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The weighted Monsef distribution

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A two-parameter weighted Monsef distribution (*WM*) is proposed in this paper. *WM* is flexible and has the property that the hazard rate function can accommodate both increasing and bathtub shapes. Most of its mathematical properties, such as probability density function, hazard function, moments and mean residual life function, are derived. The maximum likelihood method is used to estimate the distribution parameters. A simulation study is performed to examine the bias and mean square error of the maximum likelihood estimators of the parameters. Two real data sets are presented to illustrate the model flexibility in fitting some data against some other known distributions.

Key words: Weighted Monsef distribution, hazard rate function, maximum likelihood, mean residual life function.

INTRODUCTION

A challenge faced by statisticians has been to model the life expectancy of real-world phenomena using lifetime distributions (Lehmann and Casella, 1998). This challenge appears in several fields, such as survival analysis and reliability analysis of mechanical components, as well as in monitoring real data applications in biological and biomedical studies (Bjerkedal, 1960). One of the important models used to study survival data is the weighted distributions. The importance of weighted distributions occurs when some of the observations are damaged or cannot be observed due to unusual causes. In this case, the resulting values are decreased and the units have different probabilities of occurrence. Fisher and Rao first familiarized the idea of weighted distributions. Patil and Rao studied many general models that led to weighted distributions, and they presented how it arises in a natural manner in many selection issues. Patil and Rao (1978) also identified that a weighted distributions model is suitable because it offers a new concept of standard distributions and it provides methods to extend distributions, which afford more flexibility in fitting data. According to Ghitany et al. (2011), if the original observation x_0 has a probability density function (p.d.f.) "$f_0(x_0;\theta_1)$", where θ_1 is a parameter vector, and that observation x is noted according to a probability re-weighted by a weight function $w(x;\theta_2)>0$, and θ_2 is a new parameter vector, then x comes from a distribution with p.d.f (Ghitany et al., 2011).

$$f(x)=Bw(x;\theta_2)f_0(x_0;\theta_1)$$

Distributions of this type of p.d.f are known as weighted distributions. Here, B is a normalizing constant. This paper is organized as follows: In Section 2, essential shape properties of the density, hazard rate, mean residual life functions and moments of the WM model are

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presented. The moments and moments generating function are introduced in Section 2 as well. The maximum likelihood estimates (MLEs) of the unknown parameters are illustrated in Section 3. In Section 4, simulation studies are provided to examine the performance of the maximum likelihood estimators of the parameters. In Section 5, the suggested model is fitted to two available survival data sets and compared with some other related models. Finally, a conclusion is presented to conclude the paper.

Statistical measures

In this section, the shape properties of the density, hazard rate, mean residual life functions and moments of the WM are presented.

Probability density function

In this paper, a two-parameter weighted Monsef distribution (WM) is proposed with probability density function

$$f(x) = A\, x^{k-1} f_0(x;\theta), \qquad x>0, \qquad k,\theta>0, \tag{1}$$

Where *A* is a normalizing constant and:

$$f_0(x;\theta) = \frac{\theta^3}{2+\theta(2+\theta)}(x+1)^2 e^{-x\theta}, \qquad x>0, \qquad \theta>0, \tag{2}$$

is the p.d.f. of Monsef distribution, which is defined in an earlier work of Abd El-Monsef (2020) as a special case of the Erlang mixture distribution (2020). We can note that when *k = 1*, the *WM* reduces to Monsef distribution.
The p.d.f of *WM* can be written as::

$$f(x) = \frac{\theta^{k+2}}{\left(k+k^2+2k\theta+\theta^2\right)\Gamma(k)} x^{k-1}(1+x)^2 e^{-\theta x}, \quad x>0,\ \theta,k>0, \tag{3}$$

Where:

$$\Gamma(k) = \int_0^\infty x^{k-1} e^{-x}\, dx, \qquad k>0,$$

is the complete gamma function (Batir, 2005).

It is clear to see that the density function of the WM p.d.f. has the behaviors at $x=0$ and $x=\infty$, as follows:

$$f(0) = \begin{cases} \infty, & \text{if } k<1, \\ \dfrac{\theta^3}{2+\theta(2+\theta)} & \text{if } k=1, \\ 0, & \text{if } k>1, \end{cases} \qquad f(\infty)=0$$

The shape of the p.d.f can be obtained by next theorem.

Theorem 1.

The p.d.f. of the *WM* has three different shapes. These shapes can be displayed as follows:

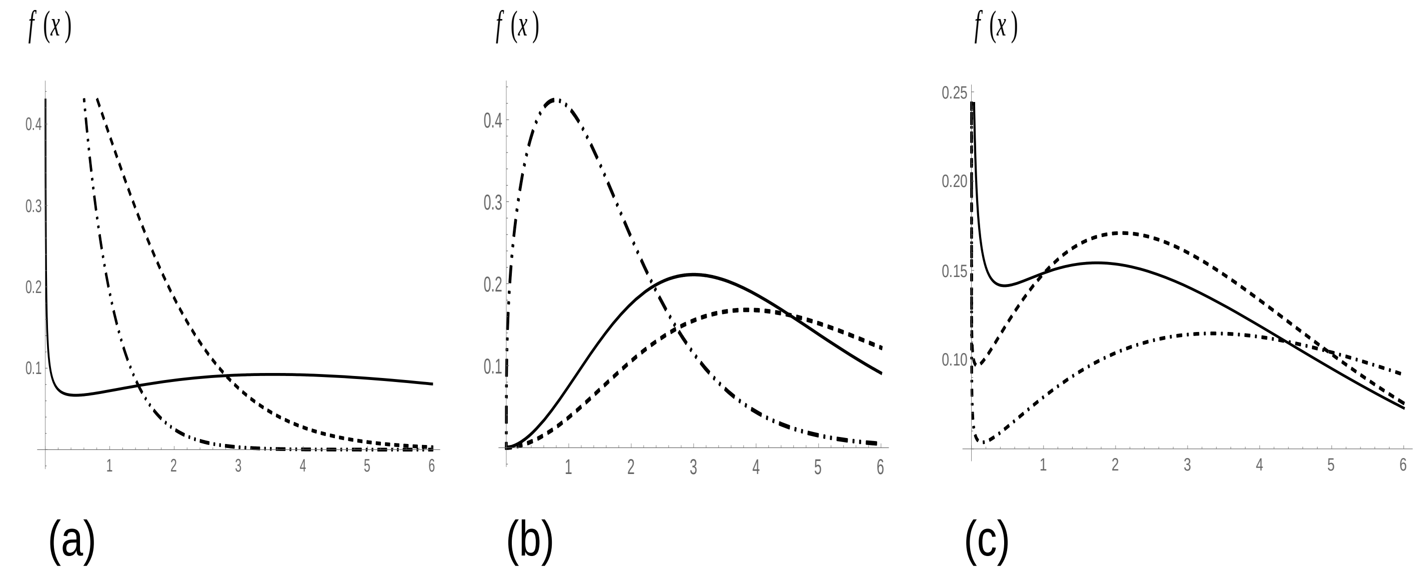


Figure 1. Behavior of probability density function of the *WM*: (a) (—) k = 0.5, θ = 0.3, (- - -) k = 0.5, θ = 2.5, (–..–..–) k = 0.8, θ = 1.4. (b) (—) k = 2.5, θ =1.0, (- - -) k = 1.3, θ = 1.5, (–..–..–) k = 2.5, θ = 0.8. (c) (—) k = 0.6, θ = 0.5, (- - -) k = 0.9, θ = 0.6, (–..–..–) k = 0.8, θ = 0.4.

(1) Decreasing if [$\theta < k < 1$: $(\theta - k-1)^2 - 4\theta(1-k) \leq 0$] or [$k \leq 1$, $\theta \geq k$].
(2) Unimodal if [$k = 1$, $\theta < 1$] or [$k > 1$, $\theta > 0$].
(3) Decreasing-increasing-decreasing if [$\theta < k < 1$: $(\theta - k-1)^2 - 4\theta(1-k) > 0$]

Proof

The first derivative of *f(x)* can be written as:

$$f'(x) = -\frac{g(x)}{x(1+x)} f(x),$$

where:

$$g(x) = \theta x^2 + (\theta - k - 1)x + (1-k)$$

Figure 1 illustrates the p.d.f. of the WM distribution. It is clear that the aforementioned conditions are satisfied by the number of values of k and θ.

Proof

The first derivative of f(x) can be written as:

Where

$$g(x) = \theta x^2 + (\theta - k - 1)x + (1-k)$$

Now we have the discriminant of $g(x)$, which is $D = (\theta - k - 1)^2 - 4\theta(1-k)$,

(i) If [$k \leq 1$, $\theta \geq k+1$], $g(x) > 0$, Also, if [$\theta < k < 1 : D < 0$], $g(x) > 0$. Consequently, $f(x)$ is decreasing.

In order to prove the last parts, we have to find the second derivative of $f(x)$, which is given by:

$$f''(x)=\frac{-1}{x(1+x)}\left\{\left[g(x)+2x+1\right]f'(x)+(2\theta x+\theta-k-1)f(x)\right\}.$$

(ii) If [k = 1, θ <1] or [k >1, θ >0], $f'(x)=0$ iff $g(x)=0$ which happens at the point x_0 where

$$x_0=\frac{-(\theta-k-1)+\sqrt{D}}{2\theta}.$$

Since, $f''(x_0)=\left(\frac{-\sqrt{D}}{x_0(1+x_0)}f(x_0)\right)<0$, $f(x)$ has a global maximum at x_0.

(iii) If [θ < k < 1: D > 0], $f'(x)=0$. if $g(x)=0$, which occurs at the two points $x_1<x_2$ where

$$x_1=\frac{-(\theta-k-1)-\sqrt{D}}{2\theta},\quad x_2=\frac{-(\theta-k-1)+\sqrt{D}}{2\theta}$$

Since, $f''(x_1)=\left(\frac{-\sqrt{D}}{x_1(1+x_1)}f(x_1)\right)>0$, $f(x)$ has a local minimum at x_1. Equally, $f''(x_2)<0$ indicates that $f(x)$ has a local maximum at x_2.

It is known that the density functions of most known two-parameter distributions, such as Gamma and Weibull distributions, are either decreasing or unimodal. The p.d.f. of the *WM* distribution has an extra shape, which can be beneficial for modeling.

Hazard rate function

The hazard (or failure) rate is more important in some aspects of continuous distribution than either the distribution function or density function.

If $F(x)$ is an absolutely continuous distribution function with density function $f(x)$, then the hazard rate function $h(x)$ is given by

$$h(x)=-\frac{d\ln(\mathrm{R(x)})}{dx}=\frac{f(x)}{R(x)}$$

is called the hazard rate function (HRF) of X (Chen et al., 2004). This function is a non-negative.
Where $R(x)$ is the survival or (reliability) function which is given by:

$$R(x)=1-F(\mathrm{x})=\Pr(\mathrm{X}>\mathrm{x})=f\int_x^\infty g(x)dx$$

The hazard rate function of the *WM* distribution is given by

$$h(x)=-\frac{\mathrm{e}^{-\theta x}x^{k-1}(1+x)^2\theta^{k+2}(x\theta)^k}{\left(k+k^2+2k\theta+\theta^2\right)\left(x^k\theta^k-(\theta x)^k\right)\Gamma(k)-x^k\theta^k\left(\theta^2\Gamma(k,\theta x)+2\theta\Gamma(1+\mathrm{k},\theta x)+\Gamma(2+\mathrm{k},\theta x)\right)},\qquad (4)$$

Behaviors of $h(x)$ at $x=0$ and $x=\infty$, separately, are given by:

$$h(0)=\begin{cases}\infty, & \text{if } k<1,\\ \dfrac{\theta^3}{2+\theta(2+\theta)} & \text{if } k=1,\\ 0, & \text{if } k>1,\end{cases} \qquad h(\infty)=\theta$$

Lemma 1

Let *X* be a non-negative continuous random variable with density function *f(x)* and hazard rate function *h(x)*. *Let* $\eta(x)=-(\mathrm{d}/\mathrm{dx})\ln f(x).$:

1. If *η(x)* is decreasing (increasing) in *x*, then *h(x)* is increasing (decreasing) in *x*.
2. If *η(x)* has a bathtub (upside-down bathtub) shape, *h(x)* has also a bathtub (upside-down bathtub) shape.

By using Glaser's consequence (Paranjpe and Rajarshi, 1986), the shape of the hazard rate of the *WM* distribution can be determined as follows.

In order to determine the shape of *h(x),* we introduce the following theorem

Theorem 2

The hazard rate function $h(x)$ of the *WM* distribution is bathtub shaped (increasing) if $0<k<1(\mathrm{k}\geq 1)$ for all $\theta>0$.

Proof

Since

$$\eta(x)=-\frac{d}{dx}\ln f(x)=-\frac{f'(x)}{f(x)}=-\frac{k-1}{x}-\frac{2}{1+x}+\theta,$$

therefore

$$\eta'(x)=\frac{k-1}{x^2}+\frac{2}{(1+x)^2}.$$

(i) For $0<k<1$, $\eta(x)=0$ indicates that $\eta(x)$ has a global minimum at $x^*=\dfrac{k-1+\sqrt{2-2k}}{1+k}$, that is, $\eta(x)$ is bathtub shaped. Thus, h(x) is also bathtub shaped.
(ii) For $k\geq 1$, $\eta'(x)>0$, that is $\eta(x)$ is increasing. Hence, h(x) is also increasing.

The hazard rate function h(x) of the WM for selected values of k and θ is illustrated in Figure 2. Here, it can be noted that the bathtub feature of the hazard rate function of WM is most convenient in modeling biological data from mortality studies.

Mean residual life function

The mean residual life function $\mu(x)=E(X-x\mid X>x)$ of the WM is given by:

$$\mu(x)=\frac{-x\theta^3\,\Gamma[k,x\theta]+(1-2x)\theta^2\,\Gamma[1+k,x\theta]-(x-2)\theta\,\Gamma[2+k,x\theta]+\Gamma[3+k,x\theta]}{\theta(\theta^2\,\Gamma[k,x\theta]+2\theta\,\Gamma[1+k,x\theta]+\Gamma[2+k,x\theta])},$$

Next is the behaviors of *μ*(*x*) at *x* = 0

$$\mu(0)=\frac{6+\theta(4+\theta)}{\theta[2+\theta(2+\theta)]}$$

The shape of mean residual life function μ(x) can be determined by using the following two lemmas.

Lemma 2: Let X be a non-negative continuous random variable with *h(x)* a hazard rate function and μ(x) a mean residual life function. If *h(x)* is decreasing (increasing), then μ(x) is increasing (decreasing) (Bryson and Siddique, 1969).

Lemma 3: Let X be a non-negative continuous random variable with p.d.f. *f(x)*, hazard rate function *h(x)* and mean residual life function μ(x). If *h(x)* has a bathtub (upside-down bathtub) shape and *f(0)*μ(0) > 1(≤1), then μ(x) has an upside-down bathtub (bathtub) shape (Gupta and Akman, 1995).

Using Lemmas 2 and 3, the shape of the mean residual life function μ(x) of the *WM* can be illustrated as follows:

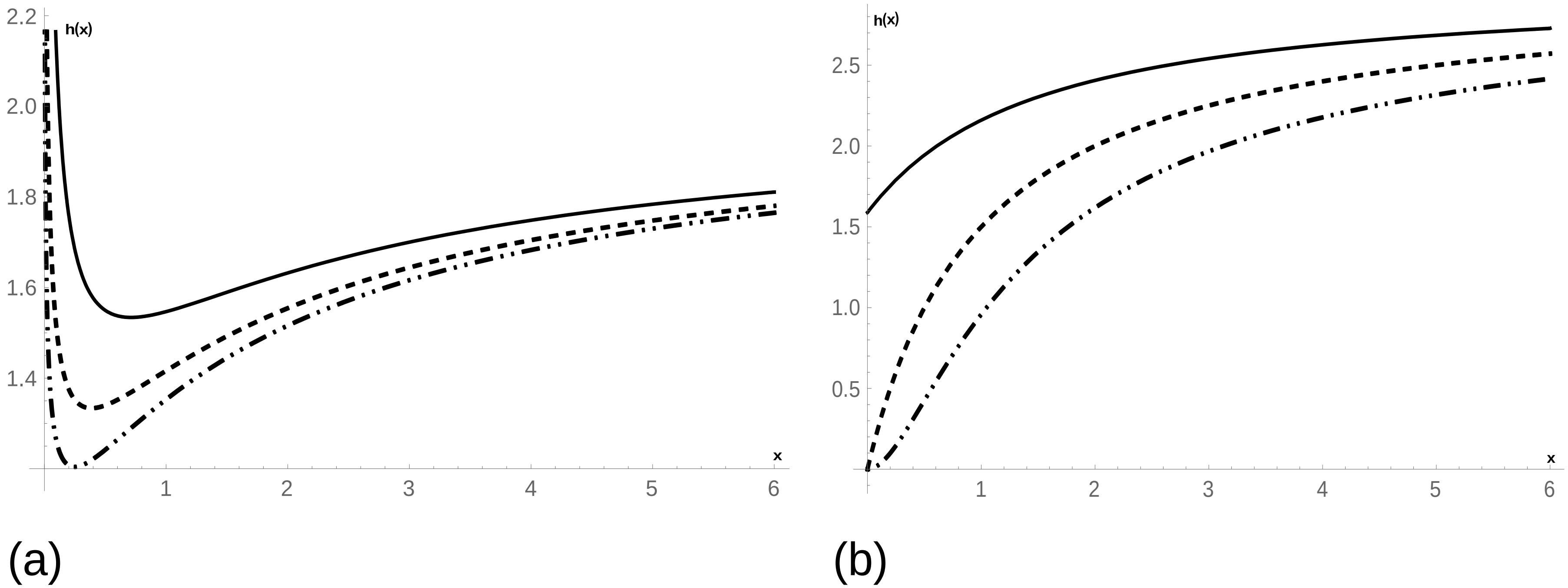


Figure 2. Behavior of Hazard rate function of *WM* distribution:. (a) *K*= 0.5 (—), *K* = 0.7 (- - -), *K* = 0.8 (..–..–..) and θ = 2.0; (b) *K* = 1.0 (—), *K* = 2.0 (- - -), *K* = 3.0 (..–..–..) and θ = 3.0

Theorem 4-3

The mean residual life function $\mu(x)$ of the *WM* is upside-down bathtub shaped (decreasing) when $0 < k < 1 (k \geq 1)$ for all $\theta > 0$.

Proof

Since $h(x)$ is bathtub shaped for $0 < k < 1$, in this case, $f(0)\ \mu(0) = \infty$, $\mu(x)$ is upside-down bathtub shaped, by Lemma 3. Finally, since $h(x)$ is increasing for $k \geq 1$, $\mu(x)$ is decreasing, by Lemma 2.
The behaviors of mean residual life function $\mu(x)$ of the *WM* are shown in Figure 3. It is clear to see that the upside-down bathtub feature is most valuable in modeling engineering reliability data.

Moments and the moment generating function

In this section, the moments about the origin and the moment generating function, are obtained.

The moments about the origin

The rth raw moment (about the origin) of the *WM* $\mu_r' = E(X^r), \mathrm{r} = 1, 2, \ldots$ is given by:

$$\mu_r' = \frac{\theta^{-r}\left[\mathrm{k}^2 + \mathrm{r} + (\mathrm{r}+\theta)^2 + \mathrm{k}(1 + 2\,\mathrm{r} + 2\theta)\right]\Gamma(\mathrm{k}+\mathrm{r})}{(\mathrm{k} + \mathrm{k}^2 + 2\,\mathrm{k}\,\theta + \theta^2)\Gamma(\mathrm{k})},$$

Thus, the mean and variance of the *WM*,

$$\mu = \frac{k+2}{\theta} - \frac{2(k+\theta)}{k + k^2 + 2k\,\theta + \theta^2}$$

$$\sigma^2 = \frac{k+2}{\theta^2} - \frac{2\left[\mathrm{k}(\mathrm{k}-1) + 2k\,\theta + \theta^2\right]}{\left[k + k^2 + 2k\,\theta + \theta^2\right]^2}$$

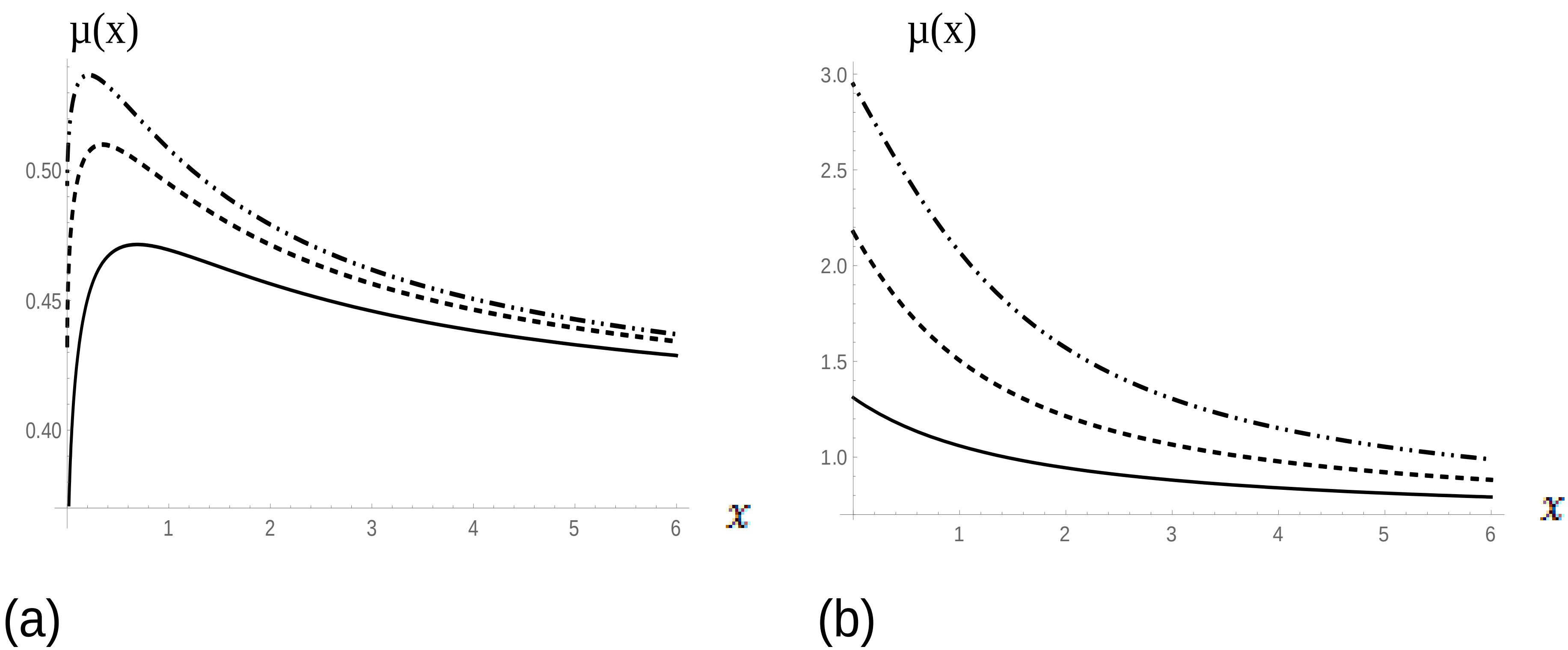


Figure 3. Behavior of mean residual life function of *WM* distribution: (a) k = 0.4 (—), k = 0.6 (- - -), k = 0.7 (..–..–..) and θ = 2.5; (b) k = 1.0 (—), k = 2.0 (- - -), k = 3.0 (..–..–..) and θ = 1.5.

The moment generating function

Here, the moment generating function of *WM* distribution is presented as:

$$M_x(t) = \frac{\theta^{k+2}(\theta - t)^{-2-k}\left(k^2 + (t-\theta)^2 + k(1 - 2t + 2\theta)\right)}{k + k^2 + 2k\theta + \theta^2}, \quad (6)$$

Estimation

In order to estimate the parameters of the proposed *WM* density function as defined in Equation 3, the loglikelihood of the sample is maximized with respect to the parameters.

Let $x_1, \ldots, x_n$ be a random sample of size n from the *WM* with p.d.f. (3) and parameters k and θ. Let $\Theta = (k, \theta)^T$ be the $r \times 1$ parameter vector. The log- likelihood function for Θ, say $\ell = \ell(\Theta)$, is given by:

$$\ell = n\left[(k+2)\ln(\theta) - \ln(k + k^2 + 2k\theta + \theta^2) - \ln(\Gamma(K))\right] + (k-1)\sum_{i=1}^{n}\ln(x_i) + 2\sum_{i=1}^{n}\ln(1 + x_i) - n\theta\bar{x},$$

Table 1. Bias and MSE for the parameters θ, k.

n	$\theta = 0.5$		$k = 0.9$		$\theta = 0.5$		$k = 1$	
	MSE	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**
20	0.03453	0.07555	0.55638	0.30094	0.03640	0.07728	0.67297	0.32921
40	0.01136	0.03529	0.17123	0.14330	0.01199	0.03624	0.21012	0.15756
60	0.00651	0.02341	0.09162	0.09193	0.00689	0.02403	0.11390	0.10121
80	0.00452	0.01633	0.06539	0.06681	0.00478	0.01679	0.08129	0.07353
100	0.00341	0.01383	0.04757	0.05481	0.00361	0.01424	0.05940	0.06059

n	$\theta = 0.2$		$k = 1$		$\theta = 0.5$		$k = 1.5$	
	MSE	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**
20	0.00744	0.03405	1.1521	0.41551	0.04284	0.08126	1.3379	0.44699
40	0.00245	0.01644	0.37081	0.20551	0.01422	0.03853	0.43452	0.21653
60	0.00140	0.01104	0.20433	0.13468	0.00824	0.02550	0.24292	0.13961
80	0.00097	0.00781	0.14463	0.09864	0.00575	0.01778	0.17486	0.10062
100	0.00073	0.00667	0.10679	0.08237	0.00436	0.01522	0.12954	0.08421

where $\bar{x}$ is the sample mean.

The maximized log-likelihood can be found by solving the nonlinear likelihood equations attained by the last differentiating.

$$U_1(k,\theta)=\frac{\partial \ell}{\partial k}=n\left[\ln(\theta)-\frac{2k+2\theta+1}{k+k^2+2k\theta+\theta^2}-\Phi(k)\right]+\sum_{i=1}^{n}\ln(x_i), \quad (7)$$

$$U_2(k,\theta)=\frac{\partial \ell}{\partial \theta}=n\left[\frac{k+2}{\theta}-\frac{2k+2\theta}{k+k^2+2k\theta+\theta^2}\right]-n\bar{x}, \quad (8)$$

where $\Phi(k)=(\mathrm{d}/\mathrm{d}k)\ln\Gamma(k)$ is the digamma function.

The MLEs $\hat{k}$ and $\hat{\theta}$ of k and θ are achieved by solving the non-linear equations $U_1(\hat{k},\hat{\theta})=0$ and $U_2(\hat{k},\hat{\theta})=0$.

SIMULATION STUDY

In this section, a simulation study is presented to examine the average bias and average mean square error (MSE) of the simulated estimates. The equation $F(x)-u=0$, where u is an observation from the uniform distribution (0,1), and F(x) is a cumulative distribution function of the *WM* distribution, is used to complete the simulation study by generating random samples following the *WM*. The simulation experiment was repeated 10,000 times each with sample sizes of 20, 40, 60, 80 and 100 for θ = (0.5, 0.5, 0.2,0.5) and k =(0.9, 1,1, 1.5). The study calculates the following measures:

From Table 1, it can be established that the MSE and the average bias decrease as the sample size increases

DATA ANALYSIS

The aim of this section is to illustrate the WM distribution by showing a successful application to two real data sets.

The carbon fibers data set

Here, the uncensored data set on the breaking stress of carbon fibers (in Gba) as reported in Cordeiro et al. (2013) is considered.

The data are (n = 66): 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 3.56, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.89, 2.88, 2.82, 2.05, 3.65, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.35, 2.55, 2.59, 2.03, 1.61, 2.12, 3.15, 1.08, 2.56, 1.80, 2.53.

The application is presented where the *WM* distribution is compared with other related models. The related models are Monsef distribution, Weibull distribution, Gamma distribution and Exponential Power distribution. To compare the goodness of fit, the information criteria *AIC = -2 log L+2c*, *BIC = -2logL + c log n* and the Kolmogrov-Smirnov statistic are used, where *c* is the number of parameters and *n* is the sample size (Alizadeh et al., 2017).

Table 2. Fitted estimates for different distributions.

Model	Parameter	-Log likelihood	AIC	BIC	K-S statistic
WM	$\hat{\theta}=2.877$ $k=6.511$	90.6934	185.387	189.766	0.13076
Monsef	$\hat{\theta}=0.8332$	114.421	230.843	233.033	0.25059
Gamma	$\hat{\theta}=0.794$ $\hat{k}=11.79$	168.049	340.099	344.478	0.55123
Weibull	$\hat{\theta}=1.4059$ $\hat{k}=4.847$	306.402	616.805	621.184	0.95640
Exponential Power	$\hat{\theta}=0.990$ $\hat{k}=0.901$ $\hat{\sigma}=0.085$	1173.14	2352.29	2358.86	0.93681

Table 3. Fitted estimates for different distributions.

Model	Parameter	-Log likelihood	AIC	BIC	K-S statistic
WM	$\hat{\theta}=0.049$ $k=0.194$	543.524	1091.05	1096.55	0.1177
Monsef	$\hat{\theta}=0.069$	551.809	1105.62	1108.37	0.1790
Exponential	$\hat{\theta}=0.023$	549.926	1101.85	1104.61	0.1349
Lindley	$\hat{\theta}=0.101$	631.848	1265.70	1268.45	0.3447

The daily ozone measurements data set

These data are used to compare the *WM* distribution with Monsef, Exponential and Lindley distributions (Lindley, 1958). The following data are the daily ozone measurements in New York between May and September 1973 as reported by Ghitany et al, (2008).

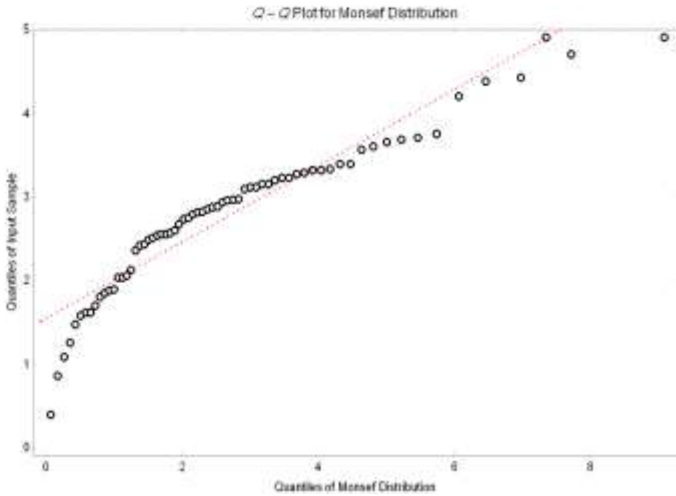
41, 36, 12, 18, 28, 23, 19, 8, 7,16, 11, 14, 18, 14, 34, 6, 30, 11, 1, 11, 4, 32, 23, 45, 115, 37, 29, 71, 39, 23, 21, 37, 20,12, 13, 135, 49, 32, 64, 40, 77, 97, 97, 85, 10, 27, 7, 48, 35, 61, 79, 63, 16, 80, 108, 20,52, 82, 50, 64, 59, 39, 9, 16, 78, 35, 66, 122, 89, 110, 44, 28, 65, 22, 59, 23, 31, 44, 21, 9,45, 168, 73, 76, 118, 84, 85, 96, 78, 73, 91, 47, 32, 20, 23, 21, 24, 44, 21, 28, 9, 13, 46,18, 13, 24, 16, 13, 23, 36, 7, 14, 30, 14, 18, 20.

The -Log likelihood, AIC and BIC values, and K-S statistic are presented in Table 3.

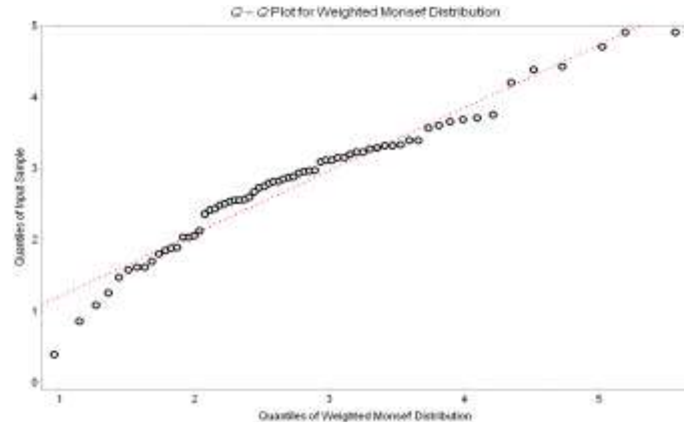
Conclusion

In the present paper, a new two-parameter distribution is introduced that was obtained from the idea of weighted distributions. Some of its mathematical properties are studied. The proposed distribution is applied on two data sets that demonstrated to provide a better fit than other related models. This is also supported by the Probability–Probability (P–P) plots presented in Figure 4 and Figure 5.

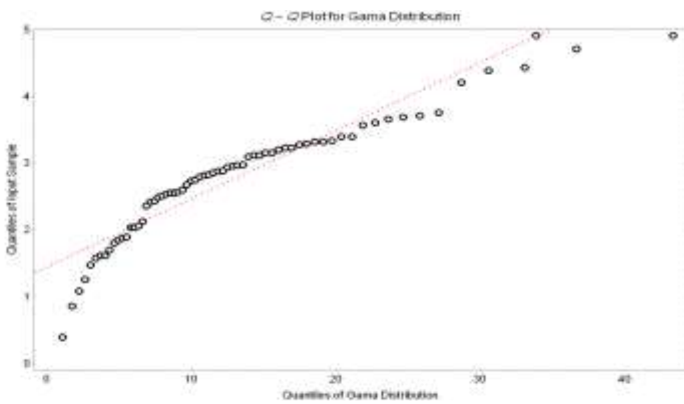
The distributional results developed in this article should find numerous applications in the physical and biological sciences and, in particular, in reliability theory and survival analysis.



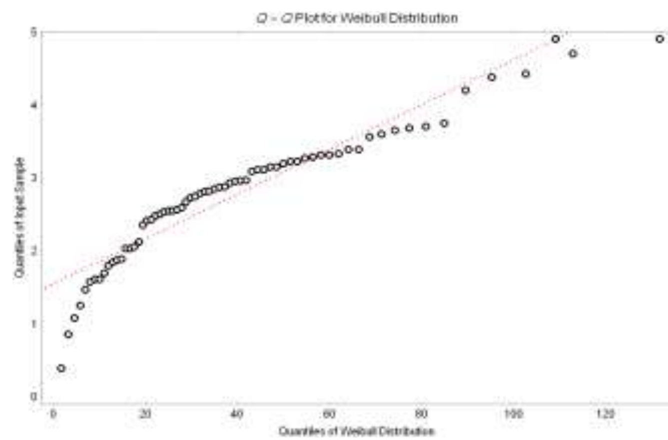
(a) Fitted Monsf Distribution Function



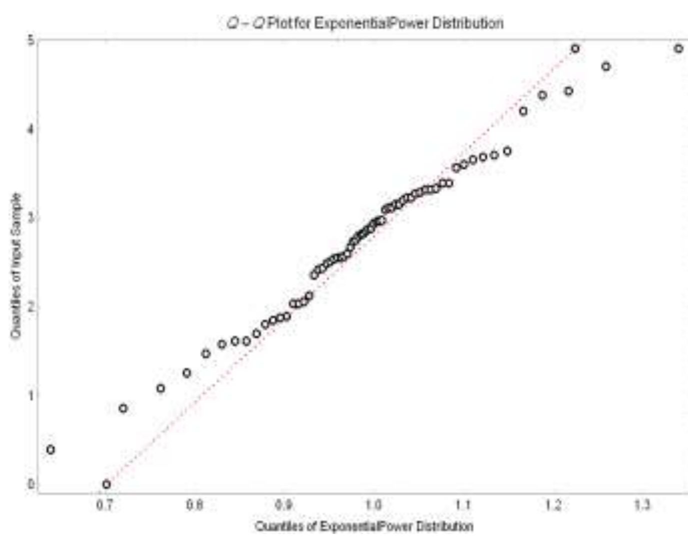
(b) Fitted *WM* Distribution Function



(c) Fitted Gamma Distribution Function

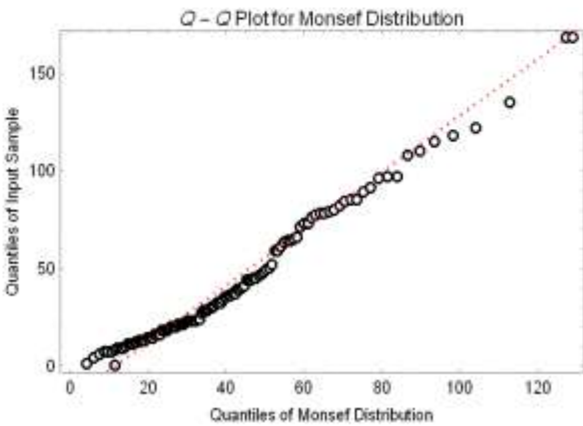


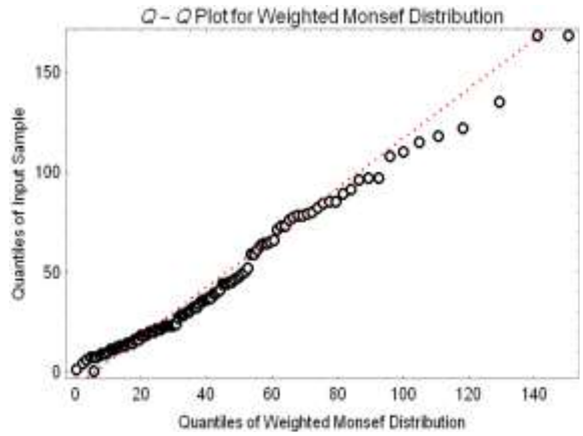
(d) Fitted Weibull Distribution Function



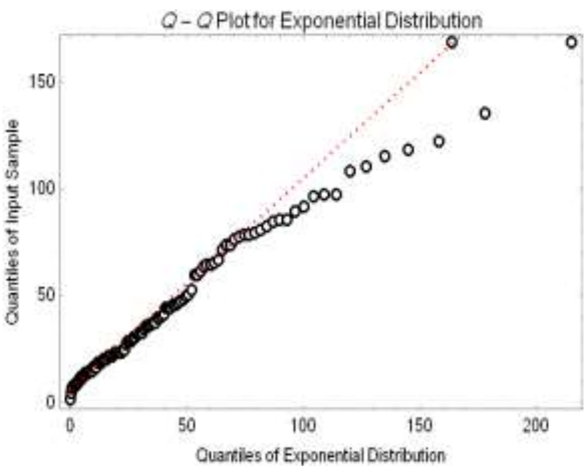
(e) Fitted Exponential Power Distribution Function

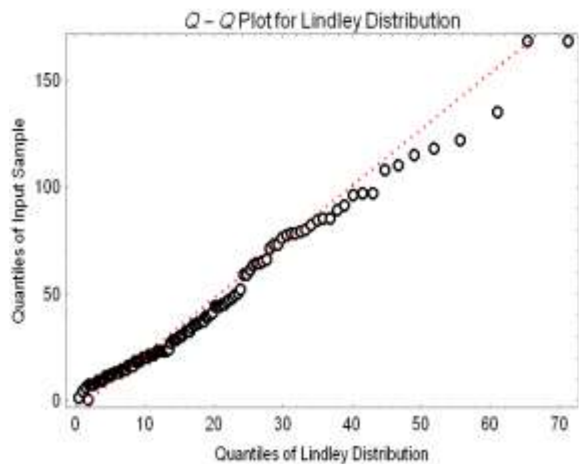
Figure 4. P–P plots for fitted (a) Monsf, (b) *WM*, (c) Gamma, (d) Weibull distributions and (e) Exponential Power Distribution.





(a) Fitted Monsf Distribution Function (b) Fitted *WM* Distribution Function





(c) Fitted Exponential Distribution Function (d) Fitted Lindley Distribution Function

Figure 5. P–P plots for fitted (a) Monsf, (b) *WM*, (c) Exponential and (d) Lindley distributions.

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