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Short Communication

Invariance of numerical range under isometric transformation

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In this paper, we look at some properties of isometries and conditions for a quasi-affinity to be unitary. Among other results it is proved that if A is a partial isometry such that either $0 \in W(A)$ or $0 \in W(A^n)$ for some positive integer n then A is unitary.

Key words: Numerical range and unitary operator.

INTRODUCTION

Let B(H) denote the Banach Algebra of bounded linear operators on a complex Hilbert space H. The numerical range of an operator T in B(H) is denoted by W(T) and is the range of values (Tx;x) where x ranges on the unit sphere in H. More precisely, $W(T) = \{(Tx;x);||x|| = 1\}$.

The numerical range like the spectrum of an operator is a subset of the complex numbers whose topological properties show some valuable information about the operator. In contrast with the spectrum, if the numerical range of an operator is real, the operator must be Hermitian Elementary properties of the numerical range are found in (Halmos, 1959; Williams, 1969).

But if the spectrum lies on the real line then very little can be drawn from this fact alone. In this paper, we investigate classes of operators for which the numerical range is invariant under similarities and then determine conditions under which some quasi-affine transforms are unitary.

Given an operator A on a Hilbert space H, W(A) is denoted as the numerical range of A and A is said to be quasi-affinity, if A is injective and has a dense range. An operator A is a quasi-affine transform of another operator B if there exists a quasi-affinity X such that XA = BX and the operators A and B are quasi similar if they are quasi-affine transforms of each other. An operator A is said to be: Unitary if $A^*A = AA^* = I$; an Isometry if $A^*A = I$; A Co-Isometry if $AA^* = I$; A Partial isometry if $A = AA^*A$. The following observations were made for Isometries.

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THEORY

Theorem 1

Let T be an operator in B(H) and A an Isometry, then $W(A^*TA) \subset W(T)$.

Proof

Let $\lambda \in W(A^*TA)$, then we have that: $\lambda = (A^*TAx;x)$ for some unit vector x = (TAx;Ax). However, $||Ax||^2 = (Ax;Ax) = (x;A^*A) = (x;x) = 1$: Hence, λ belongs to W(T).

Lemma

Let A be an Isometry such that each unit vector is of the form Ax for some x in H. then x must be a unit vector.

Proof

Let a unit vector y be of the form Ax, then $(y;y) = (Ax;Ax) = (x;A^*Ax) = (x;x) = 1$

Theorem 2

If A is an Isometry and every unit vector of H is of the form Ax, then $W(A^*TA) = W(T)$ for any bounded operators T on H.

Proof

Suppose that λ belongs to W(T), then λ = (Tx;x) for some unit vector x in H = (TAy;Ay) for some unit vector y in H = $(A^{¤}TAy;y) \in W(A^{¤}TA)$. Thus, the result, $W(T) \subset W(A^{¤}TA)$ is obtain. Hence from the previous theorem, we have that $W(T) = W(A^{¤}TA)$.

Corollary 1

If A is a Co-Isometry such that each unit vector is of the form Ax for some x in H, then for any bounded operator T, we have that $W(ATA^{¤}) = W(T)$.

Corollary 2

If a partial Isometry A is either injective or has a dense range and each unit vector is of the form Ax, then $W(A^{¤}TA) = W(T)$ or $W(T) = W(ATA^{¤})$ for all bounded operators T.

Proof

If A is a partial Isometry then we have that $A=A^{*}A \Leftrightarrow A-AA^{*}A=0 \Leftrightarrow A(I-A^{*}A)=0 \Leftrightarrow (I-AA^{*})=0$. If A is injective then $I - A^{¤}A = 0 \Leftrightarrow A^{*}A = I$. Thus A is an Isometry. If A has a dense range then again $I - AA^{*} = 0$ and so $A^{¤}A = I$. Thus A is Co-isometric. The result then follows form the previous corollary.

Remark

We note that in general that the existence of a quasi-affine inverse does not imply invertibility of an operator A. However for a partial isometry A, having a quasi-affine inverse implies that A is invertible and so is unitary (Khalagai and Otieno, 2000). In this case the following corollary to our main result is immediate

Corollary 3

If A is a partial Isometry which has quasi-affine inverse, then, for any bounded operator T on H, we have that $W(A^{¤}TA) = W(T)$. Now we make an extension to a result in (Khalagai and Otieno, 2000) on partial Isometries in the following theorem.

Theorem 3

Let A be a partial isometry then A is unitary if; (i) A is a quasi-affinity. (ii) A^{n} is a quasi-affinity.

Proof

(i) Since A is a partial isometry if $A = AA^{*}A$, we have that $A - AA^{¤}A = A(I-A^{¤}A) = (I-AA^{¤})A = 0$. Since A is a quasi-affinity $I-A^{¤}A = I-AA^{*} = 0$. Hence A is unitary. (ii) Similarly from $A = AA^{n}A$, we have that $A^{n} = AA^{*}A^{n} = A^{n}A^{*}A$. Hence, $A^{n}(I-AA^{*}) = (A^{*}A - I)A^{n}$. Since A^{n} Is a quasi-affinity, we obtain $A^{¤}A = AA^{¤} = I$ and so A is unitary.

Corollary 4

If A is a partial isometry such that either $0 \in W(A)$ or $0 \in W(A^{n})$ for some positive integer n, then A is unitary.

Proof

If $0 \in W(A)$ then A is a quasi-affinity. Hence from the previous theorem, A is unitary. Likewise, if $0 \in W(A^{n})$, then A^{n} is a quasi-affinity and so A is unitary.

Conclusion

Let L be A a right unilateral shift operator and chose a unit vector e Orthogonal to the range of A and let $Tx = (x,e)e$ then $TAx = (Ax,e)e = 0$ hence $W(A^{*}TA) = W(0) = \{0\}$. But T is a projection so that its spectrum is {0, 1} and hence its numerical range is the interval [0,1]. So generally the numerical range is not isometrically invariant.

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