One kind of construction on sunflower with two petals

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A sunflower (or ∆-system) with k petals and a core Y is a collection of sets S₁,..., Sₖ such that Sᵢ∩Sⱼ=Y for all i≠j; the sets S₁,Y,..., Sₖ\Y are petals. In this paper, we first give a sufficient condition for the existence of a sunflower with 2 petals. Let F={A,B,C} be a family of subsets of a set { a₁,...,aₘ , b₁,...,bₙ , c₁,...,cₙ } with ∑ᵢ₌₁ aᵢ = ∑ₚ₌₁ bₚ + ∑ₜ₌₁ cₜ and A={a₁,...,aₘ}, B={b₁,...,bₙ } and C={c₁,...,cₙ } are non-increasing lists of nonnegative integers. Suppose that for each r with 1 ≤ r ≤ m, ∑ᵢ₌₁ aᵢ ≤ ∑ₚ₌₁ min{bₚ,r} + ∑ₜ₌₁ min{cₜ,r}, then the family F* contains a sunflower with two petals, where F*={G₁, G₂}, G₁=G[Y∪X] and G₂=[ Z∪X] are the subgraphs induced respectively by Y∪X and Z∪X with d₁(vᵢ)=bᵢ for all vᵢ ∈Y∪X and d₂(vᵢ)=cᵢ for all vᵢ ∈Z∪X. Moreover, we generalize the consequence to the case of a much more general result.

Key words: Sunflower; family; tripartite graph.

INTRODUCTION

A non-increasing sequence π=(d₁,...,dₙ) of nonnegative integers is said to be graphic if it is the degree sequence of a graph G on n vertices and G is called a realization of π. Many characterizations of graphic lists are known, of which one of the best explicit characterizations is that by Erdos and Gallai (1960). There have been several proofs of it, including a short constructive proof in Garg et al. (2011).

A k-partite graph is one whose vertex set can be partitioned into k subsets so that no edge has both ends in any one subset. In particular, 2-partite graph and 3-partite graph are also called bipartite graph and tripartite graph respectively.

Let A = (a₁,...,aₘ) and B = (b₁,...,bₙ) be two nonincreasing sequences of nonnegative integers. The pair S=(A,B) is said to be bigraphic if there exists a simple bipartite sets X={x₁,...,xₘ} and Y={y₁,...,yₙ} such that d₁(xᵢ)=aᵢ for 1≤i≤m and d₁(yᵢ)=bᵢ for 1≤i≤n. In this case, G is referred to as a realization of S. A well-known theorem due to Gale (1957) and Ryser (1957) independently gives a characterization of S that is bigraphic. In Gale (1957), Tripathi et al. (2010) generalized that theorem and provided a good characterization that is bigraphic on lists of intervals.

A sunflower (or ∆-system) with k petals and a core Y is a collection of sets S₁,...,Sₖ such that Sᵢ∩Sⱼ=Y for all i≠j; the sets S₁,Y,..., Sₖ\Y are k petals and we require that none of them is empty. About sunflower, Erdos and Rado...
discovered the so-called Sunflower Lemma (Erdos and Rado, 1960). In this paper, we present a special tripartite graph, which reduces to a sunflower with two petals and a core. Prior to our work, only the degree sequences of bipartite graph were characterized.

A set system or a family of F is a collection of sets. Because of their intimate conceptual relation to graphs, a set system is often called a hypergraph. A family is k-uniform if all its members are k-element sets. Thus graphs are k-uniform families with k=2.

**THEOREM 1**

Let F={A,B,C}be a family of subsets of a set {a₁,…,aₘ , b₁,⋯,bₙ , c₁,⋯,cₙ } with ∑ᵢ₌₁^ₘ aᵢ + ∑ᵢ₌₁^ₙ bᵢ + ∑ᵢ₌₁^ₙ cᵢ and A={a₁,…,aₘ}, B={b₁,⋯,bₙ} and C={c₁,…,cₙ} are non-increasing lists of nonnegative integers. Suppose that for each r with 1 ≤ r ≤ m,

\[ \sum_{i=1}^{r} a_i \leq \sum_{p=1}^{q} \min\{b_p, r\} + \sum_{q=1}^{\min\{c, r\}}, \]

(1)

then the family F’ contains a sunflower with two petals, where F’={G₁, G₂, G₃} G₁={YUX} and G₂={[ ZUX]} are the subgraphs induced respectively by YUX and ZUX with d_{G₁}(vᵢ) = bᵢ for all vᵢ ∈ YUX and d_{G₂}(vᵢ) = bᵢ for all vᵢ ∈ ZUX.

The consequence of Theorem 1 can be generalized to the case of a much more general result.

**THEOREM 2**

Theorem 2 can be proved by induction on n(n≥3) and the proof technique closely follows that of Theorem 1. So a detailed proof will not be given here.

Let F={A,B₁,…,Bₖ} be a family of subsets with ∑ᵢ₌₁^ₖ aᵢ = ∑ᵢ₌₁^ₖ bᵢ + ⋯ + ∑ᵢ₌₁^ₖ cᵢ and A={a₁,…,aₘ}, B₁={b₁,…,bₖ₁},…, Bₖ={bₖₙ₋₁,…,bₖₙ} are non-increasing lists of nonnegative integers. Suppose that for each r with 1 ≤ r ≤ m,

\[ \sum_{i=1}^{r} a_i \leq \sum_{i=1}^{\min\{b_i, r\}} + \sum_{i=1}^{\min\{b_i, r\}} \]

(2)

then the family F’ contains a sunflower with two petals, where F’={G₁, G₂,…,Gₙ}, G₁={Y₁UX} ,…,Gₙ={YₙUX} are the subgraphs induced by Y₁UX ,…,YₙUX respectively, with d_{G₁}(vᵢ) = bᵢ for all vᵢ ∈ Y₁UX, i=₁,…,n.

**Proof of Theorem 1**

For convenience, let X={x₁,…,xᵢ} , Y={y₁,…,yₙ} and Z={z₁,…,zₙ} be three sets of vertices. We shall construct a special tripartite graph G, which will yield the desired sunflower. In fact, G is a realization of degree sequence π=AUBC with vertex-set XUYUZ, i.e., d(xᵢ)=aᵢ for 1≤i≤m, d(yᵢ)=bᵢ for 1≤i≤n and d(zᵢ)=cᵢ for 1≤i≤n. For convenience, we write YUZ=W and maintain that X and W are independent sets. We first construct a graph G’ with partite sets X, Y and Z satisfying d(xᵢ)=aᵢ for 1≤i≤m, d(yᵢ)=bᵢ for 1≤i≤n and d(zᵢ)=cᵢ for 1≤i≤n. Define the critical index to be the largest index r such that d(xᵢ)=aᵢ for 1≤i≤r and d(xᵢ)<aᵢ. We will iteratively remove the deficiency aᵢ−d(xᵢ) at vertex xᵢ, while maintaining d(xᵢ)=aᵢ for 1≤i≤r, d(yᵢ)=bᵢ for 1≤i≤n and d(zᵢ)=cᵢ for 1≤i≤n. Let S={xᵢ,…,xᵢ}. Note that there exists a vertex v∈N(xᵢ)∩N(yᵢ)∩N(zᵢ) for 1≤i≤r, since d(xᵢ)=aᵢ,aᵢ> d(yᵢ). To prove the theorem we have to consider two cases depending on the degree of vᵢ∈YUZ and its neighbourhood’s intersection with X.

Case 1: Suppose, for some j, vᵢ↔xᵢ for some k=r and vᵢ∈N(xᵢ) for some i≤r. If i=r, replace vᵢxᵢ with vᵢxᵢ. If i<r, replace vᵢxᵢ, vᵢxᵢ with vᵢxᵢ, vᵢxᵢ.

Case 2: Suppose, for some j, d(vᵢ)<bᵢ or d(vᵢ)<cᵢ and vᵢ∉N(xᵢ) for some i≤r. If i=r, add the edge vᵢxᵢ. If i<r, replace vᵢxᵢ with vᵢxᵢ, vᵢxᵢ.

If none of the cases above arise, an application of (1) gives:

\[ \sum_{i=1}^{r-1} a_i + d(xᵢ) = \sum_{i=1}^{r} d(xᵢ) = \sum_{j=1}^{n} \min\{d(vᵢ), r\} + \sum_{j=1}^{n} \min\{d(vᵢ), r\} \]

\[ \sum_{p=1}^{n} \min\{b_p, r\} + \sum_{q=1}^{n} \min\{c_q, r\} \geq \sum_{i=1}^{r} a_i. \]

Hence d(xᵢ)≥aᵢ. Furthermore, d(xᵢ)≥aᵢ and thus d(xᵢ)=aᵢ. Increasing r by 1 and applying the similar steps leads to the required graph G’ with partite sets X and W(that is, Y and Z) satisfying d(xᵢ)=aᵢ for 1≤i≤m, d(yᵢ)=bᵢ for 1≤i≤n and d(zᵢ)=cᵢ for 1≤i≤n. On the other hand, since W is an independent set and \[ \sum_{p=1}^{n} d(yᵢ) = \sum_{p=1}^{n} b_p + \sum_{q=1}^{n} c_q, \] we have \[ \sum_{p=1}^{n} d(yᵢ) = \sum_{p=1}^{n} b_p + \sum_{q=1}^{n} c_q = \sum_{q=1}^{n} c_q. \] That is, d(yᵢ)=b_p for 1≤p≤n and d(zᵢ)=cᵢ for 1≤i≤n. Hence we construct a tripartite graph G satisfying d(xᵢ)=aᵢ for 1≤i≤m, d(yᵢ)=b_p for 1≤p≤n and d(zᵢ)=cᵢ for 1≤i≤n. Now let F’={G₁, G₂,…,Gₙ}, G₁={Y₁UX} and G₂={ZUX} be the subgraphs
induced by \( V_1 = Y \cup X \) and \( V_2 = Z \cup X \) in \( G \), respectively, then \( G_1 \) and \( G_2 \) form a sunflower with two petals \( G-V_1 = Y \) and \( G-V_1 = Z \) and a core \( X \).

**CONFLICT OF INTERESTS**

The authors have not declared any conflict of interests.

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