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Distributional assumptions in educational assessments analysis: Normal distributions versus generalized beta distribution in modeling the phenomenon of learning

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This paper introduces the generalized beta (GB) model as a new modeling tool in the educational assessment area and evaluation analysis, specifically. Unlike normal model, GB model allows us to capture some real characteristics of data and it is an important tool for understanding the phenomenon of learning. This paper develops a contrast with the normal model, making one to observe that there are situations in which the most common assumption is that the normality of data is not always the best. The theory of educational assessment should begin to open to new statistical tools offered and add new models in order to capture one of the best features of the data and reject strong assumptions like symmetry.

Key words: Normal distribution, symmetry, unimodality, evaluation and learning analysis.

INTRODUCTION

The statistical modeling concept is changing nowadays; data were formerly needed to adapt or adjust to a specific or known model. The normal or Gaussian model became a natural assumption of many studies, particularly and strongly in educational evaluation (Fernández, 1997). Currently, it is intended that each data set has its own model, for example, asymmetry or bimodality of these. Therefore distributional analysis processes in the field of educational evaluation should adapt to new trends, as these changes will help us understand more precisely the phenomenon of learning; for instance, determine more precisely the skill levels of our students (Cabrera et al., 2010a). A statistical model is a platonic representation ideal belonging to the universe of the possible rather than the probable. Particularly, when we work with the normal model, we are associating a support characterized by the real line to this variable, which is better in many cases; however, infinity is not part of the present and of many of the variables that are essential to the evaluation process. This is usually because their supports are bounded, where direct examples are the evaluations in the metric system of most countries. Such situations are not just a problem of the educational field, but also in multiple areas, such as agriculture (Haskett et al., 1995), health (Gullco and Malcolm, 2009), engineering (Oguamanam et al., 1995 and Sulaiman et al., 1999). In short, we have a responsibility, in this perspective, to propose new models, particularly with bounded support, and ensure that the models can accurately represent the characteristics and behavior of the data, ignoring ideal characteristics like symmetry and proposing models that effectively represent the characteristics of the data (Arellano-Valle and Genton, 2005). Figure 1 represents a histogram for a set of notes of a course, a common and real situation.

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Our proposal consists of the presentation of generalized beta model that serves as a model of best features of many of our variables in the educational field; it has also various forms and flexibility, which make it an attractive model. Finally, we apply a concrete data set, that corresponds to the evaluations of three groups of students in a particular area. This is done in order to visualize goodness of fit with respect to the standard model.

Literature review

GB distribution proposed in this paper in relation to the generalization of its support is completely different to other preview GB proposals. Cordeiro et al. (2012) proposes a GB which is the composition of two different distributions, generating chances in distributional form, but not in the support; there is also Exponential GB distribution proposed by Barreto-Souza et al. (2010), which is a particular case of composition of distributions, as is the Beta-normal distribution by Eugene et al. (2002) and the life model by Nadarajah and Kotz (2005). However, all these generalizations are related with the form of the distribution, but not with the support, which is our proposal. There is another vision of GB raised by McDonald and Xu (1995), which is related to the increase of parameters in the characterization of the beta distribution; however, this proposal supports the interval of real numbers between 0 and 1, and the estimation process is extremely complicated.

The research problem and objectives

Research problem

At present, the philosophy of modeling is changing. There is a trend to suggest models with more features for data identification, such as asymmetry, which obviously has a positive impact on the measurement process, study of behavior and central aspects of the evaluation process (Cabrera et al., 2010b); however it is difficult to incorporate this point of view, because normality assumption, considering the distributional model, is the main part in data analysis in the field of education. Fernández (1997) indicates that the normal model is "as a rule" the distribution in the assessment of learning. For this reason we want to present and propose the Generalized Beta model as a tool that will lead to better response to the characteristics of our metric system, allowing an authoritative analysis and overcoming some distributional assumptions. For this reason, the issue of this paper is: "Characterizing Generalized Beta distribution as a model that best fits the metric characteristics of our evaluation scales and comparing the goodness of fit with respect to the normal model based on a set of real data".

Figure 1. Histogram for a set of notes of a course, real situation
Objectives

1. Describe formal aspects of GB distribution.
2. Characterize the GB model.
3. Promote GB distribution, as an analysis model of educational assessments.
4. Show the goodness of fit of GB compared to the normal model in a set of real educational evaluations.
5. Promote a line of research on the implications of considering models with bounded support in the field of education, specifically in the process of evaluation and analysis.

RESEARCH METHODOLOGY

The working methodology is pro-positive type, in the sense that it aims to start a new line of research in educational assessment analysis; this study considers the real characteristics of these processes, overcoming the problems of under-and overestimation generated with the assumption of normality.


Some preliminary analysis

The statistical modeling

Statistical models have been widely used in a wide range of situations, for example to solve specific problems in engineering and the different scientific areas; they form the basis of theoretical formulation of inference and much of the statistical methods (Arellano-Valle and Genton, 2005). Today, statistical modeling has methodological and technological backups that give a great viability as an area of educational development in modeling. A statistical model is a platonic conception of theoretical that, in a very generic way, can be seen as a mental constructor which aims to study and better understand how a phenomenon like cause and effect relationship underlies (Ojeda, 2003). Understanding this section is essential for understanding the meaning of this work, since one of the main objectives of education is to understand the phenomenon of learning; this phenomenon has an ideal model that perfectly explains their behavior. However, it is hard to know it; so in the modeling process, that is the process of proposing models, it should be increasingly considered characteristic elements corresponding to data we observe. In this sense, it is presented the proposed GB model, as a further step in the understanding of a phenomenon so important, as learning.

Model selection criteria

The criterion in the model selection will be based on how close they have proposed parametric models in respect to adjusted non-parametric model, assuming the latter as a reference. Non-parametric model or non-parametric estimated density will be identified by \( \hat{f}(x) \); now the density or proposed parametric model will be represented by \( \hat{f}(x) \). Then the selected model will be the one that minimizes the following relationship

\[
\|f(x) - \hat{f}(x)\|.
\]

The normal distribution

The normal distribution or normal model is known for its excellence in educational evaluation, but its use generally violates their assumptions. Its asymptotic properties make it an attractive model; however, these asymptotic results are not obtained in general with 20 observations. Theoretical assumptions exist that allow this condition, see for example the central limit theorem and its regularity conditions. There are a number of bibliographic references such as Johnson et al. (1995) and Casella and Berger (1990); therefore, we will not go into more detail. It is a symmetrical model, characterized by two parameters, \( \mu \) which represents the mean and \( \sigma^2 \) which represents variance. There is a process called standardization, which is a linear transformation, which generates a new random variable normally distributed; only now the mean is zero and variance is 1, as shown in Figure 2. The standard normal model is fully tabulated, so their probabilities are known.

CHARACTERIZATION OF GB MODEL

The generalized beta distribution which is presented in this paper is related to the generalization of its support, a different concept of generalization shown by Cordeiro et al. (2012). In probability theory and statistics, the beta distribution is a family of continuous probability distributions with support in the interval (0,1). The beta density is characterized by two positive parameters, generally indicated by \( \alpha \) and \( \beta \) or \( u \) and \( v \), which are parameters of location and scale.

The beta distribution has been applied to model the behavior of random variables limited to finite amplitude intervals in a variety of areas. Its density is

\[
f(x) = \frac{\Gamma(u+v)}{\Gamma(u)\Gamma(v)} (1-x)^{-u-1}(x)^{-v-1}; x \in (0,1)
\]

which can have various behaviors depending on the values of the parameters, from symmetric to fully asymmetric behavior, as presented in Figure 3.

We naturally want to extend these properties to different supports; for this reason, in the following sections we will present generalized beta distribution, to make later the contrast with the normal model.

Generalized beta distribution

The generalized beta distribution naturally born to give greater flexibility to support is bounded, where its density function is defined as:

\[
f(x) = \frac{\Gamma(u+v)}{(b-a)^{u+v}} \frac{1}{\Gamma(u)\Gamma(v)} (x-a)^{u-1}(b-x)^{v-1}; x \in (a,b)
\]
Where, in the same way as in the standard model, the parameters \( u \) and \( v \) are positive. The standard Beta distribution is now a situation of generalized Beta distribution, where \((a, b) = (0,1)\).

If \( X \) is a random variable with Generalized Beta distribution, then the notation is \( X \sim BG_{(a,b)}(u,v) \) or equivalently \( X \sim B(u,v,a,b) \).
Model features

To facilitate operability of the model, the parameter b is written by b = a + h; thus the density is represented by:

\[ f(x) = \frac{\Gamma(u+v)}{h^{u+v} \Gamma(u) \Gamma(v)} (x-a)^{u-1} (h-x)^{v-1}; x \in (a,a+h) \]

The following items are those that are usually presented to analyze and compare evaluations.

\[ E(X) = \frac{u}{u+v} h + a \]

Mean:

Second Moment:

\[ E(X^2) = \frac{(u+1)u}{(u+v+1)(v+u)} h^2 + \frac{u}{u+v} 2ah + a^2 \]

Variance:

\[ \text{Var}(X) = \frac{(u+1)u}{(u+v+1)(v+u)} h^2 + \frac{u^2}{(u+v)^2} h^2 \]

In the estimation process we present two of the most relevant: the first is the maximum likelihood and the second is the moments method.

In this process, assume that the parameter h or amplitude range of possible values of X is known.

In the case of the maximum likelihood estimator, simply solve the following system:

\[ \psi(u) - \psi(u+v) = \sum_{i=1}^{n} \ln \frac{x_i}{n} - \ln h \]

\[ \psi(v) - \psi(u+v) = \sum_{i=1}^{n} \ln \frac{h-x_i}{n} - \ln h \]

Estimations by moments method, represented by \( \tilde{u} \) and \( \tilde{v} \), are respectively

\[ \tilde{u} = \frac{\sum x_i^2 - \frac{1}{h} \sum x_i}{\frac{1}{h} \sum x_i^2} \quad \text{where} \quad k = \frac{\sum x_i}{n} \]

\[ \tilde{v} = \frac{\tilde{u} h - \tilde{u}}{X} \]

APPLICATION

Knowing distributional behavior of educational assessments is critical to the educational field, because it allows a rethink and evaluation of the introduction of new methodologies or criteria (Figure 4); therefore, the statistical modeling applied to education, particularly to the evaluation analysis, is essential for understanding and walking toward understanding the phenomenon of learning. Figure 4 allows, for example, the observation of the result of the launch of the registration without prerequisite disciplines, which obviously does not have a normal behavior.

The next application is based on a sample of three courses at a Chilean State University, where evaluative scale from 1 to 7 is used, with 1 being very low and 7 very high. The application seeks to show a contrast between the normal and beta generalized modeling for evaluation analysis. The courses discussed were selected randomly and some information will be kept confidential.

The process of contrast between the two models is used as a reference or ideal model; an estimated non-parametric density, generally black in color, was represented unless otherwise specified. The selection criteria were to be based on the smallest difference between the non-parametric model and the proposed one. The estimates obtained by the maximum likelihood method, are presented in Table 1. Figure 4 shows the comparison between the two fixes, where the green curve represents the Generalized Beta model; red curve, normal distribution and the black color curve, non-parametric adjustment. In Figure 5, we can see that the non-parametric asset gives evidence of the existence of two groups due to the bimodal behavior, complicating a contrast, as both models have uni-modal features; however the quality of the Generalized Beta model is higher than normal model, due to the degree of proximity to the non-parametric curve. Furthermore, it is possible to show the limitations of the standard model at asymmetric behavior, causing over and underestimations; specifically, in this case it produces an underestimation of the peaks and an overestimation of the probabilities in the tails of the distribution.

A fit that will surely surpass in quality to these two models is to use mixture models, given the bi-modality of the real curve (McLachlan and Peel, 2000); however, to get that, we need the expertise of a wide variety of models, and it is a good beginning for this. A similar situation is presented in Figure 5, except that now the notes correspond to a teacher and discipline completely different.

The estimates obtained by the maximum likelihood method are presented in Table 2.

Again, Figure 6 shows the superiority in modeling for the case of the GB distribution. Figure 7 shows another course, where difficulty to perceive differences is complex; although both have pros and cons, we need a quantum process to decide on the quality. Using the criterion of selection of models we have:

\[ \left\| \hat{f}(x) - \widetilde{f}(x) \right\| > \left\| \hat{f}(x) - f_R(x) \right\| \]

where \( \hat{f} \) and \( \widetilde{f} \) are the normal and beta generalized models proposed, respectively; thus the generalized beta model...
A POSTERIORI ANALYSIS AND EVALUATION

The presentation of the three situations displays an immediate necessity, which is adding more models to the educational field evaluation, allowing each day represents more accurately the actual behavior of the assessments. In Figure 6, the decision is complex visually; that is why we used numerical objective criterion for selection, and this criterion is not final. It is subject to improvements, as the field of non-parametric analysis is a young research line.

Conclusion

The proposed model provides a significant improvement in fit and modeling of learning assessments. This model considers information on the characteristics of the data, such as belonging to a bounded support, and it does not fall into the over estimation problems, which are really unlikely elements. The great advantage of the GB model in modeling assessments is its flexibility and adaptation to the asymmetries of the distributions, which again becomes an important aspect on how to better under-
Figure 5. Comparison of adjusted densities.

Table 2. Estimates of the parameters in the two models for the case of superiority of the GB distribution for Figure 6.

<table>
<thead>
<tr>
<th>Proposed model</th>
<th>Normal model: Red curve</th>
<th>Generalized beta model: Green curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimates of parameters</td>
<td>$\mu = 3.5$</td>
<td>$\mu = 13.3$</td>
</tr>
<tr>
<td></td>
<td>$\sigma^2 = 1.3$</td>
<td>$\nu = 9.9$</td>
</tr>
</tbody>
</table>

Figure 6. Superiority in modeling for the case of the GB distribution.
stand the phenomenon of learning.

The selection of a reliable model at this stage is critical to begin studying the impact of covariates in this model; for example, in the case of GB model, there are various tools for this type of analysis, such as Beta regression (Ferrari and Cribari-Neto, 2004). Ultimately, GB model presents interesting properties as an assessments analysis model and it could become a potential tool for good learning study. The signification of this paper is to introduce a new model in the analysis of evaluations that considers real and current characteristics of the data, for example, a bounded support.

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