African Journal of Mathematics and Computer Science Research Vol. 2(2), pp. 025-029, March, 2009
Available online at http://www.academicjournals.org/AJMCSR


*Full Length Research Paper*

# On two-sample test for detecting differences in the IFR property of life distributions

**Parameshwar V. Pandit[1]* and Nagaraj V. Gudaganavar[2]**

[1]Department of Statistics, Bangalore University, Jnanabharathi, Bangalore- 560056, India.
[2]Department of Statistics, Anjuman Arts, Science and Commerce College, Dharwad-580001, India.

Accepted 30 January, 2009

**A test proposed for testing whether one distribution is more Increasing Failure Rate (IFR) than another, based on a measure of IFR is presented in this paper. The asymptotic normality of the proposed test statistic was also established. The asymptotic null variance from the data was estimated since the variance depends on the unknown distribution. The Pitman asymptotic efficacies of the proposed test statistic are computed for various alternative IFR distributions.**

**Key words:** Two-sample test, IFR, Pitman efficacy.

## INTRODUCTION

A life is represented by a non-negative random variable X with distribution function F and survival function $\overline{F}$ =1-F. Classes of life distributions based on some notion of ageing have been introduced in the literature. The increasing failure rate (IFR) is one of the most important classes of lifetime distributions. We define IFR class below. A distribution F is said to be increasing failure rate (IFR), if $\frac{\overline{F}(x+t)}{\overline{F}(x)}$ is decreasing in x, t ≥ 0.

Proschan and Pyke (1967) proposed a test for testing exponentiality against IFR alternatives followed by Barlow and Proschan (1969), Bickel and Dorsum (1969), Bickel (1969), Ahmad (2004) and among others.

In practice, one might be interested in comparing two life distributions with respect to their ageing properties, particularly, IFR condition. Kochar (1981) and Cheng (1985) proposed several test procedures for testing equality of failure rates of two distributions. Hollandar et al. (1986) developed a test procedure for testing the null hypothesis that two life distributions F and G are equal versus the alternative hypothesis that F is more NBU than Tiwari and Zalkikar (1988) proposed a test for testing the null hypothesis that two life distributions F and G are identical versus the alternative hypothesis that F is ‘More increasing failure rate average’ than G. Recently, Lim et al. (2005) developed a class of test procedures for testing the null hypothesis that two life distributions F and G are equal against the alternative that F is ‘more NBU at specified age’ than G. However, less attention is paid in the literature for testing the null hypothesis that two life distributions F and G are identical against the alternative that F is more IFR than G.

In this paper, we develop a simple test procedure for testing the null hypothesis that two life distribution F and G are equal against the alternative that F is more IFR than G.

### The proposed two sample ‘more IFR’ test

Let $X_1,X_2,\ldots,X_m$ and $Y_1,Y_2,\ldots,Y_n$ denote two random samples from continuous life distributions F and G respectively. We want to develop test statistic for testing the null hypothesis.

$H_0$ : F = G (the common distribution is not specified)
Versus
$H_1$ : F is ‘more IFR’ than G.

Based on the two independent random samples.

Let $\mu_1$ and $\mu_2$ be the means corresponding to F and G respectively.
Consider the parameter

$$\gamma(F,G)=\gamma(F)-\gamma(G)$$

*Corresponding author. Email: panditpv@rediffmail.com.

Where

$$\gamma(F)=\frac{1}{\mu_1^2}\left(\int_o^\infty\int_o^\infty \overline{F}^2\left(\frac{x+t}{2}\right)dxdt-\int_o^\infty\int_o^\infty \overline{F}(x)\overline{F}(t)dxdt\right)$$

and

$$\gamma(G)=\frac{1}{\mu_2^2}\left(\int_o^\infty\int_o^\infty \overline{G}^2\left(\frac{x+t}{2}\right)dxdt-\int_o^\infty\int_o^\infty \overline{G}(x)\overline{G}(t)dxdt\right)$$

Here $\gamma(\mathrm{F})$ and $\gamma(\mathrm{G})$ can be considered as the measure of degree of the IFR-ness. Ahmed (2004) used this measure as basis in the construction of a test statistic for testing exponentiality against the IFR alternatives in one sample setting. If F and G belongs to IFR class, then $\gamma(F)>0(\gamma(G)>0)$ and $\gamma(F,G)$ can be taken as a measure for deciding whether F is 'more IFR' than G or not. Under $H_0$, $\gamma(F,G)=0$ and it is strictly greater than zero under $H_1$.

The following unbiased estimator for $\gamma(\mathrm{F,G})$ is proposed, which is give

$V = V_1 - V_2$

Where

$$V_1=\frac{[m(m-1)]^{-1}\ \ \Sigma\Sigma\left[2\{\min(x_i,x_j)\}^2-x_ix_j\right]}{\overline{X}^2}$$

$$V_2=\frac{[n(n-1)]^{-1}\ \ \sum_{i\neq j}\Sigma\left[2\{\min(y_i,y_j)\}^2-y_iy_j\right]}{\overline{Y}^2}$$

Here $\overline{\mathrm{X}}$ and $\overline{\mathrm{Y}}$ the sample means of X and Y samples respectively.

The asymptotic normality of the test V is presented in the following theorem.

Theorem: As N $\rightarrow$ $\infty$, $\sqrt{N}[V-\gamma(F,G)]$ is asymptotically normal with mean zero and variance $\sigma^2(F,G)=\frac{\sigma_1{}^2(F)}{\eta}+\frac{\sigma_2{}^2(G)}{1-\eta}$, where N=m+n, $\eta=\lim_{N\to\infty}\frac{m}{N}$

$$\sigma_1{}^2(F)=\frac{4}{\mu_1^4}\ \mathrm{Var}\left(\mu_1\mathrm{X}_1-2\int_0^{\mathrm{X}_1}\mathrm{xdF(x)}-2\mathrm{X}_1{}^2\,\overline{\mathrm{F}}(\mathrm{X}_1)\right)$$

and

$$\sigma_2{}^2(G)=\frac{4}{\mu_2^2}\ \mathrm{Var}\left(\mu_2\mathrm{Y}_1-2\int_0^{\mathrm{Y}_1}\mathrm{ydG(y)}+2\mathrm{Y}_1^2\,\overline{\mathrm{G}}(\mathrm{Y}_1)\right)$$

Under $H_0$ : $\sigma_1{}^2(F)=\sigma_2{}^2(G)$

$$\sigma^2=\frac{\sigma_1{}^2(F)}{\eta(1-\eta)}\ .$$

Proof: To establish asymptotic normality of the test statistic V, define

$$\mathrm{h}(\mathrm{x}_1,\mathrm{x}_2)=2\{\min(\mathrm{x}_1,\mathrm{x}_2)\}^2-\mathrm{x}_1\mathrm{x}_2$$

and set

$$h_1(x_1)=E\left[h(X_1,X_2)\middle|X_1=x_1\right]$$

$$=2\int_0^{\mathrm{x}_1}x^2\mathrm{dF}(x)-2\mathrm{x}_1{}^2\,\overline{\mathrm{F}}(\mathrm{X}_1)\text{ - }\mu_1\mathrm{x}_1$$

$$h_2(x_1,x_2)=E\left[h(X_1,X_2)\middle|X_1=x_1,X_2=x_2\right]$$

and

$$\xi_k(F)=E\left(h_1^2(x_1,\ldots,x_k)\right)-\{\gamma(F)\}^2,\ \ \mathrm{k}=1,2$$

where

$$\gamma(F)=\int_0^\infty\int_0^\infty\overline{F}^2\left(\frac{x+t}{2}\right)dxdt-\int_0^\infty\int_0^\infty\overline{F}(x)\overline{F}(t)dxdt$$

Also,

$$\xi_k(G)=E\left(h_1^2(y_1,\ldots,y_k)\right)-\{\gamma(G)\}^2,\ \ \mathrm{k}=1,2$$

is defined analogously.

Since $X_1,X_2,\ldots,X_m$ and $Y_1,Y_2,\ldots Y_n$ are independent (b assumption) (Serfling, 1980).

$$\mathrm{Var(V)}=\mathrm{Var}(\mathrm{V}_1)+\mathrm{Var}(\mathrm{V}_2)$$

$$= \binom{m-1}{2} \sum_{k=1}^{2} \binom{2}{k} \binom{m-2}{2-k} \xi_k(F)$$

$$+ \binom{n-1}{2} \sum_{k=1}^{2} \binom{2}{k} \binom{n-2}{2-k} \xi_k(G) \cdot$$

It follows from Hoeffdings (1948) U - statistics theory that the limiting distribution of $\sqrt{N}[V - \gamma(F,G)]$ is asymptotic normal with mean 0 and variance $\sigma^2$, where N = m + n is the combined sample size, $\eta = \lim_{N\to\infty} \frac{m}{N}$ and

$$= \frac{\sigma_1^2(F)}{\eta} + \frac{\sigma_2^2(G)}{1-\eta} \cdot$$

Under $H_0$, the limiting distribution of $\sqrt{N}V$ is normal with mean 0 and variance

$\frac{\sigma_1^2(F_0)}{\eta(1-\eta)} = \frac{4\xi_1(F_0)}{\eta(1-\eta)}$, where $F_0$ is the unspecified common distribution.

Here it is to be noted that the asymptotic mean of $\sqrt{N}V$ is zero, independent of unspecified common distribution $F_0$. However, the null asymptotic variance $4\xi_1(F_0)/\eta(1-\eta)$ does depend on $F_0$ through $\xi_1(F_0)$ and must be estimated from the data. To estimate $\sigma^2$, one possible way is to obtain consistent estimator for $\sigma^2$.

Since $\xi_1(F) = \mathrm{Var}[h_1(x_1)]$, we let

$$\xi_1^*(F) = \frac{1}{m-1} \sum_{i=1}^{m} \left[\hat{h}_1(x_i) - V_1\right]^2$$

where

$$\hat{h}_1(x_i) = \frac{1}{(m-1)} \sum_{\substack{j=1 \\ j\neq i}}^{m} h(x_i, x_j)$$

In similar fashion, we let

$$\xi_1^*(G) = \frac{1}{(n-1)} \sum_{j=1}^{n} \left[h_1(y_j) - V_2\right]$$

where

$$\hat{h}_1(y_j) = \frac{1}{n-1} \sum_{\substack{i=1 \\ j\neq i}}^{n} h(y_i, y_j)$$

Then it is easily verified that

$$V_1 = \frac{1}{m} \sum_{i=1}^{m} \hat{h}_1(x_i)$$

and

$$V_2 = \frac{1}{n} \sum_{j=1}^{n} \hat{h}_1(y_j).$$

$\xi_1^*(F)$ and $\xi_1^*(G)$ are consistent estimators of $\xi_1(F)$ and $\xi_1(G)$, (Puri and Sen, 1971) and thus a consistent estimator $\hat{\sigma}_N^2$ is obtained by replacing $\xi_1(F)$ and $\xi_1(G)$ by $\xi_1^*(F)$ and $\xi_1^*(G)$, respectively in the expression of $\sigma^2$.
and hence

$$\hat{\sigma}_N^2 = \left(4N\xi_1^*(F)\Big/m\right) + \left(4N\xi_1^*(G)\Big/n\right).$$

By the consistency of $\hat{\sigma}_N^2$ and Slutsky's theorem $\sqrt{N}V\hat{\sigma}_N^{-1}$ is asymptotically -N(0,1) under $H_0$.

The approximate α–level test of $H_0$ versus $H_1$ rejects $H_0$ in favour of $H_1$ if

$$\sqrt{N}V/\left\{\left(4N\xi_1^*(F)\Big/m\right) + \left(4N\xi_1^*(G)\Big/n\right)\right\}^{1/2} > z_\alpha$$

where $z_\alpha$ is the upper α–percentile point of the normal distribution. This ensures the consistency of the two sample IFR test against the class of (F,G) pairs satisfying $\gamma(F) - \gamma(G) > 0$.

**Asymptotic efficacies of the test**

We study the asymptotic performance of V, for three pairs of distribution $(F_{i,\theta}, G)$ by evaluating Pitman effi-

**Table 1.** Pitman Efficacies for Weibull distribution.

| $\theta$ | $\gamma'$ | $\sigma^2$ | Efficacy |
|---|---|---|---|
| 2 | 0.299 | 3.907 | 0.151 |
| 3 | 0.318 | 4.553 | 0.149 |
| 4 | 0.318 | 5.143 | 0.140 |
| 5 | 0.308 | 5.623 | 0.130 |
| 6 | 0.295 | 6.012 | 0.120 |
| 7 | 0.280 | 6.331 | 0.111 |
| 8 | 0.265 | 6.595 | 0.108 |
| 9 | 0.251 | 6.817 | 0.096 |
| 10 | 0.237 | 7.007 | 0.090 |

**Table 2.** Pitman Efficacies for LFR distribution.

| $\theta$ | $\gamma'(F,G)$ | $\sigma^2$ | Efficacy |
|---|---|---|---|
| 2 | 0.326 | 0.0802 | 1.328 |
| 3 | 0.230 | 0.263 | 0.201 |
| 4 | 0.177 | 0.180 | 0.174 |
| 5 | 0.143 | 0.117 | 0.155 |

cacy. Here,we assume that G is an exponential distribution with mean one. The different distributions considered here for $(F_{i,\theta})$ are given below.

1. Weibull Distribution

$$\bar{F}_{1.\theta}(x) = \exp\{-x^{\theta}\}, \theta > 0, x > \theta$$

2. Linear Failure Rate Distribution

$$\bar{F}_{2.\theta}(x) = \exp\left\{-\left(x + \theta\frac{x^2}{2}\right)\right\}, x > 0, \theta \geq 0$$

3. Makeham Distribution

$$\bar{F}_{3.\theta}(x) = \exp\{-[x + \theta(x + e^{-x} - 1)]\}, \theta \geq 0, x \geq 0$$

The Pitman asymptotic efficacy is

$$\mathrm{Eff}(V, F_{i.\theta}, G) = \sigma_0^{-2}(v)\left\{\frac{d}{d\theta}\gamma(F_{i.\theta}, G)\right\}^2_{\theta\to\theta_0}.$$

The Pitman asymptotic efficacies of the proposed test for Wiebull distribution, Linear Failure Rate distribution and Makeham distribution are $\frac{1.2711}{\lambda(1-\lambda)}$, $\frac{0.8438}{\lambda(1-\lambda)}$ and $\frac{1.366}{\lambda(1-\lambda)}$ respectively.

**Table 3.** Pitman Efficacies for Makeham distribution.

| $\theta$ | $\gamma'(F,G)$ | $\sigma^2$ | Efficacy |
|---|---|---|---|
| 1 | 0.0065 | 0.0075 | 0.0056 |
| 2 | 0.0042 | 0.0034 | 0.0052 |
| 3 | 0.0092 | 0.0564 | 0.0150 |
| 4 | 0.0110 | 0.0055 | 0.0220 |
| 5 | 0.0120 | 0.0052 | 0.0280 |
| 6 | 0.0128 | 0.0046 | 0.0355 |
| 7 | 0.0131 | 0.0038 | 0.0401 |
| 8 | 0.0130 | 0.0037 | 0.0451 |
| 9 | 0.0121 | 0.0033 | 0.0450 |
| 10 | 0.0120 | 0.0029 | 0.049 |

The Pitman efficacies of two sample test based on V is determined by specifying a common distribution with parameter $\theta$ in the null hypothesis and by considering sequence of alternatives $(F_{\theta\phi}, F_{\theta})$ where

$\phi = 1 + \frac{a}{\sqrt{N}}, a > 0$ be constant.

The Pitman asymptotic efficacy is

$$Eff(V, F_{i,\theta}, G) = \sigma_0^{-2}(V)\left\{\frac{d}{d\theta}\gamma(F_{i,\theta}, G)\right\}_{\theta\to\theta_0}.$$

The Pitman efficacies of V for different values of $\theta$ for the pair $(F_{1,\theta\phi}, F_{1,\theta})$, $(F_{2,\theta\phi}, F_{2,\theta})$ and $(F_{3,\theta\phi}, F_{3,\theta})$ are presented in Table 1, 2 and 3 respectively.

**Some remarks**

i.) A simple test procedure for testing the null hypothesis that two life time distributions are identical against the alternative that one possesses more IFRA property than another is presented. The test proposed is based on the measure considered by Ahmad (2004) for one sample problem.

ii.) As far as we know, no test has been found in the literature for the problem stated in this article. Hence, only Pitman efficacies are computed for the few alternatives by specifying the common null distribution to be exponential. However, Hollander et al. (1986), Tiwari and Zalkikar (1988) and Lim et al. (2005) consider similar procedures for testing whether the distribution is, respecttively, more new better than used(NBU), increase-ing failure rate average(IFRA) and new better than used of specified age(NBU-$t_0$) than another.

iii.) We have also computed Pitman efficacies for those alternatives with common null distributions being Weibull, Makeham and Linear failure rate distribution.

## REFERENCES

Ahmad IA (2004).A simple and more efficient new approach to life testing. Comm. Statist. Theory and Methods. 33(9): 2199-2215.

Barlow RE, Proschan F (1969). A note on tests of monotone failure rate based on incomplete data. Ann. Math. Statist. 40: 595-600.

Barlow RE, Prosechan F(1981). Statistical Theory of Reliability and life testing. Maryland, Silver Spring.

Bickel PJ (1969). Tests of monotone failure rate II. Ann. Math. Statist. 40:1250-1260.

Bickel PJ, Doksum K (1969). Test for monotone failure rate based on normalized spacings. Ann. Math, Statist. 40:1216-1235.

Hollander RM, Park DH, Proschan F (1986).Testing whether F is “more NBU” than is G.Microelectron. Reliab. 26(1): 39-44.

Lim JH, Kim DK, Park DH (2005). Tests for detecting more NBU-ness at specific age. Australian and Newziland J.Stat. 47(3): 329-337.

Proschan F, Pyke R (1967). Tests for monotone failure rate. In Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability, 3: 293-312.

Puri ML, Sen PK (1971). Nonparametric Methods in Multivariate Analysis. John Wiley, New York.

Randles RH, Wolfe DA (1979). Introduction to the Theory of Nonparametric Statistics. Wiley Series in Probability and Mathematical Statistics, John Wiley and Sons New York.

Serfling RJ (1980). Approximation theorems of Mathematical statistics. Wiley Series in Probability and Mathematical Statistics, John Wiley and Sons New York.