a chance space $(\Pi, \mathcal{B}, \mathcal{U}) \times (\Psi, \mathcal{A}, \mathcal{P})$ is defined as the function $Y_t(\pi^*, \psi^*)$.

***Ito-Liu integral***

**Definition 2.14:** (Fei, 2014) Let $Y_t = (F_t, G_t)$ be an uncertain stochastic process. For any partition $P = \{a = t_1, t_2, \ldots, t_{k+1} = b\}$ of the closed interval [a, b] with $a = t_1 < t_2 < \cdots < t_{k+1} = b$ the mesh is written as $\Delta = \max_{1\le i\le k}|t_{i+1} - t_i|$. Then the Ito-Liu integral of $Y_t$ with respect to $K_t = (B_t, L_t)$ is defined as follows,

$$\int_a^b Y_s \mathrm{d}K_s = \lim_{\Delta\to 0}\sum_{i=1}^{N} G_{t_i}\left(B_{t_{i+1}} - B_{t_i}\right) + \lim_{\Delta\to 0}\sum_{i=1}^{N} F_{t_i}\left(L_{t_{i+1}} - L_{t_i}\right)$$

provided that exists in mean square and is an uncertain random variable, where $L_t$ and $B_t$ are a one-dimensional canonical process and a one-dimensional Brownian motion, respectively. Here, $Y_t$ is called Ito-Liu integrable. In particular, when $Y_t \equiv 0$, $Y_t$ is called Liu integrable.

**Theorem 2.15 (Fei, 2014):** Let $B = (B_t)_{0\le t\le T} = (B_t^1, \ldots, B_t^m)_{0\le t\le T}^T$ and $L = (L_t)_{0\le t\le T} = (L_t^1, \ldots, L_t^n)_{0\le t\le T}^T$ be an m-dimensional standard Brownian motion and an n-dimensional canonical process, respectively. Assume that uncertain stochastic processes $Y_1(t), Y_2(t), \ldots, Y_p(t)$ are given by:

$$dY_k(t) = u_k(t)dt + \sum_{l=1}^{m} v_{kl}(t)dB_t^l + \sum_{l=1}^{n} w_{kl}(t)dL_t^l, k = 1, \ldots, p$$

Where $u_k(t)$ are all absolute integrable uncertain stochastic processes, $v_{kl}(t)$ are all square integrable uncertain stochastic processes, and $w_{kl}(t)$ are all Liu integrable uncertain stochastic processes. For $k, l = 1, \ldots, p$, let $\frac{\partial K}{\partial t}(t, y_1, \ldots, y_p), \frac{\partial K}{\partial y_k}(t, y_1, \ldots, y_p), \frac{\partial^2 K}{\partial y_k \partial y_l}(t, y_1, \ldots, y_p)$ be continuous functions. Then we have:

$$\begin{aligned} dK\big(t, Y_1(t), \ldots, Y_p(t)\big) &= \frac{\partial K}{\partial t}\big(t, Y_1(t), \ldots, Y_p(t)\big)dt \\ &+ \sum_{k=1}^{p}\frac{\partial K}{\partial y_k}(t, Y_1(t), \ldots, Y_p(t))dY_k(t) \\ &+ \frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{p}\frac{\partial^2 K}{\partial y_k \partial y_l}(t, Y_1(t), \ldots, Y_p(t))dY_k(t)dY_l(t) \end{aligned}$$

where $dB_t^k dB_t^l = \delta_{kl}dt, dB_t^k dt = dL_t^i dL_t^j = dL_t^l dt = dB_t^k dL_t^i = 0,$ for $k, l = 1, \ldots, m.$ $i, j = 1, \ldots, n.$
Here:

$$\delta_{kl} = \begin{cases} 0, & if\ k \ne l \\ 1, & otherwise. \end{cases}$$

## UNCERTAIN STOCHASTIC DIFFERENTIAL EQUATIONS

This new class of differential equation is driven by both the Brownian motion and Liu canonical process.

**Definition 3.1:** The differential equation

$$dY_t = m(t, Y_t)dt + n(t, Y_t)dL_t + r(t, Y_t)dB_t \quad (3)$$

is called an Uncertain Stochastic Differential Equation (USDE). Where Lt and Bt are one-dimensional canonical process and Brownian motion, respectively, and *m, n* and *r* are given functions.

**Remark 2:** The solution to the USDE above is called an uncertain stochastic process $Y_t$.

### Existence and uniqueness

**Theorem 3.2 (Chirima et al., 2020):** An USDE:

$$dY_t = m(t, Y_t)dt + n(t, Y_t)dL_t + r(t, Y_t)dB_t$$

satisfying the linear growth and Lipschitz conditions $|m(x,t)| + |n(x,t)| + |r(x,t)| \le L(1 + |x|), \forall\, x \in \mathbb{R}, t \ge 0$ And $|m(x,t) - m(y,t)| + |n(x,t) - n(y,t)| + |r(x,t) - r(y,t)| \le L|x - y|, \forall\, x, y \in \mathbb{R}, t \ge 0$ respectively, for some constant L, is said to have a unique solution that is sample continuous.

### Stability

**Definition 3.3:** An arbitrary USDE is said to be stable, if given any two solutions to an uncertain stochastic differential equation, $Y_t$ and $Z_t$ which satisfy the condition $\lim_{|Y_0 - Z_0|\to 0} C\{|Y_t - Z_t| > \epsilon\} = 0, \forall\, t \ge 0$

**Theorem 3.4:** (Chirima et al., 2020) If $\upsilon_t$, $\mu_t$ and $\nu_t$ are continuous functions where $\sup_{s\ge 0}\int_0^s \upsilon_t dt < +\infty$, $\int_0^{+\infty}|\mu_t|dt < +\infty, \int_0^{+\infty}|\nu_t|dt < +\infty$ then an USDE $\mathrm{d}Y_t = \upsilon_t Y_t \mathrm{d}t + \mu_t Y_t \mathrm{d}L_t + \nu_t Y_t \mathrm{d}B_t$ is stable.

## NUMERICAL EXAMPLE

Matenda and Chikodza (2018) proposed an uncertain stochastic stock model for the stock price Zt and bond price Yt:

$$\begin{cases} dY_t = aY_t dt \\ dZ_t = bZ_t dt + \sigma_2 Z_t dL_t + \sigma_1 Z_t dB_t \end{cases} \quad (2)$$

where a represents the rate of riskless interest , b represents the drift of the stock, $\sigma_2$ represents the uncertain diffusion and the random diffusion is represented by $\sigma_1$.

For an USDE of Equation (2), let $T = 1, b = 0.06, Z_0 = 40, \sigma_1 = 0.29, \sigma_2 = 0.32, N = 100$. Our aim is to use the EM method to compute the distribution of the price of stock $Z_t$ and compare our result with that exact solution and RK4 proposed by Chirima et al. (2020). Equations (2) and (1) are similar, with α-path

$$\mathrm{d}Z_t^\alpha = bZ_t^\alpha \mathrm{d}t + |\sigma_2 Z_t^\alpha|\Phi^{-1}(\alpha)\mathrm{d}t + \sigma_1 Z_t^\alpha \mathrm{d}B_t \quad (3)$$

which is an SDE α-path whose solution $Z_t$ is a stochastic contour process. $\Phi^{-1}(\alpha)$ is deterministic and represents the inverse uncertainty distribution of the canonical Liu process given by:

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}$$

In Equation (2), eliminating $\sigma_1 Z_t dB_t$ leaves us with the Liu's stock model proposed by Liu (2013).

$$dZ_t = bZ_t dt + \sigma_2 Z_t dL_t \quad (4)$$

Also, eliminating $\sigma_2 Z_t dL_t$ in Equation (2) leaves us with the Black-Scholes (1973) model.

$$dZ_t = bZ_t dt + \sigma_1 Z_t dB_t \quad (5)$$

We will proceed to solve Equation (3) by the EM method using MATLAB.

## Euler-Maruyama method with α–path

Consider the stochastic initial value problem:

$$\mathrm{d}Y(t) = m(Y(t))\mathrm{d}t + n(Y(t))\mathrm{d}L_t + r(Y(t))\mathrm{d}B(t)$$

and the SDE with α–path is given as

$$\mathrm{d}Y^\alpha(t) = m(Y^\alpha(t))\mathrm{d}t + |n(Y^\alpha(t))|\Phi^{-1}(\alpha)\mathrm{d}t + r(Y^\alpha(t))\mathrm{d}B(t) \quad (6)$$

with
$Y^\alpha(0) = Y_0^\alpha (Y_0^\alpha \text{ is an uncertain random variable})$ and $0 \le t \le T$. Before applying a numerical method to the initial value problem, we first discretize the time interval by letting $\Delta t = T/L$ for some positive integer L and $\tau_j = j\Delta t$ (Higham, 2001). Note that $Y^\alpha(\tau_j)$ will be approximated by $Y_j^\alpha$.

The Euler-Maruyama method for the SDE with α–path is given by

$$Y_j^\alpha = Y_{j-1}^\alpha + m(Y_{j-1}^\alpha)\Delta t + \left|n(Y_{j-1}^\alpha)\right|\Phi^{-1}(\alpha)\Delta t + r(Y_{j-1}^\alpha)(B(\tau_j) - B(\tau_{j-1})) \quad (7)$$

for $j = 1, 2, \dots, L$. (Higham, 2001). Taking n = 0 yields the generally known stochastic case without uncertainty in the system. Taking h = 0 yields the uncertainty case without randomness in the system as proposed by Liu (2009).

## Algorithm

The EM method below can be used in solving USDEs.

$$Y_j^\alpha = Y_{j-1}^\alpha + m(Y_{j-1}^\alpha)\Delta t + \left|n(Y_{j-1}^\alpha)\right|\Phi^{-1}(\alpha)\Delta t + r(Y_{j-1}^\alpha)(B(\tau_j) - B(\tau_{j-1}))$$

Let T be a fixed time length, N be the number of iterations, and $dt = T/N$ be the interval difference over [0, T]. We choose the step size dt as an integer multiple R ≥ 1, that is, $DT = R * dt$, and $L = N/R$.

In this case, we choose R = 1. We calculate our own discretized Brownian pathways and utilize them to produce the increments. $B(\tau_j) - B(\tau_{j-1})$ needed in Equation (7).

Step 1. Set α = 0.05 and j = 1.
Step 2. Solve Equation (6) using EM method Equation (7).
Step 3. Increase j by 1 and repeat steps 2 and 3 to the maximum value of N.
Step 4. Obtain $\mathrm{E}(Y_s^\alpha)$ by $\frac{\sum_{j=1}^N Y_j^\alpha}{N}$
Step 5. The process is repeated for α values 0.05, 0.10, 0.15… 0.95. This brings us to the inverse uncertainty distribution of $Y_t$.

From Table 1 and Figure 1, the EM method yields the least stock prices compared to RK4 and exact methods. The stock prices from RK4 prove to converge faster to those from the exact method as seen in the enlarged view of Figure 1 in Figure 2. Thus showing the RK4 method has faster rate of convergence than the EM method.

At $\alpha = 0.5$, the EM method to the USDE model gives exactly the same stock price as that for the Black-Scholes (1973) SDE model. This can be easily seen from Figures 4 and 5. It implies that at $\alpha = 0.5$, the uncertain component is eliminated from the USDE thereby turning the USDE stock model into the Black-Scholes (1973) SDE stock model, which further proves that the USDE is indeed a hybrid of the UDE and SDE. The randomness of the USDE model can also be easily seen from Figures 3, 4, 5, and 6.

The distributions represent the stock price $Z_t$ for USDE using EM, RK4 and analytical methods. An uncertain distribution is regular if it satisfies the conditions of continuity, strictly increasing, and

$$\lim_{x\to-\infty}\Phi(\mathrm{x}) = 0, \quad \lim_{x\to+\infty}\Phi(\mathrm{x}) = 1$$

(Liu 2010).

From Table 1, the results of EM and RK4 method satisfies the above conditions. The results in Figure 1 shows that the functions are increasing for both the EM

**Table 1.** Stock prices ($Z_t$) for uncertain stochastic market based on analytical method, Euler-Maruyama method and Runge-Kutta of order 4 method for α values.

| α-path | Stock Price | | |
|---|---|---|---|
| | **Analytical method** | **EM method** | **RK4 method** |
| 0.05 | 25.265 | 24.524 | 24.617 |
| 0.15 | 31.276 | 29.580 | 30.474 |
| 0.20 | 33.258 | 31.292 | 32.406 |
| 0.25 | 34.990 | 32.805 | 34.093 |
| 0.30 | 36.576 | 34.206 | 35.638 |
| 0.35 | 38.079 | 35.547 | 37.103 |
| 0.40 | 39.541 | 36.864 | 38.528 |
| 0.45 | 40.996 | 38.185 | 39.945 |
| 0.50 | 42.473 | 39.539 | 41.385 |
| 0.55 | 44.004 | 40.954 | 42.876 |
| 0.60 | 45.623 | 42.465 | 44.454 |
| 0.65 | 47.375 | 44.116 | 46.161 |
| 0.70 | 49.322 | 45.970 | 48.058 |
| 0.75 | 51.558 | 48.124 | 50.236 |
| 0.80 | 54.242 | 50.744 | 52.852 |
| 0.85 | 57.680 | 54.155 | 56.201 |
| 0.90 | 62.586 | 59.126 | 60.981 |
| 0.95 | 71.404 | 68.362 | 69.574 |

Source: Author, 2023

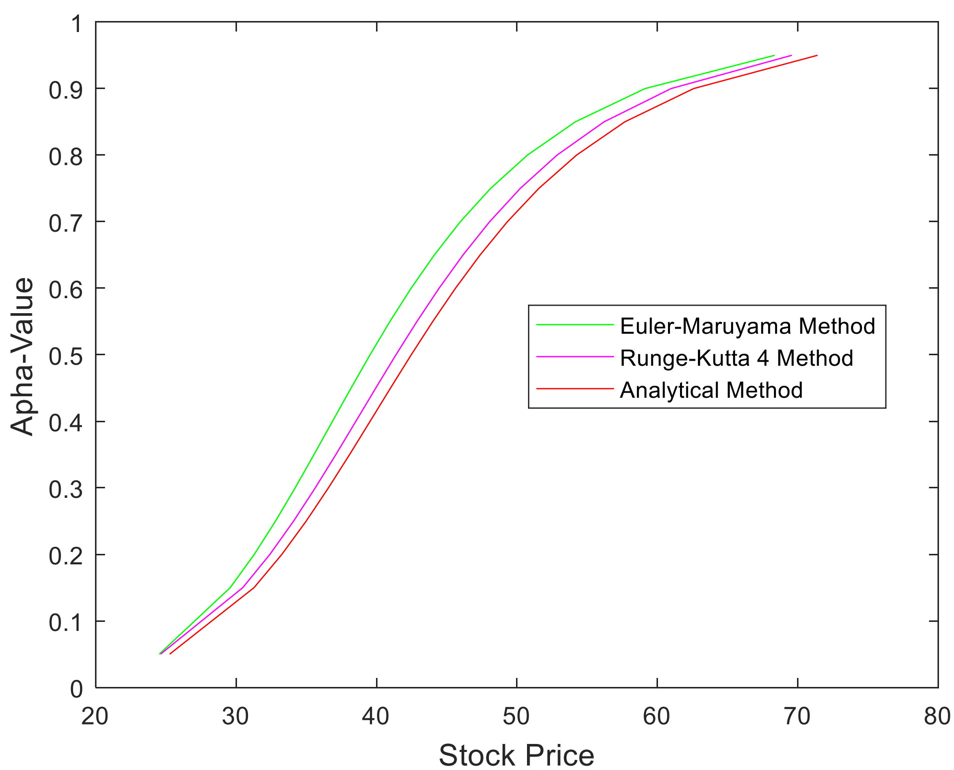


**Figure 1.** Distribution curves for stock prices in the uncertain stochastic market based on the EM, RK4 and analytical methods.
Source: Author, 2023

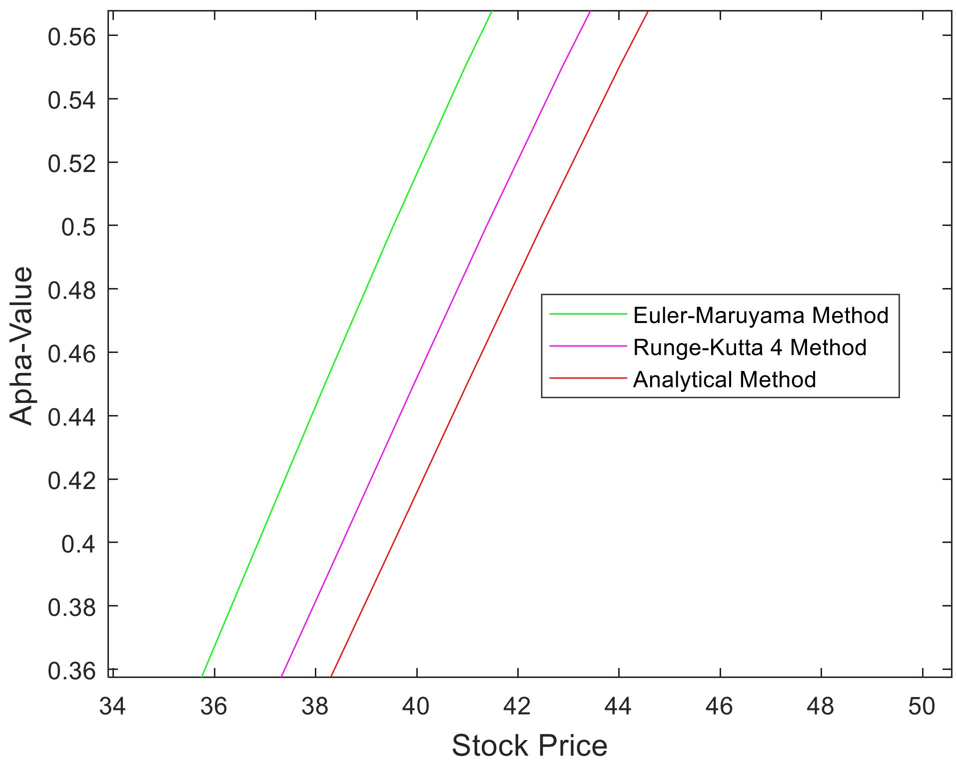


**Figure 2.** Enlarged section of Figure 1.
Source: Author, 2023

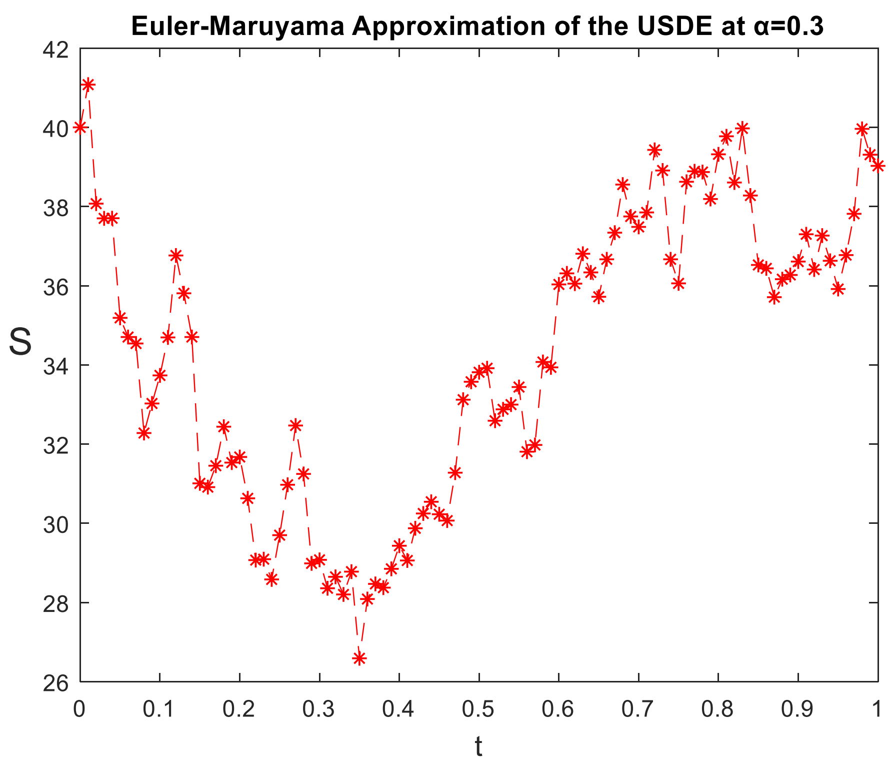


**Figure 3.** The simulation of stock price at $\alpha = 0.3$.
Source: Author, 2023

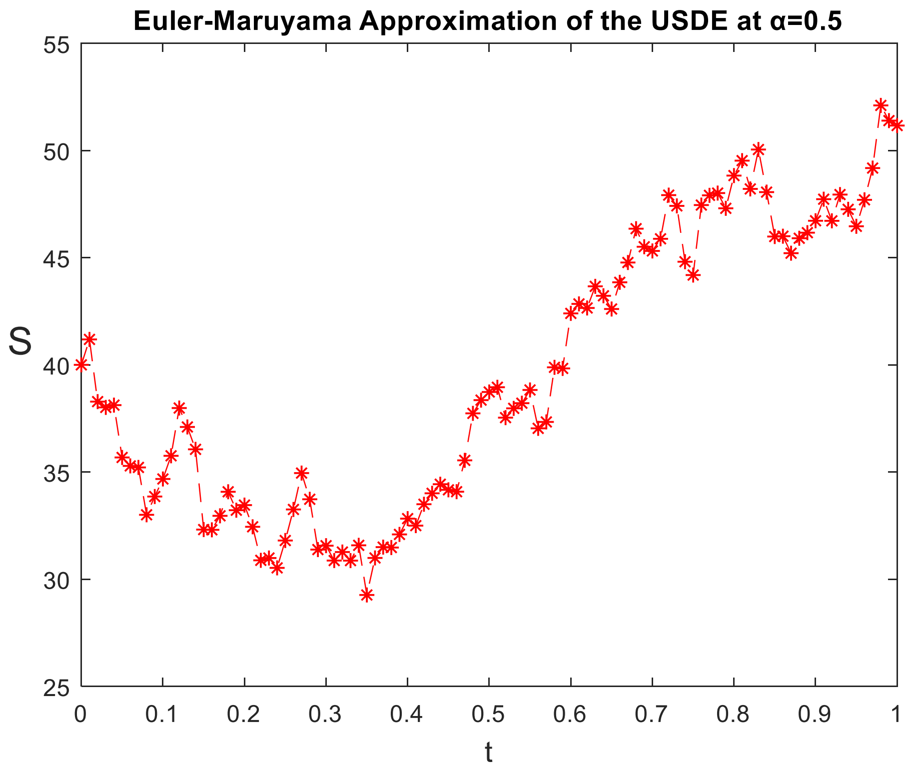


**Figure 4.** The simulation of stock price at $\alpha = 0.5$.
Source: Author, 2023

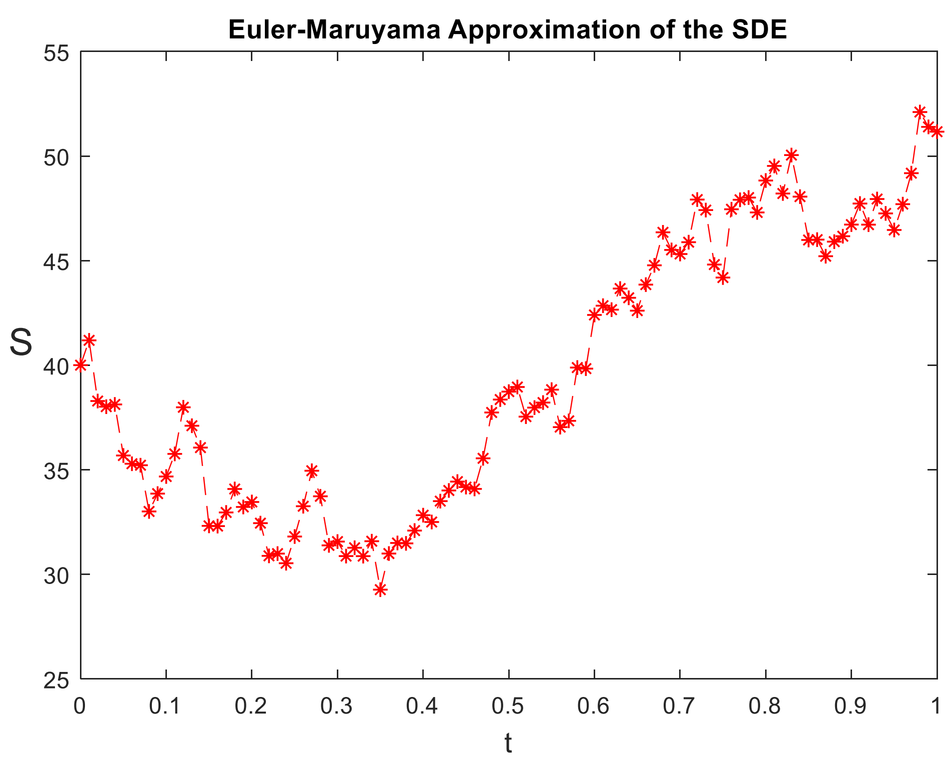


**Figure 5.** The simulation of stock price without uncertainty component.
Source: Author, 2023

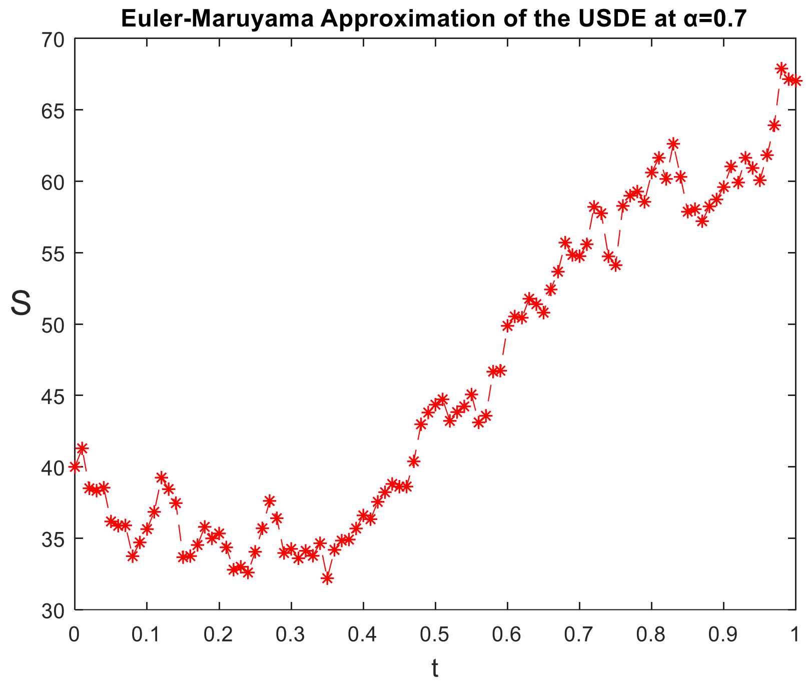


**Figure 6.** The simulation of stock price at $\alpha = 0.7$.
Source: Author, 2023

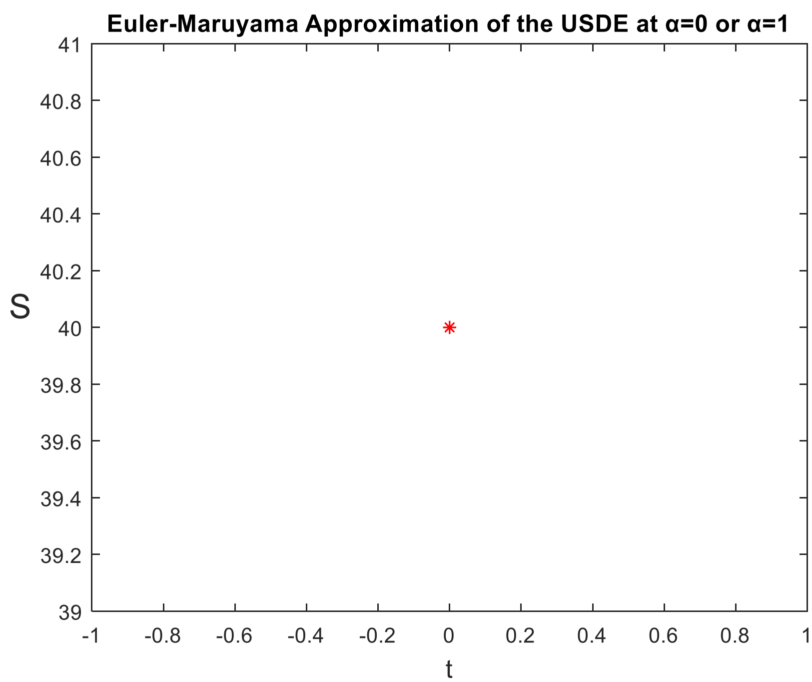


**Figure 7.** The simulation of stock price at $\alpha = 0$ or $\alpha = 1$.
Source: Author, 2023

and RK4 methods, except at $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$, which can be seen from Figure 7, further supporting the necessary and sufficient conditions by Peng and Iwamura (2013).

## Conclusion

We have demonstrated in this paper that USDEs can be converted to SDEs with α-paths and solved by the Euler-Maruyama method. The solution of the SDEs with α-path therefore becomes a stochastic contour process. The RK4 converges faster than the EM method to the exact solution. The EM method proved to yield lower stock price compared to the stock price from the RK4 and exact methods. At $\alpha = 0.5$, the USDE crashed to the SDE with the uncertain component eliminated showing that the USDE is indeed a hybrid of the UDE and SDE.

## CONFLICT OF INTERESTS

The authors have not declared any conflict of interest.

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