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Short Communication

One kind of construction on sunflower with two petals*

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A sunflower (or Δ-system) with k petals and a core Y is a collection of sets $S_1,\cdots, S_k$ such that $S_i \cap S_j = Y$ for all i≠j; the sets $S_1 \backslash Y,\cdots, S_k \backslash Y$, are petals. In this paper, we first give a sufficient condition for the existence of a sunflower with 2 petals. Let F={A,B,C} be a family of subsets of a set $\{ a_1,\cdots,a_m , b_1,\cdots,b_n , c_1,\cdots,c_n \}$ with $\sum_{i=1}^{m} a_i = \sum_{p=1}^{n} b_p + \sum_{q=1}^{n} c_q$ and $A=\{a_1,\cdots,a_m\}$, $B=\{ b_1,\cdots,b_n \}$ and $C=\{ c_1,\cdots,c_n \}$ are non-increasing lists of nonnegative integers. Suppose that for each r with $1 \le r \le m$, $\sum_{i=1}^{r} a_i \le \sum_{p=1}^{n} min\{b_p, r\} + \sum_{q=1}^{n} min\{c_q, r\}$, then the family F^* contains a sunflower with two petals, where $F^*=\{G_1, G_2\}$, $G_1=G[Y\cup X]$ and $G_2=[Z\cup X]$ are the subgraphs induced respectively by Y∪X and Z∪X with $d_{G_1}(v_j) = b_j$ for all $v_j \in Y\cup X$ and $d_{G_2}(v_j) = c_j$ for all $v_j \in Z\cup X$. Moreover, we generalize the consequence to the case of a much more general result.

Key words: Sunflower; family; tripartite graph.

INTRODUCTION

A non-increasing sequence $\pi=(d_1,\cdots,d_n)$ of nonnegative integers is said to be graphic if it is the degree sequence of a graph G on n vertices and G is called a realization of π. Many characterizations of graphic lists are known, of which one of the best explicit characterizations is that by Erdos and Gallai, (1960). There have been several proofs of it, including a short constructive proof in Garg et al. (2011).

A k-partite graph is one whose vertex set can be partitioned into k subsets so that no edge has both ends in any one subset. In particular, 2-partite graph and 3-partite graph are also called bipartite graph and tripartite graph respectively.

Let $A = (a_1, \cdots, a_m)$ and $B = (b_1, \cdots, b_n)$ be two nonincreasing sequences of nonnegative integers. The pair S=(A;B) is said to be bigraphic if there exists a simple bipartite sets $X=\{x_1, \cdots, x_m\}$ and $Y=\{y_1, \cdots, y_n\}$ such that $d_G(x_i)=a_i$ for $1\le i\le m$ and $d_G(y_i)=b_i$ for $1\le i\le n$. In this case, G is referred to as a realization of S. A well-known theorem due to Gale (1957) and Ryser (1957) independently gives a characterization of S that is bigraphic. In Gale (1957), Tripathi et al. (2010) generalized that theorem and provided a good characterization that is bigraphic on lists of intervals.

A sunflower (or Δ-system) with k petals and a core Y is a collection of sets $S_1,\cdots,S_k$ such that $S_i \cap S_j=Y$ for all i≠j; the sets $S_1\backslash Y ,\cdots, S_k\backslash Y$ are k petals and we require that none of them is empty. About sunflower, Erdos and Rado

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discovered the so-called Sunflower Lemma (Erdos and Rado, 1960). In this paper, we present a special tripartite graph, which reduces to a sunflower with two petals and a core. Prior to our work, only the degree sequences of bipartite graph were characterized.

A set system or a family of F is a collection of sets. Because of their intimate conceptual relation to graphs, a set system is often called a hypergraph. A family is k-uniform if all its members are k-element sets. Thus graphs are k-uniform families with k=2.

THEOREM 1

Let F={A,B,C}be a family of subsets of a set { $a_1,\cdots,a_m$, $b_1,\cdots,b_n$, $c_1,\cdots,c_n$ } with $\sum_{i=1}^{m} a_i = \sum_{p=1}^{n} b_p + \sum_{q=1}^{n} c_q$ and A={$a_1,\cdots,a_m$}, B={$b_1,\cdots,b_n$} and C={$c_1,\cdots,c_n$} are non-increasing lists of nonnegative integers. Suppose that for each r with $1 \le r \le m$,

$$\sum_{i=1}^{r} a_i \le \sum_{p=1}^{n} min\{b_p, r\} + \sum_{q=1}^{n} min\{c_q, r\}, \quad (1)$$

then the family F^* contains a sunflower with two petals, where F^*={ G_1 ,G_2}, G_1=G[Y∪X] and G_2=[Z∪X] are the subgraphs induced respectively by Y∪X and Z∪X with $d_{G_1}(v_j) = b_j$ for all v_j ∈Y∪X and $d_{G_2}(v_j) = b_j$ for all v_j∈Z∪X.

The consequence of Theorem 1 can be generalized to the case of a much more general result.

THEOREM 2

Theorem 2 can be proved by induction on n(n≥3) and the proof technique closely follows that of Theorem 1. So a detailed proof will not be given here.

Let F={A,B_1, B_2,…,B_n} be a family of subsets with $\sum_{i=1}^{m} a_i = \sum_{i=1}^{k_1} b_i^1 + \cdots + \sum_{i=1}^{k_n} b_i^n$ and A={$a_1,\cdots,a_m$}, B_1={ $b_1^1,\cdots, b_{k_1}^1$},…, B_{n-1}={$b_1^{n-1},\cdots, b_{k_{n-1}}^{n-1}$} and B_n={$b_1^n,\cdots, b_{k_n}^n$ } are non-increasing lists of nonnegative integers. Suppose that for each r with $1 \le r \le m$,

$$\sum_{i=1}^{r} a_i \le \sum_{i=1}^{k_1} \min\{b_i^1, r\} + \cdots + \sum_{i=1}^{k_n} \min\{b_i^n, r\}, \quad (2)$$

then the family F^* contains a sunflower with two petals, where F^*={G_1 ,G_2,…,G_n}, G_1=G[Y_1∪X] ,…,G_n=[Y_n∪X] are the subgraphs induced by Y_1∪X ,…,Y_n∪X , respectively, with $d_{G_i}(v_j)= b_j^i$ for all v_j ∈Y_i∪X, i=1,…,n.

Proof of Theorem 1

For convenience, let X={$x_1,\cdots,x_m$} ,Y={ $y_1,\cdots,y_n$ } and Z={ $z_1,\cdots,z_n$ } be three sets of vertices. We shall construct a special tripartite graph G, which will yield the desired sunflower. In fact, G is a realization of degree sequence π=A∪B∪C with vertex-set X∪Y∪Z, i.e., $d(x_i)=a_i$ for 1≤i≤m, $d(y_j)=b_j$ for 1≤j≤n and $d(z_j)=c_j$ for 1+n≤j≤2n. For convenience, we write Y∪Z=W and maintain that X and W are independent sets. We first construct a graph G' with partite sets X, Y and Z satisfying $d(x_i)=a_i$ for 1≤i≤m, $d(v_j)\le b_j$ for 1≤j≤n and $d(v_j)\le c_j$ for 1+n≤j≤2n. Define the critical index to be the largest index r such that $d(x_i)=a_i$ for 1≤i<r and $d(x_r)<a_r$. We will iteratively remove the deficiency a_r- $d(x_r)$ at vertex x_r , while maintaining $d(x_i)=a_i$ for 1≤i<r, $d(v_j)\le b_j$ for 1≤j≤n and $d(v_j)\le c_j$ for 1+n≤j≤2n. Let S={$x_{r+1},\cdots,x_m$}. Note that there exists a vertex v∈[N(x_i)\ N(x_r)]∩W for 1≤i<r, since $d(x_i)=a_i\ge a_r> d(x_r)$. To prove the theorem we have to consider two cases depending on the degree of v_j∈Y∪Z and its neighbourhood's intersection with X.

Case 1: Suppose, for some j, $v_j \leftrightarrow x_k$ for some k>r and $v_j \notin N(x_l)$for some l≤r. If l=r, replace v_jx_k with v_jx_r. If l<r, replace vx_l、v_jx_k with vx_r、v_jx_l .

Case 2: Suppose, for some j, $d(v_j)<b_j$ or $d(v_j)<c_j$ and $v_j \notin N(x_l)$for some l≤r. If l=r, add the edge v_jx_r. If l<r, replace vx_l with vx_r、v_jx_l .

If none of the cases above arise, an application of (1) gives:

$$\sum_{i=1}^{r-1} a_i + d(x_r) = \sum_{i=1}^{r} d(x_i)$$
$$= \sum_{j=1}^{n} min\{d(v_j), r\} + \sum_{j=n+1}^{2n} min\{d(v_j), r\}$$
$$= \sum_{p=1}^{n} min\{b_p, r\} + \sum_{q=1}^{n} min\{c_q, r\} \ge \sum_{i=1}^{r} a_i .$$

Hence $d(x_r)\ge a_r$. Furthermore, $d(x_r)\le a_r$ and thus $d(x_r)=a_r$. Increasing r by 1 and applying the similar steps leads to the required graph G' with partite sets X and W(that is, Y and Z) satisfying $d(x_i)=a_i$ for 1≤i≤m, $d(y_p)\le b_p$ for 1≤p≤n and $d(z_q)\le c_q$ for 1+n≤q≤2n. On the other hand, since W is an independent set and $\sum_{p=1}^{n} d(y_p) + \sum_{q=1}^{n} d(z_q) = \sum_{p=1}^{n} b_p + \sum_{q=1}^{n} c_q,$ we have $\sum_{p=1}^{n} d(y_p) = \sum_{p=1}^{n} b_p , \sum_{q=1}^{n} d(z_q) = \sum_{q=1}^{n} c_q.$ That is, $d(y_p)=b_p$ for 1≤p≤n and $d(z_q)=c_q$ for 1≤q≤n. Hence we construct a tripartite graph G satisfying $d(x_i)=a_i$ for 1≤i≤m, $d(y_p)=b_p$ for 1≤p≤n and $d(z_q)=c_q$ for 1+n≤q≤2n. Now let F*=G, G_1=G[Y∪X] and G_2=[Z∪X] be the subgraphs induced by $V_1 = Y \cup X$ and $V_2 = Z \cup X$ in G, respectively, then G_1 and G_2 form a sunflower with two petals G- V_1=Y and G- V_1=Z and a core X.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

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