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L-stable implicit trapezoidal-like integrators for the solution of parabolic partial differential equations on manifolds

Johnson Oladele Fatokun[1]* and Akpan, I. P.[2]

[1]Department of Mathematical Sciences and Information Technology, Federal University, Dutsin-Ma, P. M. B.5001, Dutsin-Ma, Nigeria.
[2]Department of Mathematical Sciences, Nasarawa State University, Keffi, Nigeria.

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A new Trapezoidal-type scheme is proposed for the direct numerical integration of time-dependent partial differential equations. The evolving system of ordinary differential equations after discretization is usually stiff, so it is desirable for the use of a numerical method in solving it to have good properties concerning stability. The method proposed in this article is L-stable and at least of algebraic order two. It is increasingly common to encounter partial differential equations (PDEs) posed on manifolds, and standard numerical methods are not available for such novel situations. Here, an L-Stable implicit Trapezoidal-like numerical integrator was developed for solving partial differential equations on manifolds. This approach allows the immediate use of familiar finite difference methods for the discretization and numerical solution. Presented here are the motivation and details of the method, illustration of its numerical convergence and stability properties for a general case. Numerical experiments illustrate the performance of the new method on different stiff systems of ODEs after discretizing in the space variable of some PDE problems. Results show accuracy of maximum error for various space and time steps.

Key words: Trapezoidal-type method, partial differential equations, L-stability, implicit surfaces, manifolds, stiff systems.

INTRODUCTION

The development of numerical schemes for the solution of parabolic partial differential equations has attracted considerable attention in the past few decades. This can be explained by the fact that many industrial processes involving heat transfer and diffusion in biological systems can be modeled with parabolic partial differential equations. The Method of Lines (MOL) is one of the predominant approaches for solving time dependent parabolic heat partial differential equations (Hayes and Russell, 2007).

A well-known numerical approach to solve a time dependent PDE, whose solutions vary both in time and in space, is the Method of Lines (Trefethen, 2000). Schiesser and Grifiths (2009) demonstrated the applicability of the Method of Lines in solving problems in physics, fluid dynamics, reactor models as well as automatic control.

In this approach, a semi-discrete approximation to the PDE is constructed by setting a regular grid in space. The spatial independent variables that have boundary conditions are discretized, thereby, generating a coupled system of ordinary differential equations (ODEs) in the

*Corresponding author. E-mail: jfatokun@fudutsinma.edu.ng.

time dependent variable t, which is associated with the initial value. Numerical approximations to the solutions of the PDE are obtained by marching forward in time on the grid. It is at this point that existing and generally well established numerical methods for initial value ODEs can be used to compute numerical solutions to the PDE. Ashi (2008) considered the linear diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = v\frac{\partial^2 u(x,t)}{\partial x^2}, t_0 \le t \le T, x_0 \le x \le x_q \quad (1)$$

where $u(x,t)$ is the dependent variable, t and x are the independent variables representing time and one-dimensional space respectively, v is a real positive constant, subject to an initial condition at $t_0 = 0$, $u(x,t=0) = u^0(x)$ and two boundary conditions, corresponding to boundaries of a physical system, $u(x=x_0,t) = f(t), u(x=x_q,t) = g(t)$, where $f(t) and\ g(t)$ are given boundary values of u for all t.

In using the Method of Lines procedure to solve the parabolic partial differential equation (1), Ashi (2008) discretized $u(x,t)$ in space with q+1 points, of which q-1 are interior points, on a uniform grid with step size h as follows $u(x_n,t) \approx u_n(t),\ 0 \le n \le q,$ where the index n designates a position along the grid in x. Approximating the spatial derivative $\frac{\partial^2 u(x,t)}{\partial x^2}$ in Equation (1), by the second-order centered finite difference approximation:

$$\left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{x=x_n} \approx \frac{u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)}{h^2} + O(h^2). \quad (2)$$

Substituting Equation (2) into (1), a system of q-1 approximating ODEs evolves thus:

$$\begin{aligned} &u_0(t) = f(t),\\ &\frac{du_1(t)}{dt} = \frac{v\big(u_2(t) - 2u_1(t) + u_0(t)\big)}{h^2},\\ &\frac{du_2(t)}{dt} = \frac{v\big(u_3(t) - 2u_2(t) + u_1(t)\big)}{h^2}\\ &\frac{du_{q-1}(t)}{dt} = \frac{v\big(u_q(t) - 2u_{q-1}(t) + u_{q-2}(t)\big)}{h^2},\\ &u_q(t) = g(t), \end{aligned} \quad (3)$$

Subject to the initial conditions

$$u_n(t=0) = u^0(x_n),\ \ 0 \le n \le q. \quad (4)$$

The system (3) and the initial conditions (4) constitute the complete method of lines approximation of Equation (1). Jator (2011) in considering the PDE of the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad (x,t) \in [0,1] \times [0 < t < T], \quad (5)$$

Subject to the initial/boundary conditions

$$u(x,o) = f(x), x \in [0,1],\ \ u(0,1) = u(1,t) = 0,\ \ t \ge 0 \quad (6)$$

applied the finite difference approach by constructing a rectangular mesh in the semi-infinite rectangle [0, 1] and [0, ∞] and drawing lines parallel to the x-axis and t-axis with mesh spacing Δx and $\Delta t,$ and sought the approximate solutions to Equation (5) at mesh points $(x_m, t_n), m = 1,2,\ldots,M, t = 1,2,\ldots,$ where $x_m = m\Delta x, t_n = t_n$. Letting $u(t) \in \mathbb{R}^m$ be the exact solution of the resulting semi-discrete problem

$$\frac{du(t)}{dt} = Au(t),\ \ U(0) = \varphi \quad (7)$$

where $u(t) = [u_1(t), u_2(t), \ldots, u_M(t)]^T$

$\varphi = [\varphi(x_1), \varphi(x_1), \ldots, \varphi(x_1)]^T,$

$$\text{And } A = \frac{1}{(\Delta x)^2}\begin{pmatrix} -2 & 1 & 0 & 0 & . & . & . & 0 & 0 \\ 1 & -2 & 1 & 0 & . & . & & 0 & 0 \\ . & & & & & & & & . \\ . & & & & & & & & . \\ . & & & & & & & & . \\ . & & & & & & 1 & -2 & 1 \\ 0 & . & . & . & & & 0 & 1 & -2 \end{pmatrix} \quad (8)$$

The PDE (5) and (6) is reduced to a system of ODE and a numerical solution is obtained using a leaping- type class of algorithms involving the hybrid Adams-type methods as well as the hybrid backward differentiation type formulas.

Motivation and development of the new scheme

According to Lambert (1993), the eigenvalues of the matrix, A in Equation (8), are known to be $\lambda_j = \frac{\left[-2+2\cos\left(\frac{j\pi}{(M+1)}\right)\right]}{(\Delta x)^2}, j = 1,2,\ldots,M$ which are real and lie in the interval $\left(-\frac{4}{(\Delta x)^2}, 0\right)$ of the negative real axis.

The system of ordinary differential equations that arises in the Method of Lines solution of time-dependent partial differential equations is usually stiff (Ramos and Vigo-Aguiar, 2007), because stiff systems have more stringent stability restrictions than non-stiff systems (Curtiss and Hirschfelder, 1952; Hairer and Wanner, 1996). It is desirable for the numerical method used in solving this system (8) to have good stability properties (Suli and Mayers, 2003; Gear, 1971). Two of such properties are A-stability and L-stability.

Development of the new L-stable method

Consider the parabolic PDE

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial t^2}u(x,t), \quad (x,t) \in [0,1] \times [0 < t < T] \tag{9a}$$

Subject to the initial/boundary condition

$$u(x,0) = \varphi(x), \quad 0 \le x \le 1; \quad u(0,t) = u(1,t) = 0, t \ge 0 \tag{9b}$$

Let the theoretical solution $U(x,t)$ of Equation (9a) subject to the initial/boundary conditions in Equation (9b) at a fixed value of x be approximated by the interpolating function

$$U(.,t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} + Ce^{\lambda_3 t} \tag{10}$$

By imposing the following constraints on Equation (10),

$$u^{n+j} = U(.,t_{n+j}); \quad for\ j = 0,1 \tag{11}$$

$$u'^{n+j} = U'(.,t_{n+j}); \quad for\ j = 0,1 \tag{12}$$

the following system of equations is obtained thus:

$$u^n = U(.,t_n) = Ae^{\lambda_1 t_n} + Be^{\lambda_2 t_n} + Ce^{\lambda_3 t_n} \tag{13}$$

$$u'^n = U'(.,t_n) = A\lambda_1 e^{\lambda_1 t_n} + B\lambda_2 e^{\lambda_2 t_n} + C\lambda_3 e^{\lambda_3 t_n} \tag{14}$$

$$u'^{n+1} = U'(.,t_{n+1}) = Ae^{\lambda_1 t_n}e^{\lambda_1 l} + Be^{\lambda_2 t_n}e^{\lambda_2 l} + Ce^{\lambda_3 t_n}e^{\lambda_3 l} \tag{15}$$

Where $t_{n+1} - t_n = l$

Solving Equations (13) - (15) simultaneously yields:

$$Ae^{\lambda_1 t_n} = \frac{1}{D}[(\lambda_2\lambda_3 e^{\lambda_3 l} - \lambda_2\lambda_3 e^{\lambda_2 l})u^n + (\lambda_2 e^{\lambda_2 l} - \lambda_3 e^{\lambda_3 l})u'^n + (\lambda_3 - \lambda_2)u'^{n+1}] \tag{16}$$

$$Be^{\lambda_2 t_n} = \frac{1}{D}[(\lambda_1\lambda_3 e^{\lambda_1 l} - \lambda_1\lambda_3 e^{\lambda_3 l})u^n + (\lambda_3 e^{\lambda_3 l} - \lambda_1 e^{\lambda_1 l})u'^n + (\lambda_1 - \lambda_3)u'^{n+1}] \tag{17}$$

$$Ce^{\lambda_3 t_n} = \frac{1}{D}[(\lambda_1\lambda_2 e^{\lambda_2 l} - \lambda_1\lambda_2 e^{\lambda_1 l})u^n + (\lambda_1 e^{\lambda_1 l} - \lambda_2 e^{\lambda_2 l})u'^n + (\lambda_2 - \lambda_1)u'^{n+1}] \tag{18}$$

$$D = \lambda_1\lambda_2(e^{\lambda_2 l} - e^{\lambda_1 l}) + \lambda_1\lambda_3(e^{\lambda_1 l} - e^{\lambda_3 l}) + \lambda_2\lambda_3(e^{\lambda_3 l} - e^{\lambda_2 l}) \tag{19}$$

where *D* is the determinant of the linear system of Equations (13) – (15). From Equation (11) it follows that

$$u^{n+1} = U(.,t_{n+j}) = Ae^{\lambda_1 t_n}e^{\lambda_1 l} + Be^{\lambda_2 t_n}e^{\lambda_2 l} + Ce^{\lambda_3 t_n}e^{\lambda_3 l} \tag{20}$$

By substituting Equations (16) – (19) into Equation (20), the discrete scheme

$$u^{n+1} - \frac{P}{D}u^n = \frac{1}{D}[Qu'^n + Ru'^{n+1}] \tag{21}$$

is obtained, where

$$P = (\lambda_2\lambda_3 e^{\lambda_3 l} - \lambda_2\lambda_3 e^{\lambda_2 l})e^{\lambda_1 l} + (\lambda_1\lambda_3 e^{\lambda_1 l} - \lambda_1\lambda_3 e^{\lambda_3 l})e^{\lambda_2 l} + (\lambda_1\lambda_2 e^{\lambda_2 l} - \lambda_1\lambda_2 e^{\lambda_1 l})e^{\lambda_3 l} \tag{22}$$

$$Q = (\lambda_2 e^{\lambda_2 l} - \lambda_3 e^{\lambda_3 l})e^{\lambda_1 l} + (\lambda_3 e^{\lambda_3 l} - \lambda_1 e^{\lambda_1 l})e^{\lambda_2 l} + (\lambda_1 e^{\lambda_1 l} - \lambda_2 e^{\lambda_2 l})e^{\lambda_3 l} \tag{23}$$

$$R = (\lambda_3 - \lambda_2)e^{\lambda_1 l} + (\lambda_1 - \lambda_3)e^{\lambda_2 l} + (\lambda_2 - \lambda_1)e^{\lambda_3 l} \tag{24}$$

Equation (21) with the corresponding definitions (22) – (24) is the proposed numerical integrator.

The case of two real eigenvalues

Let the theoretical solution to Equation (9a) subject to the conditions in (9b) be approximated by the interpolating function

$$U(.,t) = A(I - e^{\lambda_1 t}) - B(I - e^{-\lambda_2 t}) + C \tag{25}$$

where $A, B, and\ C$ are m-topples with real entries, I is the identity matrix, while λ_1 and λ_2 are eigenvalues of matrices $A, and\ B$.

By demanding that the interpolating function (25) coincides with the theoretical solution at the end points of the interval $[t_n, t_{n+1}]$, the following equations are obtained:

$$u^n = A(I - e^{\lambda_1 t_n}) - B(I - e^{-\lambda_2 t_n}) + C \tag{26}$$

$$u^{n+1} = A(I - e^{\lambda_1 t_{n+1}}) - B(I - e^{-\lambda_2 t_{n+1}}) + C \tag{27}$$

Subjecting the interpolating function in Equation (25) to the following derivative constraints

$$U'(.,t_{n+j}) = u'^{n+j}, for\ j = 0, 1. \tag{28}$$

and from Equation (25) we obtain the following pairs of equations:

$$u'^n = U'(.,t_n) = -\lambda_1 Ae^{\lambda_1 t_n} - \lambda_2 Be^{-\lambda_2 t_n} \tag{29}$$

$$u'^{n+1} = U'(.,t_{n+1}) = -\lambda_1 Ae^{\lambda_1 t_n}e^{\lambda_1 l} - \lambda_2 Be^{-\lambda_2 t_n}e^{-\lambda_2 l} \tag{30}$$

Solving Equations (29) and (30) simultaneously for $Ae^{\lambda_1 t_n}\ and\ Be^{-\lambda_2 t_n}$ and by appropriate substitution into Equations (26) and (27) yields:

$$u^{n+1} - u^n = Ru'^n + Su'^{n+1} \quad (31)$$

where

$$R = \left[\frac{(1-e^{\lambda_1 l})e^{-\lambda_2 l}}{\lambda_1(e^{\lambda_1 l}-e^{-\lambda_2 l})} - \frac{(e^{-\lambda_2 l}-1)e^{\lambda_1 l}}{\lambda_2(e^{\lambda_1 l}-e^{-\lambda_2 l})}\right] \quad (32a)$$

And

$$S = \left[\frac{(e^{-\lambda_2 l}-1)}{\lambda_2(e^{\lambda_1 l}-e^{-\lambda_2 l})} - \frac{(1-e^{\lambda_1 l})}{\lambda_1(e^{\lambda_1 l}-e^{-\lambda_2 l})}\right] \quad (32b)$$

Stability analysis of the method

To establish the order of accuracy and the stability properties of the numerical integrator, we shall prove the following two proposed theorems:

Theorem 1

The numerical integrator (21) is A-stable.

Proof

Applying the Schur's method on the new scheme (21), it follows that:

$$\alpha_0 = -P_1;\ \alpha_1 = 1;\ \beta_0 = Q_1;\ \beta_1 = R_1;\ q = l\lambda \quad (33)$$

where

$$P_1 = \frac{P}{D}, Q_1 = \frac{Q}{D}\ and\ R_1 = \frac{R}{D} \quad (34)$$

From the general linear multistep method we have that for Equation (33)

$$\Rightarrow (-P_1 - Q_1 q) + (1 - R_1 q)r = 0 \quad (35)$$

$$\Rightarrow r = \frac{P_1 + Q_1 q}{1 - R_1 q} \quad (36)$$

$$\therefore |r| = \frac{P_1 + Q_1 q}{1 - R_1 q} < 1 \quad (37)$$

Replacing P_1 by $\frac{P}{D}$, Q_1 by $\frac{Q}{D}$, and R_1 by $\frac{R}{D}$

$$\Rightarrow -1 < \frac{P + Qq}{D - Rq} < 1 \quad (38)$$

$$Rq - D < P + Qq < D - Rq$$
$$Rq - D < P + Qq\ and\ P + Qq < D - Rq$$
$$(R - Q)q < P + D\ \ and\ (Q + R)q < D - P$$

$$q < \frac{P + D}{R - Q}\ and\ q < \frac{D - P}{Q + R}$$
$$q < \frac{P + D}{R - Q}\ and\ q < 1\ iff P, Q, R > 0$$
$$q < 1 for\ R - Q > 0, \quad and\ P + D > 0 \Rightarrow R - Q > P + D$$
$$\because q < 1 \Rightarrow q < 0\ iff\ P > 0, Q > 0\ and\ R > 0 \quad (39)$$

The region of absolute stability of the method (21) is $(-\infty, 0)$.

The method retains the A-stability property of the Trapezoidal rule.

Theorem 2

The numerical integrators (21) and its invariant (31) are L-stable.

Proof

Substituting the scalar test equation,

$$u' = \lambda u \quad (40)$$

into Equation (31) yields

$$u^{n+1} = u^n + \lambda R u'^n + \lambda S u'^{n+1}$$
$$\Longrightarrow (1 - \lambda S)u^{n+1} = (1 + \lambda R)u^n$$
$$\Longrightarrow u^{n+1} = \left(\frac{1 + \lambda R}{1 - \lambda S}\right)u^n$$

Hence

$$\rho(\lambda l) = \left(\frac{1+\lambda R}{1-\lambda S}\right) \quad (41)$$

Substituting for *R* and *S* in Equation (41), simplifying and taking the limit, using L 'Hopitals' rule we obtain $\lim_{\lambda l \to -\infty} \rho(\lambda l) = 0$

hence, the poof of the theorem. Thus, the integrators (21) and its invariants (31) are L-stable.

Analysis of order of accuracy of the methods

Using Taylor series to expand both hand sides of Equation (31) we obtain

$$u^n + u'^n + \frac{l^2}{2!}u''^n + \frac{l^3}{3!}u'''^n + \frac{l^4}{4!}u^{ivn} + \frac{l^5}{5!}u^{vn} + \ldots$$
$$= u^n + (R + S)u'^n + Slu''^n + S\frac{l^2}{2!}u'''^n + S\frac{l^3}{3!}u^{ivn} + S\frac{l^4}{4!}u^{vn} + \ldots \quad (42)$$

Expanding $e^{\lambda_1 l}$ and $e^{-\lambda_2 l}$ by Maclaurin series and substituting into Equations (32a) and (32b) we obtain

$$R = \frac{1}{\mu(\lambda_1,\lambda_2)}\left\{\frac{1}{2}(\lambda_1+\lambda_2)l^2 + \left[\frac{(\lambda_2-\lambda_1)}{2} + \frac{(\lambda_1{}^2-\lambda_2{}^2)}{3}\right]l^3 + \left[\frac{(\lambda_1{}^3+\lambda_2{}^3)}{3} - \frac{(5\lambda_1\lambda_2{}^2-\lambda_1{}^2\lambda_2)}{12}\right]l^4 + \left[\frac{\lambda_1{}^4}{30} - \frac{\lambda_1{}^3\lambda_2}{24} - \frac{\lambda_1{}^2\lambda_2{}^2}{12} + \frac{\lambda_1\lambda_2{}^2}{12} + \frac{\lambda_1\lambda_2{}^3}{24} - \frac{\lambda_2{}^4}{30}\right]l^5 + \left[\frac{\lambda_1{}^5}{144} - \frac{\lambda_1{}^4\lambda_2}{80} + \frac{7\lambda_1{}^3\lambda_2}{144} + \frac{\lambda_1{}^2\lambda_2{}^3}{144} - \frac{\lambda_1\lambda_2{}^4}{80} + \frac{\lambda_2{}^5}{144}\right]l^6 + \left[\frac{\lambda_1{}^6}{840} - \frac{\lambda_1{}^5\lambda_2}{360} + \frac{\lambda_1{}^4\lambda_2{}^2}{360} - \frac{\lambda_1{}^2\lambda_2{}^4}{360} - \frac{\lambda_1\lambda_2{}^5}{840}\right]l^7 + \ldots\right\} \quad (43)$$

and

$$S = \frac{1}{\mu(\lambda_1,\lambda_2)}\left\{\frac{(\lambda_1+\lambda_2)l^2}{2!} + \frac{(\lambda_1{}^2-\lambda_2{}^2)l^3}{3!} + \frac{(\lambda_1{}^3+\lambda_2{}^3)l^4}{4!} + \frac{(\lambda_1{}^4-\lambda_2{}^4)l^5}{5!} + \frac{(\lambda_1{}^5+\lambda_2{}^5)l^6}{6!} + \frac{(\lambda_1{}^6-\lambda_2{}^6)l^7}{7!} + \ldots\right\} \quad (44)$$

where

$$\mu(\lambda_1,\lambda_2) = (\lambda_1+\lambda_2)l + \frac{(\lambda_1{}^2-\lambda_2{}^2)l^2}{2!} + \frac{(\lambda_1{}^3+\lambda_2{}^3)l^3}{3!} + \frac{(\lambda_1{}^4-\lambda_2{}^4)l^4}{4!} + \frac{(\lambda_1{}^5+\lambda_2{}^5)l^5}{5!} + \frac{(\lambda_1{}^6-\lambda_2{}^6)l^6}{6!} + \frac{(\lambda_1{}^7+\lambda_2{}^7)l^7}{7!} + \ldots \quad (45)$$

Adding equations (43) and (44) we obtain

$$R+S = \frac{1}{\mu(\lambda_1,\lambda_2)}\left\{(\lambda_1+\lambda_2)l^2 + \left[\frac{(\lambda_2-\lambda_1)}{2} + \frac{(\lambda_1{}^2-\lambda_2{}^2)}{2}\right]l^3 + \left[\frac{(9\lambda_1{}^3+7\lambda_2{}^3-10\lambda_1\lambda_2{}^2+2\lambda_1{}^2\lambda_2)}{24}\right]l^4 + \left[\frac{(\lambda_1{}^4-\lambda_2{}^4-\lambda_1{}^3\lambda_2-2\lambda_1{}^2\lambda_2{}^2+2\lambda_1\lambda_2{}^2+\lambda_1\lambda_2{}^3)}{60}\right]l^5 + \left[\frac{(6\lambda_1{}^5+6\lambda_2{}^5-9\lambda_1{}^4\lambda_2+35\lambda_1{}^3\lambda_2+5\lambda_1{}^2\lambda_2{}^2-9\lambda_1\lambda_2{}^4)}{720}\right]l^6 + \left[\frac{(7\lambda_1{}^6-\lambda_2{}^6-14\lambda_1{}^5\lambda_2+14\lambda_1{}^4\lambda_2{}^2-14\lambda_1{}^2\lambda_2{}^4-6\lambda_1\lambda_2{}^5)}{5040}\right]l^7 + \ldots\right\} \quad (46)$$

Substituting Equation (46) for $R+S$ and Equation (45) for S in Equation (42), comparing coefficients of corresponding terms on both sides and letting the coefficients of $l, l^2 \, and \, l^3$ vanish identically, the local Truncation Error of the method is obtained as

$$L.T.E. = \frac{1}{\mu(\lambda_1,\lambda_2)}[(\lambda_1+\lambda_2)]\frac{l^3}{2}u'''n \quad (47)$$

where

$$\mu(\lambda_1,\lambda_2) = (\lambda_1+\lambda_2)l + \frac{(\lambda_1{}^2-\lambda_2{}^2)l^2}{2!} + \frac{(\lambda_1{}^3+\lambda_2{}^3)l^3}{3!} + \frac{(\lambda_1{}^4-\lambda_2{}^4)l^4}{4!} + \frac{(\lambda_1{}^5+\lambda_2{}^5)l^5}{5!} + \frac{(\lambda_1{}^6-\lambda_2{}^6)l^6}{6!} + \frac{(\lambda_1{}^7+\lambda_2{}^7)l^7}{7!} + \ldots$$

the method is at least of order two.

NUMERICAL EXPERIMENT

Problem 1

The problem is given by

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = 0, \qquad u(1,t) = 0 \\ u(x,0) = 2Sin(\pi x), \end{cases}$$

Where $0 \le x \le 1, 0 \le t \le 1,$ and the exact solution given by $u(x,t) = 2e^{-\pi^2 t}\sin(\pi x).$

Problem 2

Consider the time-dependent Burger's equation

$$\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\frac{\partial u^2}{\partial x},$$

Where $0 \le x \le 1, 0 \le t \le 1,$ and the exact solution is given by $u(x,t) = \left(1 + e^{(0.5x-0.25t)/\mu}\right)^{-1}$

This form of the Burger's equation was solved by Mazzia and Mazzia (1997) and Ramos and Vigo-Aguiar (2007) as a test problem. All the computations were performed in Maple.

Computations of absolute and relative errors

Absolute errors

The absolute errors of the various methods were computed by the use of the formula:

$$|u_{ij} - u(x_i,t_j)|$$

Where the numerical solution at the grid point $(x_i,t_j)\ is\ u_{ij}$ and the exact solution at the same grid point is $u(x_i,t_j)$.

Relative errors

Relative errors of the various methods were computed by use of the formula:

$$\left|\frac{|u_{ij} - u(x_i,t_j)|}{|1 + u(x_i,t_j)|}\right|$$

Where the numerical solution at the grid point $(x_i,t_j)\ is\ u_{ij}$ and the exact solution at the same grid point is $u(x_i,t_j)$.

RESULTS

On implementation of the new method on these problems the numerical results and errors were computed in comparison with the standard Trapezoidal method and other recent methods in literature. All computations were carried out using Maple 15. The results are as shown in the Tables 1 to 5 and Figures 1 to 6.

DISCUSSION AND CONCLUSION

In this paper, the construction of an implicit Trapezoidal

Table 1. Solutions and approximations by the new scheme and standard trapezoidal method for Problem 1.

t	Theoretical Solution	New scheme	Standard Trapezoidal
0.0	0.03926738494	0.03926738494	0.03926738494
0.00625	0.03691836955	0.03691836943	0.03691763312
0.01250	0.03470987496	0.03470987445	0.03470844637
0.01875	0.03263349425	0.03263349391	0.03263140759
0.02500	0.03068132504	0.03068132459	0.03067861940
0.03125	0.02884593659	0.02884593605	0.02884262901
0.03750	0.02712034297	0.02712034234	0.02711650826
0.04375	0.02549797614	0.02549797543	0.02549362703
0.05000	0.02397266096	0.02397266017	0.02396782822
0.05625	0.02253859171	0.02253859084	0.02253330427
0.06250	0.02119030995	0.02119030902	0.02118459502

Table 2. Absolute Errors of the new scheme and standard trapezoidal method for Problem 2.

t	New scheme	Standard Trapezoidal Rule
0	0	0
0.00625	1.2×10^{-10}	7.3643×10^{-7}
0.01250	5.1×10^{-10}	1.42859×10^{-6}
0.01875	3.4×10^{-10}	2.08666×10^{-6}
0.02500	4.5×10^{-10}	2.70564×10^{-6}
0.03125	5.4×10^{-10}	3.28758×10^{-6}
0.03750	6.3×10^{-10}	3.83471×10^{-6}
0.04375	7.1×10^{-10}	4.34911×10^{-6}
0.05000	7.9×10^{-10}	4.83274×10^{-6}
0.05625	8.7×10^{-10}	5.28744×10^{-6}
0.06250	9.3×10^{-10}	5.71493×10^{-6}

Table 3. Relative errors of the new scheme and standard trapezoidal method for Problem1.

t	New scheme	Standard Trapezoidal
0	0	0
0.00625	$1.157275283 \times 10^{-10}$	$7.102101972 \times 10^{-7}$
0.01250	$4.928917877 \times 10^{-10}$	$1.380667214 \times 10^{-7}$
0.01875	$3.292552508 \times 10^{-10}$	$2.020716946 \times 10^{-7}$
0.02500	$4.366043986 \times 10^{-10}$	$2.625098500 \times 10^{-7}$
0.03125	$5.248599237 \times 10^{-10}$	$3.195405533 \times 10^{-7}$
0.03750	$6.133653221 \times 10^{-10}$	$3.733457356 \times 10^{-7}$
0.04375	$6.923465639 \times 10^{-10}$	$4.240973553 \times 10^{-7}$
0.05000	$7.715049728 \times 10^{-10}$	$4.719598661 \times 10^{-7}$
0.05625	$8.508236333 \times 10^{-10}$	$5.170895301 \times 10^{-7}$
0.06250	$9.107019435 \times 10^{-10}$	$5.596341783 \times 10^{-7}$

integrator for the numerical solution of the stiff system of ODE arising from the MOL discretization of parabolic PDE is presented. The method is shown to have good stability properties needful for the integration of stiff systems of ODE. In Tables 1, 2, and 3, the new scheme performs better in accuracy and stability than the standard Trapezoidal scheme for both Problems 1 and 2. In Tables 4 and 5, the results of the new Scheme

Table 4. Relative errors of the various methods at various space and time steps for Problem 1.

Δx	**Δt**	**New scheme**	**Mazzia and Mazzia (transverse) Order 4**	**Mazzia and Mazzia (longitudinal) Order 4**	**Ramos and Vigio-Aguiar (longitudinal)**
5.00×10^{-2}	5.00×10^{-2}	$1.477666948 \times 10^{-5}$	8.5×10^{-4}	2.7×10^{-4}	2.2577×10^{-4}
2.50×10^{-2}	2.50×10^{-2}	$3.857983984 \times 10^{-7}$	6.9×10^{-5}	1.4×10^{-5}	5.4749×10^{-5}
1.25×10^{-2}	1.25×10^{-2}	$6.872971978 \times 10^{-9}$	3.3×10^{-6}	7.7×10^{-7}	1.3426×10^{-7}
6.25×10^{-3}	6.25×10^{-3}	$1.157275283 \times 10^{-10}$	2.7×10^{-7}	4.2×10^{-8}	3.3197×10^{-8}

Table 5. Relative errors of the various methods at various space and time steps for Problem 2.

Δx	**Δt**	**New scheme**	**Mazzia and Mazzia (transverse) Order 4**	**Mazzia and Mazzia (longitudinal) Order 4**	**Ramos and Vigio-Aguiar (longitudinal)**
2.5×10^{-2}	5.00×10^{-2}	$1.22535242 \times 10^{-5}$	3.6×10^{-2}	7.3×10^{-2}	1.4952×10^{-2}
1.25×10^{-2}	2.5×10^{-2}	$3.764622138 \times 10^{-5}$	7.4×10^{-3}	1.2×10^{-2}	3.2521×10^{-3}
6.25×10^{-3}	1.25×10^{-2}	$1.615575772 \times 10^{-6}$	6.9×10^{-4}	1.1×10^{-3}	3.5238×10^{-4}
3.125×10^{-3}	6.25×10^{-3}	$1.225339584 \times 10^{-5}$	4.1×10^{-5}	7.1×10^{-5}	2.4773×10^{-5}

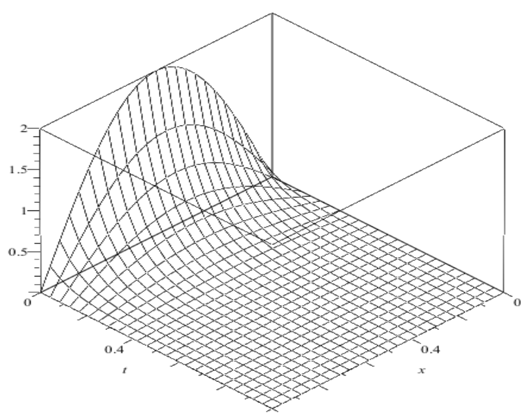

Figure 1. A 3 dimensional plot of the surface $u(x,t) = 2e^{-\pi^2 t}\sin(\pi t)\ at\ 0 \le x \le 1, 0 \le t \le 1$ for Problem 1

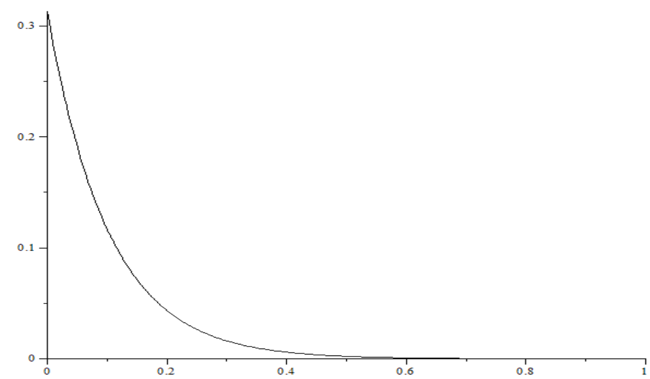

Figure 3. The graph of $u(x,t) = 2e^{-\pi^2 t}\sin(\pi t)\ at\ x = 5.0\ x\ 10^{-2}, 0 \le t \le 1$.

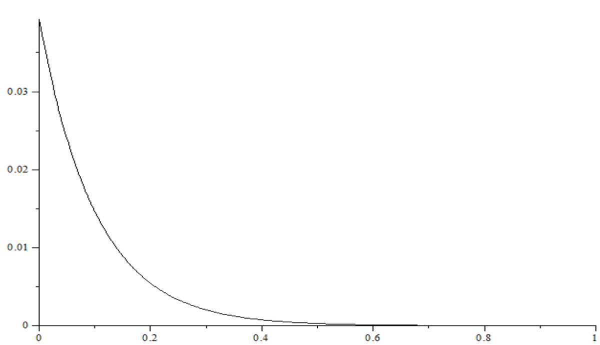

Figure 2. The graph of $u(x,t) = 2e^{-\pi^2 t}\sin(\pi t)\ at\ x = 6.25\ x\ 10^{-3}, 0 \le t \le 1$ for Problem 1.

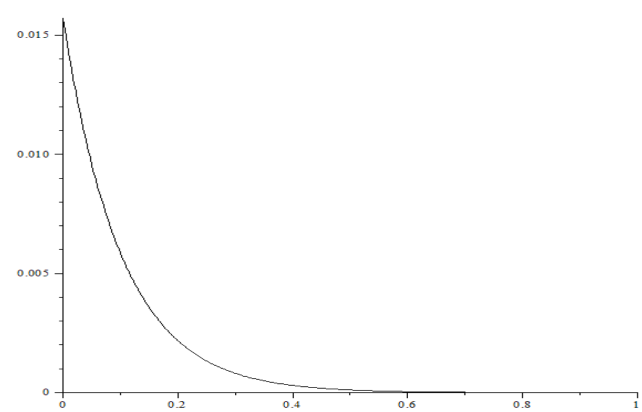

Figure 4. The graph of $u(x,t) = 2e^{-\pi^2 t}\sin(\pi t)\ at\ x = 2.5\ x\ 10^{-3}, 0 \le t \le 1$ for Problem 1.

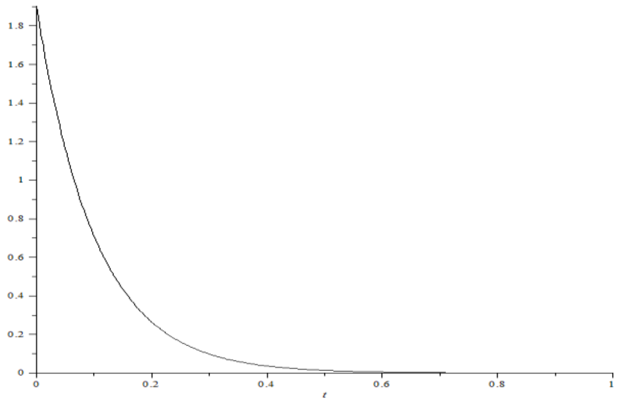

Figure 5. The graph of $u(x,t) = 2e^{-\pi^2 t}\sin(\pi t)\ at\ x = 0.4, 0 \le t \le 1$ for Problem 1.

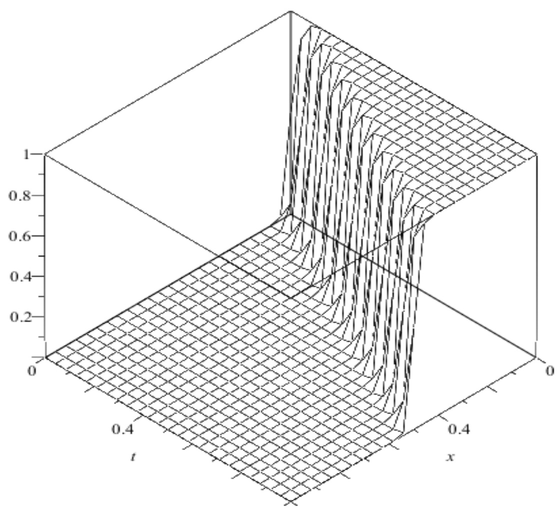

Figure 6. A 3 dimensional plot of the surface $u(x,t) = \left(1 + e^{(0.5x-0.25t)/\mu}\right)^{-1} at\ 0 \le x \le 1, 0 \le t \le 1, \mu = 0.005$ for Problem 2.

compares favourably with both the transverse and longitudinal methods of Mazzia and Mazzia (1997) as well as the longitudinal method of Ramos and Vigio-Aguiar (2007), showing that the method is considered to give more stable and accurate results at various space and time steps for both Problems 1 and 2. Though the knowledge of the first two eigenvalues of the discretization matrix is required for the implementation of this scheme, it does not pose a major problem as these values can easily be computed by the method of Shampine (1994) as demonstrated in this paper.

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