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Full Length Research Paper

The heat equation and the dynamics of labor and capital migration prior and after economic integration

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In the article, the dynamic approach was implemented to the task of Pareto allocation of the factors for a region after economic integration. The regional flows of labor and value added of sectors were found to have opposite directions to each other, with their dynamics obeying the heat equation, while the dynamics of the capital to the equation of attenuated oscillations. In case of economic integration the spatial dynamics of the flows of labor and the value added sectors of one region was found to be influenced by the aggregate allocation of labor and value added of sectors of all the integrated states. The non-homogeneous heat equation for the dynamics of the factors was solved using economic origin of the aggregate wages and value added providing expressions for the factors' dynamics capable to be used in practical computations.

Key words: Heat equation, dynamics of labour, capital migration, economic integration.

INTRODUCTION

Modeling of international economic integration has important task of defining the final allocation of capital and labor flows serving as tools in equalizing a level of economic development in integrated area. Due to economic integration, the factors strive towards maximization of their respective returns such as soundness of wages and rates of return. Hence, reallocation of the factors after economic integration may be treated as a best one, that is as Pareto-optimal allocation. One of the definitions of Pareto-optimal allocation of the factors is the one if there is no any better allocation.

Often larger "density" of capital (Zhang, 1991.) is located in demographically less developed areas, while labor is more dense in financially less capitalized territories. Economic integration reveals this disparity and stimulates flows of respective factors: usually labor and capital move in opposite directions (Figure 1). Historically such flows were observed after the fall of Berlin wall, and that is usually the case when there is economic integration (Puu, 1997) (Formation of NAFTA led to the development of Mexico by mainly the US capital; expansion of the EU in 2004 created the basis for development of Central European new member states by their Western allies in the union).

The dynamics of the value added was not extensively considered in previous research except one of the author (Dalimov, 2008.) At the same time, the dynamics of its components being price and output has been analyzed for instance in (Zhang, 1991.)

In this article we deduct an equation responsible for regional dynamics of the GDP, defining the GDP by adding one more term – the labor's income, to previously introduced (Dalimov, 2008.) definition of the GDP as a difference between the value added of sectors and aggregate investment. An analysis is split between the dynamics of the capital, labor and the values added of sectors prior and after international economic integration.

The model for regional economic development

We start from definition of gross domestic product Y consisting of sectors' products obtained due to investment I of capital, minus aggregate wages $W\overline{L}$:

$$Y = \sum_{i=1}^{n} P_i Q_i - I - W\overline{L} \qquad (1)$$

Where $L = L(x,z,t)$ - quantity of population, x, z - geographic coordinates; $W = 12w(x,z,t)$ is a volume of annual wages; w –monthly wages; n – number of sectors in economy; P_i and Q_i are price and output for the sector *i*.

Migration of labor $\overline{L}$ takes place much more rapidly (Butkovskiy, 1982). Quantities of population *L* and labor $\overline{L}$ are connected by expression: $\overline{L} = L - L_{under\text{-}aged} - L_{retired}$, where the two last terms stand for quantity of under-aged and retired persons. But migration flows of population *L* and labor $\overline{L}$ match, that is, $gradL = grad\overline{L}$, since $gradL_{under\text{-}aged} = gradL_{retired} = 0$) than general demographic changes:

$$\nabla\overline{L} \equiv \frac{\partial\overline{L}}{\partial x} + \frac{\partial\overline{L}}{\partial z} \equiv grad\overline{L} >> \frac{\partial\overline{L}}{\partial t}. \qquad (2)$$

We may also apply the other two conditions for capital and labor both tending to geographic areas where their gradients strive to become zero. If we denote the moment of reaching the final reallocation of the factors by *T*, then we have:

$$\underset{t=T}{grad}\,\overline{L} \to 0; \underset{t=T}{gradK} \to 0, \qquad (3)$$

Where $K = K(x,z,t)$ is a function of capital. Spatial dynamics of the wages will be neutralized at the moment *T*:

$$\underset{t\to T}{grad}\,w \to 0 \qquad (4)$$

The task is formulated as a search of reallocation of the factors under terms (2)-(4). Direction of the flow of capital is determined by the rate of return. To link the GDP with the rate of return which seems to be higher in less developed areas, assume that the GDP is equal to a rent of the capital in the area for considered period as multiplication of an average rate of return *r* per amount of capital:

$$Y = rK. \qquad (5)$$

By inputting expression (5) to (1) we have:

$$rK = \sum_{i=1}^{n} P_i Q_i - I - W\overline{L}. \qquad (6)$$

Hence, we have obtained an expression for investment:

$$I = \sum_{i=1}^{n} P_i Q_i - rK - W\overline{L}. \qquad (7)$$

By using (7) and an expression for temporal derivative of the GDP $\dot{Y} = \frac{dY}{dt}$ [4]:

$$\dot{Y} = I - sY$$

We obtain direct link between different components of the value added:

$$rK(1+s) + \dot{Y} = \sum_i PQ - W\overline{L}. \qquad (8)$$

Assume changes of the GDP as such as close to equilibrium where we may consider derivative of the GDP as $\dot{Y} = 0$, making the last obtained expression as the following:

$$rK(1+s) = \sum_i PQ - W\overline{L}, \qquad (9)$$

Now we will use an expression for the temporal derivative of the GDP:

$$\dot{Y} = I - sY - X_n, \qquad (10)$$

Where net exports $X_n = X - M = \mu\nabla^2 Y$ and $\dot{X}_n = \mu\nabla^2 Y - X_n$, $X; M$ are exports and imports respectively. Differentiation of (10) over time leads to the following:

$$\begin{aligned}\ddot{Y} &= \dot{I} - s\dot{Y} - \dot{X}_n = \\ &= \frac{\partial}{\partial t}\left(\sum_{i=1}^{n} P_i Q_i - W\overline{L} - Y\right) - s\dot{Y} + (\mu\nabla^2 Y - X_n) = \\ &= \frac{\partial}{\partial t}\left(\sum_{i=1}^{n} PQ - W\overline{L}\right) + \sum_{i=1}^{n} P_i Q_i - W\overline{L} - (2+s)\dot{Y} - (1+s)Y + \mu\nabla^2 Y.\end{aligned} \qquad (11)$$

Thus, we obtained differential equation responsible for the regional dynamics of the GDP including the dynamics of the sectors' value added, labor and wages:

$$\ddot{Y} + (2+s)\dot{Y} + (1+s)Y - \mu\nabla^2 Y = \frac{\partial}{\partial t}\left(\sum_{i=1}^{n} PQ - W\overline{L}\right) + \sum_{i=1}^{n} P_i Q_i - W\overline{L}. \qquad (12)$$

Now we will attempt to replace the GDP in (12) by capital using (5).

a) First let's estimate temporal derivatives of the GDP:

$$\dot{Y} = \dot{r}K + r\dot{K}, \ddot{Y} = \ddot{r}K + 2\dot{r}\dot{K} + r\ddot{K}. \qquad (13)$$

Economically speaking, average rate of return changes in time less than the capital: $\dot r << \dot K$, it leads to the conclusion that:

$$\ddot r << \ddot K, \tag{14}$$

Hence, the scale of temporal derivatives for the GDP is mainly defined by temporal derivatives of the capital:

$$\dot Y \approx r\dot K; \; \ddot Y \approx r\ddot K. \tag{15}$$

b) In estimating the spatial derivative of the GDP one may see that spatial derivatives of rent and capital seem to be the magnitudes of the same order:

$$K\nabla r \approx r\nabla K. \tag{16}$$

This follows from intuitive economic representation on relatively simultaneous striving of respective gradients to be equalized in integrated area:

$$\frac{\nabla r}{r} \approx \frac{\nabla K}{K}. \tag{17}$$

To discuss spatial allocation of labor instead of considering rate of return, let's express the latter from (9) in the form:

$$r = \frac{\sum_{i=1}^{n} P_i Q_i - W\bar L}{K(1+s)}. \tag{18}$$

Substitution of (5) under the sign of laplasian $\Delta \equiv \nabla^2$ with the use of (18) provides the following:

$$\Delta Y = \Delta(rK) = \frac{\Delta\sum_{i=1}^{n} P_i Q_i - \Delta W\bar L}{1+s}. \tag{19}$$

Replacement of the results obtained in (13)-(19) into (12) leads to new equation without explicit presence of the GDP:

$$r(\ddot K + (2+s)\dot K + (1+s)K) \approx \mu\frac{\Delta\sum_{i=1}^{n} P_i Q_i - \Delta W\bar L}{1+s} + \frac{\partial}{\partial t}\Big(\sum_{i} P_i Q_i - W\bar L\Big) + \sum_{i=1}^{n} P_i Q_i - W\bar L \tag{20}$$

Let's divide both sides of (20) on r, then replace its value from (18) to the given expression and separately group the terms having capital and labor. It leads to the following:

$$\frac{\ddot K + (2+s)\dot K + (1+s)K}{K} = (1+s)\left(1 + \frac{\frac{\partial}{\partial t}\Big(\sum_{i=1}^{n} P_i Q_i - W\bar L\Big) + \mu\frac{\Delta\sum_{i=1}^{n} P_i Q_i - \Delta W\bar L}{1+s}}{\sum_{i=1}^{n} P_i Q_i - W\bar L}\right) = \lambda \tag{21}$$

Variables $\bar L$ and K look formally independent from each other, allowing equalizing both sides to constant λ. This gives us two separate equations for the dynamics of capital, on the one hand, and value added of sectors, labor and wages on the other hand:

$$\begin{cases} \ddot K + (2+s)\dot K + (1+s-\lambda)K = 0 & (22) \\ \frac{\partial}{\partial t}\Big(\sum_{i=1}^{n} P_i Q_i - W\bar L\Big)(1+s) = (\lambda-1-s)\Big(\sum_{i=1}^{n} P_i Q_i - W\bar L\Big) - \mu\Delta\Big(\sum_{i=1}^{n} P_i Q_i - W\bar L\Big) & (23) \end{cases}$$

First one is solved in the following way:

$$K = e^{(-1-0.5s)t}(K_1\cos\omega_0 t + K_2\sin\omega_0 t) \tag{24}$$

Where natural frequency of the oscillations of the capital is equal to $\omega_0^2 = 1 + s - \lambda$; *A* and *B* are constants. Thus, capital makes temporarily attenuating oscillations, with the frequency depending from migration of labor as well. In other words, oscillations of the capital under certain values of s, λ may become periodic.

Equation (23) is simplified if we introduce the following identity:

$$u = \sum_{i=1}^{n} P_i Q_i - W\bar L. \tag{25}$$

By denoting $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \equiv u_{xx} + u_{zz}; \frac{\partial u}{\partial t} = u_t$ and

regrouping the terms we obtain the following:

$$\boxed{u_t = \alpha(u_{xx} + u_{zz}) + \beta u} \tag{26}$$

Where $\alpha = -\frac{\mu}{1+s}; \beta = \frac{\lambda}{1+s} - 1.$

Remarkably, equation (26) is identical to the heat equation. In other words, the dynamics of both labor and value added as components of the function u within a region is similar to the dynamics of a heat or gas diffusing in some space.

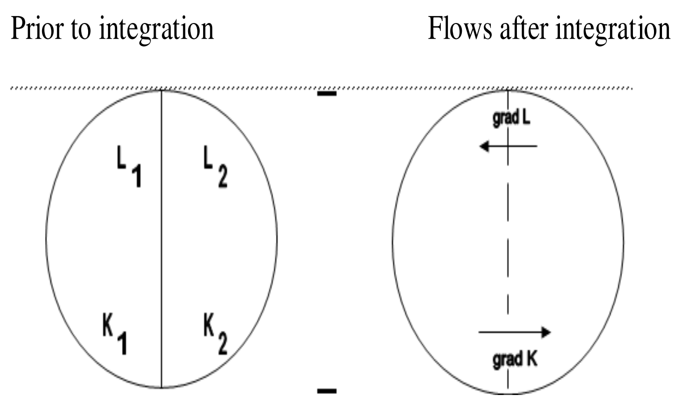

Figure 1. Labor and capital allocation.
Note: L_1, L_2 and K_1 K_2 are respective allocations of labor and capital ion two previously separated economic areas.

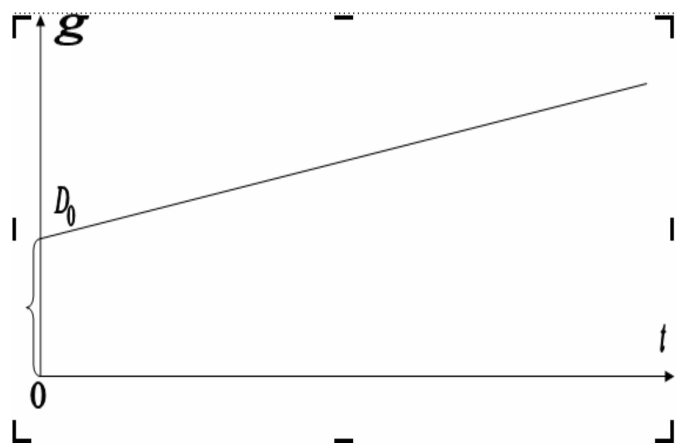

Figure 2.Wages temporal dynamics line.

Directions of labor and the value added sectors are included in the function u at expression (25) with opposite signs, which means that the flows of labor and value added are opposite to each other. Economically, it is then the exact dynamics discussion started with (Figure 1). The other conclusion following from (25) and (26) is that it is the value added and not the capital, which moves against the flow of labor after economic integration. That means that entities which produce a value added move to less developed regions to set-up businesses.

Equation (26) is among most studied in physics (Butkovskiy, 1982; Carslaw and Jaeger, 1984; Ionkin and Morozova, 2000; Schachinger and Schnizer 1980; Tikhonov and Samarskii, 1990) and needs specific terms (proper boundary or temporary conditions for the function u) to set in case of computations for the migration of labor, capital and sectors' value added allocation within a region.

Solution of the heat equation for regional dynamics of economic factors

One may try analytically to solve equation (26) directly inputting, expressions for labor and the value added sectors to (26) deriving from their economic origin.
Dynamically aggregate wages may be represented as a straight line (Figure 2):

$$W\overline{L} \equiv g(x;z;t) = Ct + D_0, \tag{27}$$

Where C is growth rate of the cost of labor, and D_0 is given by initial condition $g(x;z;t=0) = D_0$.

Expression (27) is based on the idea that the wages increase in time due to various factors, mainly due to rise of the level of welfare and inflation of a national currency. It is assumed that C, D are temporarily constant, but spatially it is not exactly so: dependence $C = C(x,z)$ reflects the fact that wages usually differ within a region. For instance, wages in urban areas quite differ from the ones in rural areas, with wages in metropolitan areas being much higher due to higher concentration of businesses and their activities compared to other less developed and populated areas.

Dynamics of the value added of sectors may be represented as periodic function (Ionkin and Morozova, 2000).

$$\sum_{k=1}^{n} P_k Q_k = \sum_{k=1}^{n} P_{0k} Q_{0k} \sin^2 \omega_k t = \sum_{k=1}^{n} B_k (1-\cos 2\omega_k t), \tag{28}$$

Where n is a number of sectors; $P_{0k};Q_{0k}$ - initial price and output in sector k ; $B_k(x,z) \equiv \dfrac{P_{0k}Q_{0k}}{2}$. Substitution of (27)-(28) to (25) provides the following:

$$u = \sum_{k=1}^{n} B_k (1-\cos 2\omega_k t) - Ct - D_0 \tag{29}$$

Using (29), one may find respective derivatives of the function u present in equation (26):

$$\begin{cases} u_t \equiv \dfrac{\partial u}{\partial t} = 2\sum_{k=1}^{n} B_k \omega_k \sin 2\omega_k t - C \quad ; & (30) \\ u_{xx} + u_{zz} \equiv \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial z^2}\right)u \equiv \Delta u = \sum_{k=1}^{n} (1-\cos 2\omega_k t)\Delta B_k - t\Delta C & (31). \end{cases}$$

By placing (29)-(31) into (26) we obtain:

$$2\sum_{k=1}^{n} B_k \omega_k \sin 2\omega_k t - C = \alpha\left[\sum_{k=1}^{n}(1-\cos 2\omega_k t)\Delta B_k - t\Delta C\right] + \beta\left[\sum_{k=1}^{n} B_k (1-\cos 2\omega_k t) - Ct - D_0\right] \tag{32}$$

The regrouping shows that we have different time scale of the variables:

$$2\sum_{k=1}^{n} B_k \omega_k \sin 2\omega_k t - C = \sum_{k=1}^{n}(1-\cos 2\omega_k t)\left[\alpha\Delta B_k + \beta B_k\right] - t\left[\alpha\Delta C + \beta C\right] - \beta D_0 \ . \tag{33}$$

By dividing both parts on time t and taking the limit under substantially large t we may neglect quickly diminishing terms and obtain an approximate equation $\alpha\Delta C + \beta C \approx 0$, or:

$$\Delta C + \omega_{0C}^2 C \approx 0 \tag{34}$$

Where a natural frequency of the growth rate of labor's income is equal to:

$$\omega_{0C}^2 \equiv \frac{\beta}{\alpha} \tag{35}$$

Solution of equation (34) may be selected in the following form:

$$C = C_l e^{i\frac{\omega_{0C}(x+z)}{\sqrt{2}}} \tag{36}$$

Where C_l is constant. By placing expression (36) into (33) and making necessary simplifications we obtain:

$$2\sum_{k=1}^{n} B_k \omega_k \sin 2\omega_k t - \sum_{k=1}^{n}(1-\cos 2\omega_k t)\left[\alpha\Delta B_k + \beta B_k\right] = C - \beta D_0 \ . \tag{37}$$

This expression represents an equation with the only variable B_k which becomes clear if we rewrite (38) in the following form:

$$\sum_{k=1}^{n}\left[\Delta B_k + \frac{N_k}{M_k} B_k\right] M_k = C - \beta D_0 \tag{38}$$

Where

$$M_k \equiv \alpha(\cos 2\omega_k t - 1); N_k \equiv 2\omega_k \sin 2\omega_k t + \beta(\cos 2\omega_k t - 1)$$

To solve equation (38) we will use a method of variation of constants by considering first its homogeneous case :

$$\sum_{k=1}^{n}\left[\Delta B_k + \frac{N_k}{M_k} B_k\right] M_k = 0 \tag{39}$$

One may see that equation (39) is true in case if the following is valid:

$$\Delta B_k + \omega_{0B}^2 B_k = 0 \tag{40}$$

Where $\omega_{0B}^2 \equiv \dfrac{N_k}{M_k}$ - the natural frequency of spatial oscillations of the value added. Explicit calculation of ω_{0B}^2 provides the following:

$$\omega_{0B}^2 \equiv \frac{N_k}{M_k} = \frac{2\omega_k \sin 2\omega_k t + \beta(\cos 2\omega_k t - 1)}{\alpha(\cos 2\omega_k t - 1)} = \frac{2\omega_k \sin 2\omega_k t}{\alpha(\cos 2\omega_k t - 1)} + \frac{\beta}{\alpha} \ . \tag{41}$$

By using (35) one may see that the frequencies in the spatial dynamics of labor ω_{0C} and the value added of sectors ω_{0B} are linked together with the frequency of temporal oscillations of the value added sectors ω_k :

$$\omega_{0B}^2 = \frac{2\omega_k \sin 2\omega_k t}{\alpha(\cos 2\omega_k t - 1)} + \omega_{0C}^2 \tag{42}$$

Notably, the frequency ω_{0B} is not constant since it depends on time. Equation (40) has the same type of homogeneous solution as we found for (34):

$$B_k = B_{kq} e^{i\frac{\omega_{0B}(x+z)}{\sqrt{2}}} \tag{43}$$

Implementing the method of variation of constants, the essence of which is letting the constant B_{kq} from homogeneous solution to depend from arguments $B_{kq} = B_{kq}(x,z)$; then substituting it to homogeneous solution (43) and

$$B_k = B_{kq}(x,z) e^{i\frac{\omega_{0B}(x+z)}{\sqrt{2}}} \tag{44}$$

further replacing it into the original equation (38). The sequence of taking respective derivatives produces the following:

$$\frac{\partial B_k}{\partial x} = e^{i\frac{\omega_{0B}(x+z)}{\sqrt{2}}}\left[\frac{\partial B_{kq}}{\partial x} + i\frac{\omega_{0B} B_{kq}}{\sqrt{2}}\right] \ ;$$

$$\frac{\partial^2 B_k}{\partial x^2}=e^{i\frac{\omega_{0B}(x+z)}{\sqrt{2}}}\left[\frac{\partial^2 B_{kq}}{\partial x^2}+i\omega_{0B}\sqrt{2}\frac{\partial B_{kq}}{\partial x}-\frac{\omega_{0B}^2 B_{kq}}{2}\right] \quad (45)$$

Substitution of the derivatives to expression (40) leads to the following simplification:

$$\Delta B_k+\omega_{0B}^2 B_k=\nabla(\nabla B_{kq}+i\omega_{0B}^2\sqrt{2}B_{kq}) . \quad (46)$$

The form of the right part of (46) shows that we have to represent the right part of (38) as a sequence with the same index k using for instance, an expression (42):

$$\omega_{0C}=\pm\sqrt{\omega_{0B}^2-\frac{2\omega_k\sin 2\omega_k t}{\alpha(\cos 2\omega_k t-1)}} \quad (47)$$

Substitution of (42) to equation (34) and expression (36) leads to the form of the magnitude C we were seeking:

$$C=\sum_{k=1}^{n}C_k\exp\left[\pm i\sqrt{\frac{\omega_{0B}^2-\dfrac{2\omega_k\sin 2\omega_k t}{\alpha(\cos 2\omega_k t-1)}}{2}}(x+z)\right]\equiv\sum_{k=1}^{n}C_k\exp[\pm i\Omega_k(x+z)], \quad (48)$$

Where $\Omega_k\equiv\pm\sqrt{\frac{\omega_{0B}^2-\dfrac{2\omega_k\sin 2\omega_k t}{\alpha(\cos 2\omega_k t-1)}}{2}}$. Economic reasoning says that we may use only positive frequency

$$\Omega_k\equiv\sqrt{\frac{\omega_{0B}^2-\dfrac{2\omega_k\sin 2\omega_k t}{\alpha(\cos 2\omega_k t-1)}}{2}}$$

since we do not consider changes with negative coordinates $(x;z)$ indicating in practice geographic location. Finally, we replace respective terms in the equation (38) by (46) and (48):

$$\nabla\sum_{k=1}^{n}\left[\nabla B_{kq}+i\omega_{0B}\sqrt{2}B_{kq}\right]M_k=\sum_{k=1}^{n}C_k\exp[i\Omega_k(x+z)]-\beta D_0 . \quad (49)$$

To take off a sign of the first gradient $\nabla\equiv\frac{\partial}{\partial x}+\frac{\partial}{\partial z}$, we have to take an integral from both parts of the equality (49):

$$\iint\partial x\partial z\nabla\sum_{k=1}^{n}\left[\nabla B_{kq}+i\omega_{0B}\sqrt{2}B_{kq}\right]M_k=\iint\partial x\partial z\left(\sum_{k=1}^{n}C_k e^{i\Omega_k(x+z)}-\beta D_0\right) \quad (50)$$

Integration leads to the following:

$$\sum_{k=1}^{n}\left[\nabla B_{kq}+i\omega_{0B}\sqrt{2}B_{kq}\right]M_k=-\sum_{k=1}^{n}\frac{C_k}{\Omega_k^2}e^{i\Omega_k(x+z)}-\beta D_0 xz+D_1 x+D_2 . \quad (51)$$

Since $\exp(iy)=\cos y+i\sin y$, than one may see that both parts of the equation (51) contain real and imaginary terms which must be equal to each other. This leads to the following two equations:

For real part:

$$\sum_{k=1}^{n}M_k\nabla B_{kq}=-\sum_{k=1}^{n}\frac{C_k}{\Omega_k^2}\cos[\Omega_k(x+z)]-\beta D_0 xz+D_1 x+D_2 . \quad (52)$$

By repeating the integration as it was done in (50)-(51) we obtain:

$$\sum_{k=1}^{n}B_{kq}M_k=-\iint\partial x\partial z\left(\sum_{k=1}^{n}\frac{C_k}{\Omega_k^2}\cos[\Omega_k(x+z)]-\beta D_0 xz+D_1 x+D_2\right). \quad (53)$$

Using the trigonometric equality equality $\cos(x+z)=\cos x\cos z-\sin x\sin z$, we derive the following:

$$\sum_{k=1}^{n}B_{kq}M_k=-\sum_{k=1}^{n}\frac{C_k}{\Omega_k^4}\cos[\Omega_k(x-z)]-\beta D_0\frac{x^2z^2}{4}+D_1\frac{x^2}{2}+D_2 x+D_3; \quad (54)$$

For imaginary part:

$$\sqrt{2}\sum_{k=1}^{n}\omega_{0B}B_{kq}M_k=-\sum_{k=1}^{n}\frac{C_k}{\Omega_k^2}\sin[\Omega_k(x+z)] \quad (55)$$

Consider a case when all the constants of the integration in (54) are equal to nil: $D_0=D_1=D_2=D_3=0$. Besides, terms in left and right with an index k also must be equal to each other. Hence, we have the final system of the equations where the only unknown is B_{kq} :

$$\begin{cases} B_{kq}M_k=-\dfrac{C_k}{\Omega_k^4}\cos[\Omega_k(x-z)]; & (56\\ \sqrt{2}\omega_{0B}B_{kq}M_k=-\dfrac{C_k}{\Omega_k^2}\sin[\Omega_k(x+z)] & (57)\end{cases}$$

That leads to obvious conclusion:

$$B_{kq}=-\frac{C_k}{M_k\Omega_k^4}\cos[\Omega_k(x-z)]=-\frac{C_k}{\omega_{0B}\Omega_k^2 M_k\sqrt{2}}\sin[\Omega_k(x+z)]$$

. (58)

By placing (58) to (44) we finally obtain:

$$B_k=-\frac{C_k}{M_k\Omega_k^4}\cos[\Omega_k(x-z)]e^{i\frac{\omega_{0B}(x+z)}{\sqrt{2}}} \quad (59)$$

Thus, the solution of the heat equation (26) is the following:

$$u=\sum_{k=1}^{n}\frac{C_k}{M_k\Omega_k^4}\cos[\Omega_k(x-z)]e^{i\frac{\omega_{0B}(x+z)}{\sqrt{2}}}[\cos 2\omega_k t-1]-t\sum_{k=1}^{n}C_k e^{i\Omega_k(x+z)}.$$

(60)

It may be rewritten in another form:

$$u(x,z,t)=\sum_{k=1}^{n}C_k\left[\frac{\cos\Omega_k[(x-z)]}{M_k\Omega_k^4}e^{i\frac{\omega_{0B}(x+z)}{\sqrt{2}}}[\cos 2\omega_k t-1]-t\sum_{k=1}^{n}e^{i\Omega_k(x+z)}\right]$$

(61)

Expressions for the regional dynamics of the value added sectors and the labor's income may be obtained by neglecting imaginary parts in (61):

$$\sum_{k=1}^{n}P_kQ_k=\sum_{k=1}^{n}\frac{C_k\left(\cos\left[(\Omega_k-\frac{\omega_{0B}}{2})x-(\Omega_k+\frac{\omega_{0B}}{2})z\right]+\cos\left[(\Omega_k+\frac{\omega_{0B}}{2})x-(\Omega_k-\frac{\omega_{0B}}{2})z\right]\right)}{2M_k\Omega_k^4}[\cos 2\omega_k t-1]$$

(62)

$$W\bar{L}=t\sum_{k=1}^{n}C_k\cos\Omega_k(x+z) \quad (63)$$

Both last expressions contain magnitudes available from statistics, making the use of expressions (62)-(63) in practice possible. One may now check the validity of the condition (3) set fpr stable allocation of labor in region (or the "end" of intra-regional migration).
By placing an expression (63) to (3) we obtain:

$$grad\,W\bar{L}(t=T)=-\Omega_k T\sum_{k=1}^{n}C_k\sin\Omega_k(x+z)=0 \quad (64)$$

This may only be valid when $\Omega_k \equiv \pm\sqrt{\dfrac{\omega_{0B}^2-\dfrac{2\omega_k\sin 2\omega_k t}{\alpha(\cos 2\omega_k t-1)}}{2}}=0$, which is true if $\omega_{0C}=\pm\sqrt{\omega_{0B}^2-\dfrac{2\omega_k\sin 2\omega_k t}{\alpha(\cos 2\omega_k t-1)}}=0$ due to expression (42). By definition, ω_{0C} is the frequency of spatial oscillations of labor, and it is equal to zero when there are no spatial changes of labor (no migration) suggesting that the last condition has an economic sense.

International economic integration

Consider unification of previously separated areas after abolishment of trade restrictions, including specific case of a common market having no barriers for intra-movement of labor and capital. Using (12), we may write the following equation responsible for the dynamics of the GDP for the state l after economic integration of m regions where $l\in[1;m]$:

$$\ddot{Y}_l+(2+s_l)\dot{Y}_l+(1+s_l)Y_l-\mu_l\nabla^2 Y_l=\sum_{k=1}^{m}\sum_{i=1}^{n}(P_{lki}Q_{lki}+\frac{\partial}{\partial t}P_{lk}$$

(65)

and i denotes summation by sectors. One may see that we have added summation over k number of states, with $k\in[1;m]$. Repeating steps similar to (13)-(17), we obtain:

$$r_l(\ddot{K}_l+(2+s_l)\dot{K}_l+(1+s_l)K_l)\approx\mu_l\frac{\Delta(\sum_{i=1}^{n}P_{li}Q_{li}-W_l\bar{L}_l)}{1+s_l}+$$

$$\sum_{k=1}^{m}(\sum_{i=1}^{n}(P_{lki}Q_{lki}+\frac{\partial}{\partial t}P_{lki}Q_{lki})-(W_{lk}\bar{L}_{lk}+\frac{\partial}{\partial t}W_{lk}\bar{L}_{lk})). \quad (66)$$

Let's divide both sides on r_l using (18) and separately group terms having the capital and labor's income:

$$\frac{\ddot{K}_l+(2+s_l)\dot{K}_l+(1+s_l)K_l}{K_l}=(1+s_l)\frac{\sum_{k=1}^{m}\sum_{i=1}^{n}P_{lki}Q_{lki}+\frac{\partial}{\partial t}\sum_{k=1}^{m}\sum_{i=1}^{n}P_{lki}Q_{lki}-\sum_{k=1}^{m}(W_k\bar{L}_k+\frac{\partial}{\partial t}W_k\bar{L}_k)+\mu_l\frac{\Delta(\sum_{i=1}^{n}P_{li}Q_{li}-W_l\bar{L}_l)}{1+s_l}}{\sum_{i=1}^{n}P_{li}Q_{li}-W_l\bar{L}_l}=\lambda_l$$

(67)

Expression (67) shows that the dynamics of the capital has not changed: it still produces attenuated oscillations since it obeys the expressions (22) and (24): one just needs to add an index l for the capital.

At the same time, the dynamics of labor became much more complex:

$$(\sum_{k=1}^{m}(\sum_{i=1}^{n}P_{ki}Q_{ki}-W_k\bar{L}_k)+\frac{\partial}{\partial t}\sum_{k=1}^{m}(\sum_{i=1}^{n}P_{ki}Q_{ki}-W_k\bar{L}_k))(1+s_l)-\lambda_l(\sum_{i=1}^{n}P_{li}Q_{li}-W_l\bar{L}_l)+\mu_l\Delta(\sum_{i=1}^{n}P_{li}Q_{li}-W_l\bar{L}_l)=0 \quad (68)$$

By introducing the variable $u_k=\sum_{i=1}^{n}P_{ki}Q_{ki}-W_k\overline{L}_k$ to (68) we may compactly rewrite as:

$$(\sum_{k=1}^{m}u_k+\frac{\partial}{\partial t}\sum_{k=1}^{m}u_k)(1+s_l)=\lambda_l u_l-\mu_l\Delta u_l. \quad (69)$$

By dividing both sides of equation (69) on $1+s_l$ and by regrouping its terms leads us to the following expression:

$$au_l-b\Delta u_l=\sum_{k=1}^{m}(u_k+\frac{\partial}{\partial t}u_k), \quad (70)$$

Where $a\equiv\frac{\lambda_l}{1+s_l};b\equiv\frac{\mu_l}{1+s_l};a,b=const.$

Expression (70) shows that spatial dynamics of labor and the value added of one specified region (*l*) come under direct influence of aggregate allocation of labor and value added sectors of all the integrated states plus their temporal dynamics.

In search of the final allocation of labor and capital in some specific country one may see that it is influenced by the temporal change of them in all the integrated states and their current allocation, and not by their spatial dynamics. In other words, migration of labor within a region is motivated by existing level of the wages in all the integrated area, including its change over the time.

Solution of the heat equation for regional dynamics of the factors after economic integration

We may attempt analytically to find a function u_l of the region l considering an economic integration of m regions. Using an expression (29) one may rewrite equation (70) in explicit form:

$$a\left[\sum_{q_l=1}^{n_l}B_{q_l}(1-\cos 2\omega_{q_l}t)-C_l t-D_{l0}\right]-b\left[\sum_{q_l=1}^{n_l}(1-\cos 2\omega_{q_l}t)\Delta\right.$$
$$=\sum_{k=1}^{m}\left[\sum_{q_k=1}^{n_k}B_{q_k}(1-\cos 2\omega_{q_k}t)-C_k t-D_{k0}+2\sum_{q_k=1}^{n_k}B_{q_k}\omega_{q_k}\sin 2\omega\right. \quad (71)$$

Here k is indexing the regions, and q_k and q_l are indexing the sectors in respective regions.

We may use the same method as in section 3 of the article by dividing both parts on time t and taking limits under $t\rightarrow\infty$. It provides the following:

$$aC_l+b\Delta C_l=-\sum_{k=1}^{m}C_k. \quad (72)$$

Division of both sides of equation (72) to b leads to the following:

$$\Delta C_l+\omega_{0C_l}^2C_l=-\frac{\sum_{k=1}^{m}C_k}{b}, \quad (73)$$

Where $\omega_{0C_l}^2=\frac{a}{b}=\frac{\lambda_l}{\mu_l}$.

Using expression (48), we rewrite (73) in the following form:

$$\Delta C_l+\omega_{0C_l}^2C_l=-\sum_{k=1}^{m}\sum_{q_k=1}^{n_k}\frac{C_{kq_k}}{b}\exp\left[\pm i\Omega_{kq_k}(x+z)\right] \quad (74)$$

Consider the solution of the equation (74) in the same form as the right part of the equation (74), that is;

$$C_l=\sum_{k=1}^{m}\sum_{q_k=1}^{n_k}C_{lkp_k}\exp\left[\pm i\Omega_{kq_k}(x+z)\right]. \quad (75)$$

Substitution of expression (75) to equation (74) leads to the following:

$$\sum_{k=1}^{m}\sum_{q_k=1}^{n_k}(\omega_{0C_l}^2-\Omega_{kp_k}^2)C_{lkq_k}\exp\left[\pm i\Omega_{kq_k}(x+z)\right]=-\sum_{k=1}^{m}\sum_{q_k=1}^{n_k}\frac{C_{kp_k}}{b}. \quad (76)$$

It allowed us to see that:

$$C_{lkq_k}=\frac{C_{kq_k}}{b(\omega_{0C_l}^2-\Omega_{kq_k}^2)} \quad (77)$$

Hence, solution of equation (74) is the following:

$$C_l=\sum_{k=1}^{m}\sum_{q_k=1}^{n_k}\frac{C_{kq_k}}{b(\omega_{0C_l}^2-\Omega_{kq_k}^2)}\exp\left[\pm i\Omega_{kq_k}(x+z)\right] \quad (78)$$

Since C_l represents the rate of spatial changes of labor of a unified region, one may see that it is affected by values of the rates of spatial allocation of labor of unified states prior to unification. Expression (78) allows us to determine respective dynamics of labor's income:

$$W_l\overline{L}_l = t\sum_{k=1}^{m}\sum_{q_k=1}^{n_k}\frac{C_{kq_k}}{b(\omega_{0C_l}^2-\Omega_{kq_k}^2)}\exp\left[\pm i\Omega_{kq_k}(x+z)\right] \tag{79}$$

Looking back to equation (71), one may see that the only unknown variable left in it is B_{q_l} since now we know all the other parameters. By placing expressions (48) and (78) to the equation (71) and regrouping similar terms we obtain the following:

$$\sum_{q_l=1}^{n_l}\left[(aB_{q_l}-b\Delta B_{q_l})(1-\cos 2\omega_{q_l}t)\right]=$$
$$=\sum_{k=1}^{m}\sum_{q_k=1}^{n_k}\left[B_{q_k}(1-\cos 2\omega_{q_k}t+2\omega_{q_k}\sin 2\omega_{q_k}t)-(1+t\left[1-\frac{a+\Omega_{kq_k}^2}{\omega_{0C_l}^2-\Omega_{kq_k}^2}\right])C_{q_k}\exp\left[\pm i\Omega_{q_k}(x+z)\right]\right] \tag{80}$$

Implementing an expression (59) for B_{q_k} finally leads to the following equation:

$$\sum_{q_l=1}^{n_l}\left[(aB_{q_l}-b\Delta B_{q_l})(1-\cos 2\omega_{q_l}t)\right]=$$
$$=\sum_{k=1}^{m}\sum_{q_k=1}^{n_k}\left[B_{q_k}(1-\cos 2\omega_{q_k}t+2\omega_{q_k}t\sin 2\omega_{q_k}t)-(1+t\left[1-\frac{a+\Omega_{kq_k}^2}{\omega_{0C_l}^2-\Omega_{kq_k}^2}\right])C_{q_k}\exp\left[\pm i\Omega_{q_k}(x+z)\right]\right] \tag{81}$$

Solution of the last equation analytically is not possible due to different indexing of series on the right and left sides of the equation, but one can make several qualitative conclusions. The right part of (81) is represented by magnitudes $B_{q_k}=\frac{P_{0q_k}Q_{0q_k}}{2}$ and respective frequencies ω_{q_k} of oscillations of the value added sectors of the unified states, with both of them available from statistics. We have also temporarily dependent frequency Ω_{kq_k} which may also be calculated. So the first conclusion is that we can numerically compute needed values B_{q_l} for specific sectors, taking the time t equal to certain reasonable values.

Second conclusion is that spatial allocation and actual values of the value added of the specific sectors become linked to the magnitudes of the value added of the sectors of all the unified states as well as to the values of the wages.

Conclusions

In the article the dynamic approach was implemented to the task of Pareto allocation of the factors for a region prior and after economic integration. The flows of capital and labor are opposite to each other, both tending to areas with respective maximum rates of return. The modeling consists of defining the GDP as having components of the capital, value added sectors and labor. Then discussion was concentrated on deducting an equation responsible for the dynamics of the regional economic development, with the GDP as a main variable. There are two main outcomes of analytical modeling:

- In case of regional economic development the dynamics of the flows of labor and value added of sectors obey the heat equation; while the dynamics of capital was found to correspond to equation of attenuated oscillations

- In case of international economic integration the type of the capital dynamics has not changed while the dynamics of the flows of labor and the value added of sectors became more complex showing that spatial dynamics of labor and the value added of one region are linked with aggregate allocation of labor and value added of sectors of all the integrated states.

The future development of the outcomes obtained in the article should both be oriented towards practical implementation of deducted formulas and to further analyzed the research in the subject due to its innovative character. For instance, it seems expedient to check deducted results using statistics of previously created economic unions which may show feasibility of the method which is to be "tuned" and then implemented for any future use concerning the creation of prospective economic unions.

REFERENCES

Butkovskiy AG (1982) Green's Functions and Transfer Functions Handbook. Halstead Press–John Wiley & Sons, New York,.

Carslaw HS, Jaeger JC (1984). Conduction of Heat in Solids. Clarendon Press, Oxford,.

Dalimov RT (2008). Modelling international economic integration. Victoria: Trafford,

Ionkin NI, Morozova VA (2000). The two-dimensional heat equation with non-local boundary conditions. Differential Equations, July 2000; MAIK Nauka – Springer; 36(7): 982-987.

Puu T (1997) Non-linear economic dynamics. Berlin: Springer,.

Schachinger E, Schnizer B (1980). General solution of the two-dimensional heat equation for two concentric domains of different material. Heat and Mass Transfer, , Springer Berlin / Heidelberg; 4(1): 7-13.

Tikhonov AN, Samarskii AA (1990) Equations of Mathematical Physics. Dover Publ., New York,.

Zhang WB (1991) Synergetic economics. Springer-Verlag,