Educational Research and Reviews Vol. 6(5), pp. 436-446, May 2011
Available online at http://www.academicjournals.org/ERR
ISSN 1990-3839 ©2011 Academic Journals

Full Length Research Paper

Prospective teachers' inclination to single representation and their development of the function concept

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Accepted 1 April, 2011

This study investigates prospective teachers' understanding of the connections between algebraic and graphical representations of the functions and their development of the concept via process-object conceptions in each of these situations. The results indicated that most of the participants were dependent upon an algebraic expression to think about a function and use it in the processes of derivative and integral. Development of a process conception of function appears to be more complicated than it is considered. The participants successfully employed this quality of thinking when the situation was familiar to them; yet they had great struggle to cope with a function process when they were given an unfamiliar situation. The research findings also suggest that the possession of an object conception of function could enable one to move freely between algebraic and graphical representations. On the other way around, students' capability at shifting between algebraic and graphical representations of the functions appear to be crucial stage to develop an object conception of function.

Key words: Prospective teachers, function concept, algebraic and graphical representations, connections between the representations, process and object conceptions.

INTRODUCTION

Representations are traditionally conceived as something that stands for something else (Janvier, 1987). They are an inherent part of mathematics and one cannot presume to have studied mathematics without having apprehended such symbolism (Janvier et al., 1987). Representations play a vital role in teaching and learning mathematics. They allow illustration of mathematical ideas in a coherent and consistent way and provide a common tool that the members of teaching-learning community make use of to express their opinions, to share their ideas with the others, and to reflect collectively upon the mathematical concepts being studied. From learning perspective, representations facilitate acquisition and retention of mathematical notions being learned. Without such symbolism one could hardly pursue the learning process suggested by the constructivist theory. Advocates of constructivist theory argue that human mind does not hold abstract notions; rather it possesses symbolism that contains distilled meaning of mathematical concepts (Gray et al., 1999). Students possess mathematical ideas in their minds in some sort of representational codes (example, graphs, diagrams, and analogy); they recall and decipher these codes and reach out the information when solving problems in the domain. It is conjectured, therefore, that representations could lessen the cognitive load on human mind and increase its capacity to attain and preserve a large amount of information in a relatively small space. In addition to their role in storing mathematical notions, representations provide aid to the individuals in their thinking. For instance, the image of a circle is not necessarily just storing some aspects of the circle, but it is also used as a cognitive tool to think about the essence and the properties of this concept.

Educators make distinction between 'external representations' and 'internal representations' (Goldin, 2001). External representations refer to all symbolic organisations (algebraic expressions, graphs, diagrams, etc.) that are used with the aim of representing externally a certain mathematical reality. They are spoken, written or some sort of visible entities; for instance, a curve or a set-diagram on a paper would be a representation of a function for an individual. Internal representations, on the other hand, refer to mental images of a mathematical concept developed by an individual through his/her interactions with and reflections upon the external representations (Goldin, 1998). They are closely associated with the human mind; it is therefore one's internal representations of a mathematical concept could differ from those of others.

An understanding of the connections between representations and between pieces of knowledge as well as the ability to translate amongst representations are considered to be defining aspect of conceptual knowledge and problem solving (Hiebert and Lefevre, 1986; Goldin, 1998; Hiebert and Carpenter, 1992). Educators agree on that a meaningful learning of mathematics could be attained when a variety of representations have been developed and the functioning relationships are established amongst them (Goldin, 2001). It is suggested therefore that students should be given an opportunity to experience the concept across the representations (Keller and Hirsch, 1998). With particular reference to the notion of function Thompson (1994) states that one could develop a subjective sense of the function – the so called "core concept of function" – as he sees invariance in the concept as it moves across the representations. The underlying philosophy is simply that well-chosen representation could convey part of the meaning of a mathematical concept, but establishing links among the representations could give a more coherent and unified message.

Various representations are used in teaching and learning mathematics. Among them algebraic expressions and Cartesian graphs dominate the mathematics curricula all over the world. Nevertheless, despite their importance in teaching and learning mathematics the role of representations is underestimated (Kaput, 1987). Algebraic expressions and Cartesian graphs are generally used as isolated constructs; and traditional approaches to teaching elementary and advanced calculus are largely restricted to mechanical manipulations with the algebraic expressions without connection to their corresponding graphs. It is reported, however, that students who receive such traditional instructions are likely to develop ill-structured knowledge in which they attribute properties of a mathematical concept to the formal representations, graphs or expressions, and not to the concept itself (Schwarz and Dreyfus, 1995). Those who lack an understanding of the connections between algebraic and graphical representations would have enormous difficulties in understanding mathematical notions including the concept of function (Even, 1998).

This study examines prospective teachers' capability at shifting between algebraic and graphical representations of the functions and explores their vertical development in each of these situations. In addition, the study aims to highlight the relationship between the students' understanding of the connections among the representations of the function and their vertical development of the concept via process-object conceptions. Having cited the purpose of the study we now turn upon the notion of function and illustrate theoretical positions introduced to interpret cognitive processing of this concept.

The concept of function and the theoretical frameworks

Simply defining a function is a relation that does matching between the elements of two sets. It could be also conceived as a process that transforms inputs to outputs. The concept has two fundamental properties: the univalence and the arbitrariness conditions. The former suggests that every element in the domain must have a unique image in the co-domain while the latter suggests that a function could do matching (or transformation) in an arbitrary manner, not necessarily through an algebraic or an arithmetical rule. Because of its central importance and unifying aspect within the mathematics curriculum (Yerushalmy and Schwartz, 1993), the epistemology of the function concept and the cognitive processing of this notion have been well researched (Eisenberg, 1991; Sfard, 1991; Dubinsky and Harel, 1992; Thompson, 1994). These studies indicated primarily that not only high schools students but also pre-service and in-service teachers lack a proper understanding of the function concept.

Theoretically speaking the notion of function is a simple mathematical idea, yet it is the variety of representations (graphs, algebraic expressions, tables, etc.) and the range of sub-notions (constant function, absolute value functions, piecewise functions, etc.) associated with the concept that make it difficult to learn (Eisenberg, 1991). Many researchers have indicated that students have a limited view, that is, a weak concept image of function (Tall and Vinner, 1981). For instance, students' mental images of graphical representation of functions are largely restricted to smooth and continues line or curve; because of this limited view most students reject the graphs of functions in strange shapes. It is reported that students display a predominant reliance on the use of and the need for algebraic expressions when dealing with the function concept (Breidenbach et al., 1992; Schwingendorf et al., 1992; Tall and Vinner, 1981). Many students could not differentiate an algebraic function from an equation. In addition, Tall and Bakar (1992) reported that half of the university students in their sample rejected the graph of a constant function because these students possessed a misconnection that a function has variables x and y; and when x changes accordingly y should change. Furthermore, most students lack the ability to see interrelations between algebraic and graphical representations of the concept. Only 14% of 152 prospective teachers who took part in Even's (1998) study were capable of shifting from algebraic to graphical representation of a quadratic function so that they could work out the number of solutions that the corresponding quadratic equation had.

Several theoretical models have been introduced to interpret cognitive processing of the function concept and these include the notions of "concept image and concept definition" (Tall and Vinner, 1981), "structural and operational conception of function" (Sfard, 1991), "multi-representational approach" (Thompson, 1994; Keller and Hirsch, 1998) and "action-process-object conceptions of function" (Duubinsky and Harel, 1992; Breidenbach et al., 1992; Cottrill et al., 1996). Each of these contributes to an understanding of the students' way of thinking and scrutinises the sources of their difficulties and the misconceptions associated with the function concept. Each also provides us with some insight into the classroom practices and the presentations of the function concept in mathematics curriculum.

In the present study two theoretical models were utilised to interpret the data collected. A multi-representational model was used to examine the students' horizontal growth of the function concept across algebraic and graphical representations and to explore their understanding of the connections between the two. Advocates of this model suggest, fundamentally, that any one of the representations cannot explicitly describe all properties of the function concept (Thompson, 1994). A graph would incite a structural conception because it denotes a function as a unified entity in which properties and components of the function have been combined (Sfard, 1991). On the other hand, an algebraic expression could be construed both operationally (as a process transforming inputs to outputs) and structurally (as a static relation between two magnitudes). One may develop a much better understanding of the concept providing that he/she sees qualitative relations between these two representations.

The notions of process and object conceptions were used to interpret students' vertical development of the function concept in the algebraic and graphical situations. A process conception is attained through interiorising actions (actions refer to repeatable mental or physical manipulations performed to transfer objects, such as numbers and sets, into new ones) (Dubinsky and Harel, 1992; Breidenbach et al., 1992). The possessor of a process conception would think of a function process in terms of inputs and outputs without necessarily performing all the operations of a function process in sequence (Dubinsky and Harel, 1992; Breidenbach et al., 1992). Once attained, a process can be manipulated in various ways; it can be reversed or combined with other processes. For instance, possessors of a process conception would compose two constant functions without any disruption which might be caused by the absence of explicit formulas in the situations. Constant reflection upon a process may lead to its eventual encapsulation as an object (Cottrill et al., 1996). With this quality of understanding one may use a function in the process of derivative or integral as if it was a single entity. From the graphical perspective, an object conception enables one to manipulate a graph of a function (example, sketching the graph of $y=f'(x)$ when the graph of $f(x)$ is given) without dealing with the graph point by point or shifting to its algebraic form.

RESEARCH METHOD AND DATA ANALYSIS

This study employed a qualitative inquiry (Yin, 2003) to produce rich and realistic data concerning the research case at hand. The research sample included 42 prospective high school teachers. At the time the study was carried out when the participants were about the finish their master program (master without a thesis) in mathematics education; so they had taken all the subject-matter and pedagogical modules including analysis, linear algebra, teaching methods and school experience. Data were obtained from written exam and semi-structured interviews. First, students were given a written exam which included 15 open-ended questions. These questions aimed, in general, to explore students' flexibility at moving between algebraic and graphical representations of the function and their vertical development of the concept in each of these situations via process-object conceptions of function. They were designed in an open-ended form to encourage students to express their actual thinking. Semi-structured interviews were conducted with three students after the written exam. The interviewees were selected based on their performances in the exam (one from low achievers, one from medium achievers and one from high achievers). The line of inquiry developed in accord with the participants' answers. Aspects of clinical interview (Gingsburg, 1981) were considered to delve into the students' thinking. Interviews were audio-taped and the annotated field notes were taken for consideration.

Data were analysed in the light of theoretical notions explained above. Several methods and techniques were used. Content analysis method (Miles and Huberman, 1994) was utilised to distinguish meaning embedded in the students' written and spoken expressions. The method of discourse analysis aimed not to interpret the students' understanding of the functions according to their responses to a single situation but to construe it from a larger perspective. Quantitative methods were applied to convert qualitative results obtained from an analysis of the students' exam papers into descriptive statistics. First, students' exam papers were examined thoroughly and a summary of their answers to each question was written up. Then, codes were established to identify the sort of conceptions and the method of solutions that they employed while resolving each question. Some of these codes included, for instance:

Alg-Appr. (Employs algebraic approach), Alg-to-Graph. (Shifts from algebra to graphical representation), Proc-Con. (Indicates process conception), Obj-Con. (Indicates an object conception), Graph-to-Alg. (Shifts from graph to algebra), Cont-Misc. (Reveals continuity misconception).

This process was repeated a couple of times and, then, a method of pattern coding was applied to collect units of meaning inferred from the text under more general categories. As it was the manner in the analysis of the students' exam papers, interviews were fully transcribed and considered line by line. First, a summary of

Table 1. Statistical analysis of the students' responses to Task 2.

	Frequency	Percent	Valid percent	Cumulative percent
No response	13	31.0	31.0	31.0
Correct (graphical approach)	7	16.7	16.7	47.6
Correct (algebraic approach)	1	2.4	2.4	50.0
Incorrect (algebraic manipulation)	21	50.0	50.0	100.0
Total	42	100.0	100.0	

students' answers to each question was written up. These documents were read thoroughly a couple of times and, then, codes were established to identify the sort of conception and the methods of solution that the interviewees used to resolve each question. Some of these codes included, for instance:

Alg-Man (Algebraic manipulation), Point-Appr. (Displays pointwise approach), Obj-Con. (Displays an object conception), Graph-Appr. (Employs graphical approach).

Repeated on different copies of the text eventually a method of pattern coding was used and the units of meaning were collected under more general categories. These categories are presented and discussed in the coming sections.

RESULTS

An analysis of the data sets indicated that most of the participants were dependent upon an algebraic expression to think about a function. A few of them demonstrated mental flexibility and shifted from algebraic expressions to graphs to resolve the problems that they were given. They were relatively successful at shifting from graph to algebra when the graph was familiar to them, such as a graph of linear function passing through the origin. Concerning the vertical development of the function concept it appears that most of the participating teachers had attained a process conception of function. Nevertheless, the quality of their conceptions fluctuated in accord with the representations and the problem situations. They were competent to deal with a process of an algebraic function but had great struggle to interpret a function process represented with a Cartesian graph. In this study, students' understanding of the connections between algebraic and graphical representations of functions and their vertical development of this concept in each of these situations will be illustrated through their responses to 9 questions from the written exam. Qualitative data will be presented to complement the results of written exam as well.

Task 1: The first question asked the students to give a definition of function and write down a couple of examples that comes to their mind first.

In response, all the students apart from one defined the concept as a relation that matches every element of the domain to a unique element in the co-domain. In their explanation there was no indication that they associated the concept with the algebraic or arithmetical rules. Nevertheless, answering to the second part of this question three quarter of the participants provided algebraic expressions as the examples of functions. This indicated that their mental images of functions were largely restricted to algebraic expressions and suggested, to some extent, that they needed algebraic expressions to think about functions. Students' inclination to algebraic expressions was substantiated by their responses to the tasks that could be easily resolved by shifting from algebra to graph. Consider the second question:

Task 2: How many functions are there on IR satisfying the constraints $f(-2)=-4$, $f(1)=2$ and $f(3)=10$? Give reasons to your answer.

Statistical analysis of the questionnaire results is provided in Table 1.

Only 17% of the students displayed a graphical approach. These students plotted (-2,-4), (1, 2), (3, 10) in the Cartesian space and, then, sketched a couple of curves through these points to justify their arguments that they could sketch infinite curves through these points and each of them represents a function. Writing down an algebraic piecewise function one student claimed that he would produce infinite functions by changing the rules on the sub-domains. More importantly, half of the participants tried to find an algebraic function with a single rule over the whole domain, but they failed to do so. This was an indication of their dependence upon an explicit algebraic description to think about a function and, this observation was validated by the students' approaches to the following problem.

Task 3: A function $y=ax^2+bx+c$ gives out 0 for the inputs -3 and 5. Work out the narrowest solution set of the inequality $ax^2+bx+c<0$; give reasons to your answer (adopted from Even, 1998).

Shifting from algebraic to graphical representation could make the resolution of this problem more accessible, yet only 12% of the students employed this strategy (Table 2).

As it is seen in Figure 1 these students sketched $\cup$ or $\cap$ shaped parabolas intersecting x-axis at (-3, 0) and (5, 0); then they concluded that the narrowest solution set for the given inequality includes an interval of (-3, 5). The vast

Table 2. Statistical analysis of the students' responses to Task 3.

	Frequency	Percent	Valid percent	Cumulative percent
No response	4	9.5	9.5	9.5
Correct (algebraic approach)	14	33.3	33.3	42.9
Correct (graphical approach)	5	11.9	11.9	54.8
Incorrect (algebraic manipulation)	19	45.2	45.2	100.0
Total	42	100.0	100.0	

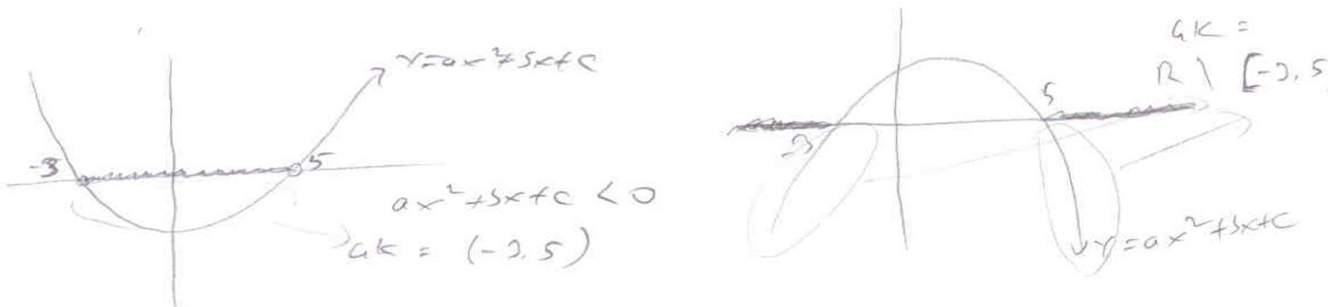

Figure 1. Tolga's written explanation to Task 3 from written exam.

Table 3. Statistical analysis of the students' responses to Task 4.

	Frequency	Percent	Valid percent	Cumulative percent
No response	17	40.5	40.5	40.5
Correct (global approach)	13	31.0	31.0	71.4
Incorrect (pointwise approach)	12	28.6	28.6	100.0
Total	42	100.0	100.0	

majority of them (88.5%) displayed algebraic approaches. Having examined in a table the rots {(-3,0), (5,0)} and the sign of coefficient of x^2 in comparison to each other 33% of them obtained the solution set as an interval of (-3, 5). Almost half of the participant (45%) made manipulations to work out the values of a, b and c so that they could get the precise form of ax^2+bx+c and resolve the problem, yet they failed to do so.

The interview results complemented the students' performances in the written exam. Only one interviewee, Berk, employed graphical approaches and resolved previous two questions correctly while the other two, Aysun and Alper, displayed algebraic methods and failed to get the correct answer. Students' answers to following question confirmed once again that they had difficulties in shifting from algebraic to graphical representations (Table 3).

Task 4: Sketch the graph of

$$f(x)=\begin{cases}1, if\ x\, an\, irrational\, number,\\ 0, if\ x\, is\, a\, rational\, number.\end{cases}$$

, and explain how did you draw it? (adopted from Even, 1998).

31% of them sketched the graph without any confusion which might be caused by the density of rational and irrational numbers in the sub-domains. Their comments indicated that these students were capable of visualising the graph of function and sketching it in a global way. One of the interviewees, Berk, displayed the same quality of understanding and he said:

"The elements in the domain are not countable...and we do not know how many irrational numbers are there between two rational numbers... Yet, we understand it from the function [algebraic expression] that the images of x will be placed on the x-axis if the input is a rational numbers and it will get the value of 1 if x is an irrational numbers. ... Therefore the graph will be in two pieces and each will be made of disjointed points. ... These points [images] are placed on the x-axis and on the line of x=1… (Figure 2)".

29% of the participants seemed to approach graphing pointwise. They tried to sketch the graph point-by-point as if the elements of the sub-domains were countable and well-ordered. Interestingly, all who approached to

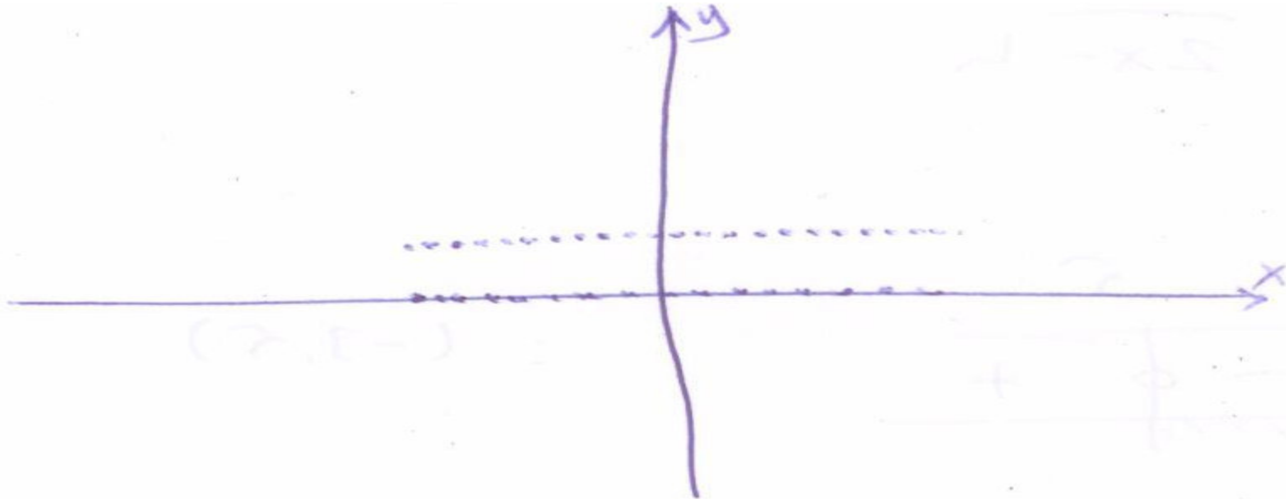

Figure 2. Berks's written explanation to Task 4.

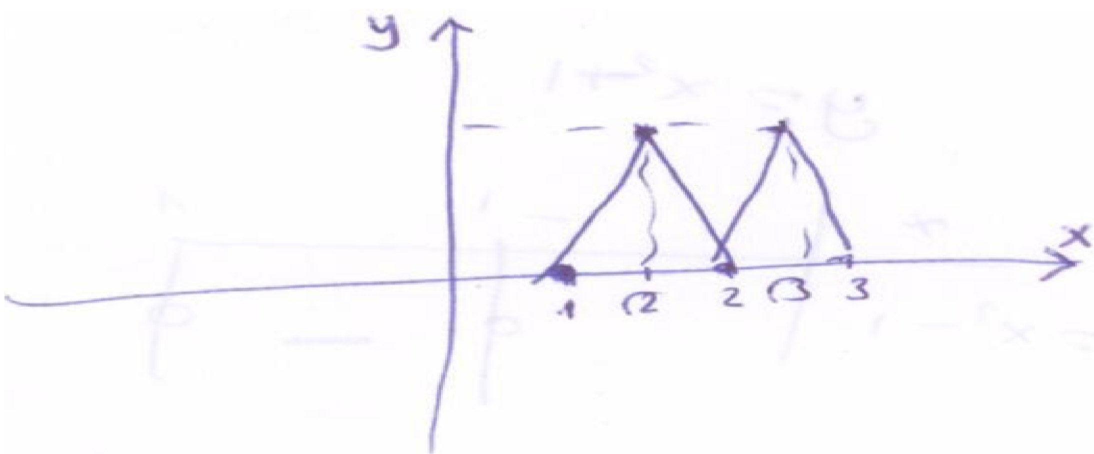

Figure 3. Aysun's written explanation to Task 4.

graphing pointwise demonstrated a continuity misconception by combining disjointed points that they placed on x-axis and the line of *y=1* with lines or curves. The same method was used by two interviewees, Aysun and Alper, who marked the images of a couple of inputs in the Cartesian space and, then, they joined these points with lines as seen in Figure 3.

Students' vertical development of the function concept fluctuated in accord with the representations and the problem contexts. Most appeared to possess a process conception in the algebraic situations; for instance, 90% of them composed two processes when the functions were not defined with explicit formulas, *f(x)=-5* and *g(x)=3*. 83% of them examined the process of $f(x)=\frac{x^3+1}{x^2-9}$ in the light of concept definition and claimed that this relation is not a function because it does not produce images for -3 and 3. They suggested that it does represent a function if the domain is redefined as IR-{-3, 3}. Students' achievement declined when they were asked to examine a function process in a graph made of discrete points (Figure 4).

57% of them rejected the situation because they thought that a graph of a function should be a continuous line or curve. 36% of them appeared to possess a process conception, since these students rejected the situation calming that it omits some elements in the domain, but also they suggested that if the domain is redefined to include certain elements the graph denotes a function.

Students' performances declined further when they were asked to operate a graph of a function in further processes. Two questions were used to investigate students' capability in establishing graphical connections between a function and its derivative and integral. In

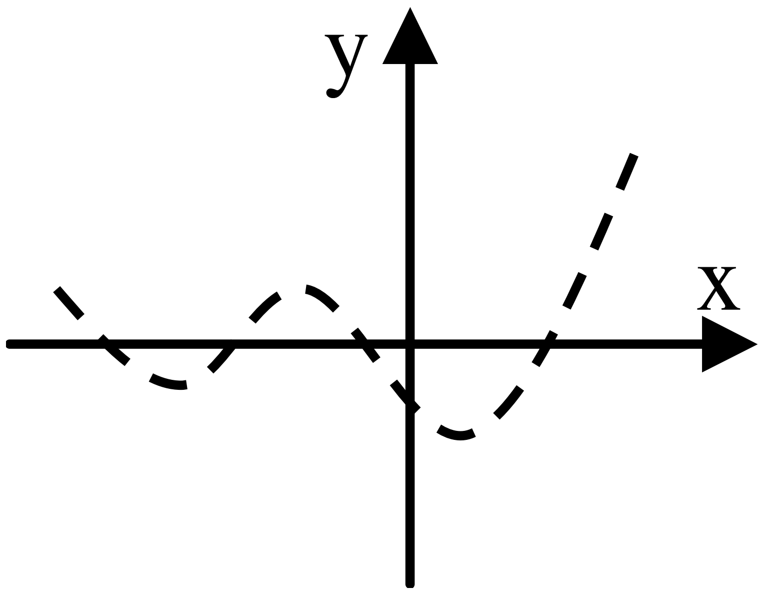

Figure 4. A function process in a graph made of discrete points.

Table 4. Statistical analysis of the students' responses to Task 8.

	Frequency	Percent	Valid percent	Cumulative percent
No response	10	23.8	23.8	23.8
Correct (global approach)	6	14.3	14.3	38.1
Correct (shift to algebra)	25	59.5	59.5	97.6
Incorrect (algebraic manipulation)	1	2.4	2.4	100.0
Total	42	100.0	100.0	

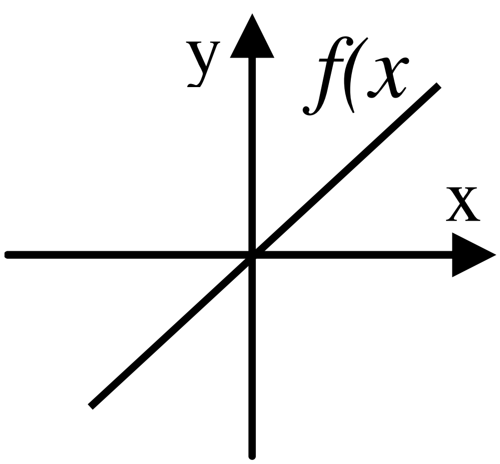

Figure 5. A graph of f(x).

responding to the following task 14% of the students displayed a global approach and sketched the graph of $y=\int f(x)d(x)$ without dealing with the graph of *f(x)* point by point or shifting to its algebraic form (Table 4). It is conjectured that these students had attained an object conception of function.

Task 8: A graph of *f(x)* is given on the right side (Figure 5). Sketch the graph of $y=\int f(x)d(x)$, and explain how did you obtain it?

60% of the participants obtained, first, the corresponding expression (such that *y=ax*), then they took its integral and obtained a quadratic function, and finally they shifted back to the graphical context and sketched a parabola. It is conjectured that these students were in need of an algebraic formula to resolve the problem. In the interviews, one student, Berk, sketched the graph of $y=\int f(x)d(x)$ in a global way and the remaining two (Aysun and Alper) shifted to algebraic expression. Berk gave the following explanation:

"This is a linear function, because it is passing through the origin. I am not concerned with its price form. … As we get its integral we obtain a quadratic function, and we

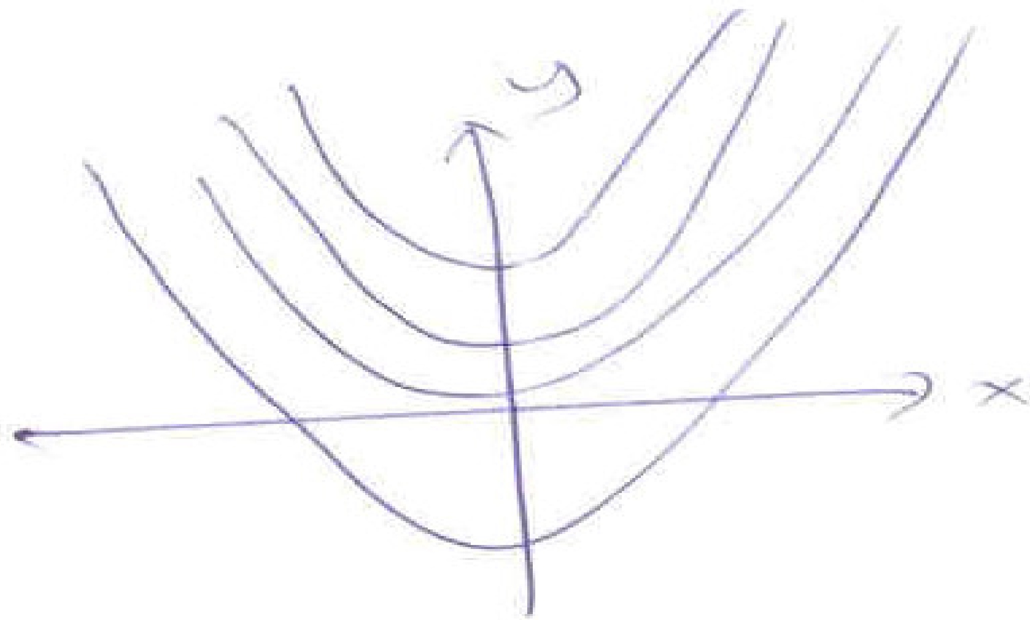

Figure 6. Berk's written explanation to Task 7.

Table 5. Statistical analysis of the students' responses to Task 9.

	Frequency	Percent	Valid percent	Cumulative percent
No response	8	19.0	19.0	19.0
Correct (global approach)	6	14.3	14.3	33.3
Correct (shift to algebra)	18	42.9	42.9	76.2
Incorrect (algebraic manipulation)	10	23.8	23.8	100.0
Total	42	100.0	100.0	

*know that a graph of a quadratic function is a parabola. …. This parabola would be something like that… "(*Figure 6)

Students' responses to the following task (Table 5) confirmed again that most of them were unable to use a graph of a function in further process as if it was a single entity.

Task 9: A graph of a third power function is given on the right side (Figure 7). Sketch the graph of $y = \frac{df(x)}{d(x)}$, and explain how did you obtain it?

Consistent with the result of previous task 14% of the participants displayed a global approach while 67% of them used algebraic methods – 43% correct and 24% incorrect. One of the interviews, Berk, obtained the graph of *f′(x)* in a global way while Aysun and Alper used the corresponding algebraic expression as a mediator. Consider the following exchange:

Alper: … I cannot take its [f(x)] derivative; it is just a graph…I need…its algebraic form; then it is quite easy to take its derivative and to sketch the graph of new function. …

Int: OK, carry on then.

Alper: It is a third power function passing through origin and located in the second and forth region. … This means that it has no constant term and the coefficient of x^3 is a negative number; we do not know exactly what it [coefficient] is, but this is not important. … Its derivative is a quadratic function [takes its derivative]…umm…and the graph should be something like this [sketches a parabola in Figure 8]".

Quite clearly, Alper is unable to interpret the impact of derivative upon a function in the graphical context – a lack of an object conception of function. He needs to use

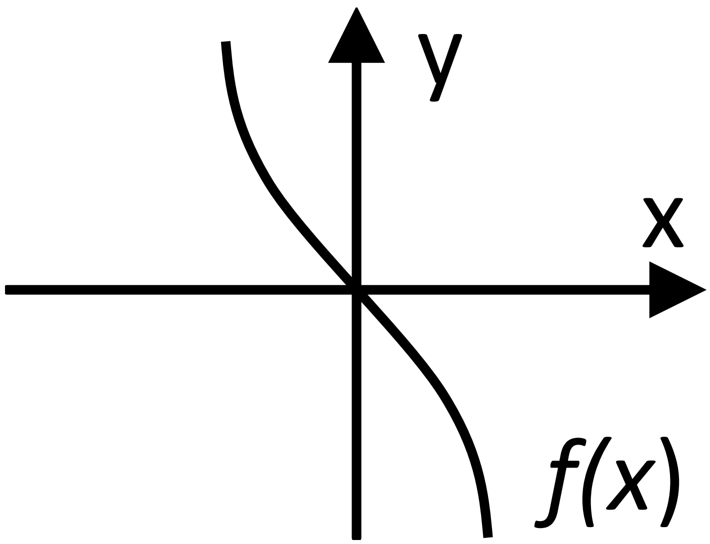

Figure 7. A graph of a third power function.

the algebraic form of the function as a cognitive tool so that he would shift from a graph of third power function to that of its derivative function – an indicator of a process conception of function.

So far the prospective teachers' understanding of the connections between algebraic and graphical representations of the functions and their vertical development in each situation were illustrated presenting quantitative and qualitative data in a harmonic way. To make the interpretation easier a summary of interview results is provided in Table 6. Since three of the interviewees revealed the same quality of understanding, a process conception, in responding to Tasks 1, 5, 6 and 7 their answers to these problems are not considered in Table 6.

Table 6 shows that only one student, Berk, employed graphical approaches to resolve the problems given algebraically; and he was consistent in his interpretation of the graphs of functions as a unified entity and use it in the processes of derivative and integral, so this student appeared to possess an object conception of function within the graphical context. Aysun and Alper were unable to shift from algebraic expressions to their corresponding graphs to resolve the problems. They displayed a pointwise approach to sketch a graph of piecewise function and shifted from graph to algebra to use a function in the process of derivative and integral. So, their understanding appears to be restricted to a process conception of function in the graphical situations.

DISCUSSION AND CONCLUSION

One cannot teach what one does not know; therefore, teachers are expected to have strong subject-matter understanding of the mathematical concepts that they will teach. The purpose of this study was to examine two crucial aspects of prospective teachers' subject-matter understanding of the function concept: their flexibility in moving between algebraic and graphical representations of the functions and their vertical development of this notion via process-object conceptions in each of these situations. The study also aimed to highlight how these two aspects of teachers' subject-matter understanding are interconnected. Nevertheless, the study has limitation in that algebraic tasks used in this research lacked the quality to investigate the development of the students' object conception of function.

Having said this, the research produced evidences which have implications for pedagogical consideration and teacher training programs in Turkey. The results indicated that most of the participants had attained a possess conception in the algebraic situations. Nearly 90% of them were able to combine the processes of two constant functions despite the absence of explicit formula in the situation. 82% of them interpreted an algebraic relation, $f(x) = \frac{x^3+1}{x^2-9}$, in the light of concept definition and detected the elements for which the function is not defined. These students were flexible in their thinking, because looking at the problem from different perspective they suggested that this relation would represent a function providing that the domain set is re-defined as IR-{-3, 3}. Nevertheless, their success declined to 36% when they were dealing with a function process represented with graph made of disjointed points. This phenomenon indicates that one's understanding of the function concept does not develop in a linear way; instead it could fluctuate in accord with the representation systems. In addition, the evidence suggests that students' process conception of function in the algebraic situations grows faster compared to its development in the graphical contexts. This might have relation, especially in Turkish context, with their background training in which students are mostly engaged with the algebraic functions, and the graphical representations are disregarded to a greater extent.

Process conception is deemed to be a critical stage in the development of the students' understanding of the function concept. Breidenbach et al. (1992) assert that a proper understanding of this notion should include a process conception if it is to go beyond mechanical manipulations with the algebraic expressions. It is reported that a process conception would help students to overcome various misconceptions associated with the functions (for example, confusion of one-to-one and uniqueness to the right condition) and enable them to make manipulation with functions when there is no explicit formula in the situations (for example, composing two constant functions) (Harel and Dubinsky, 1992). Our findings validate, to some extent, these assertions. Nevertheless, although almost all the participants within this research displayed a strong process conception to

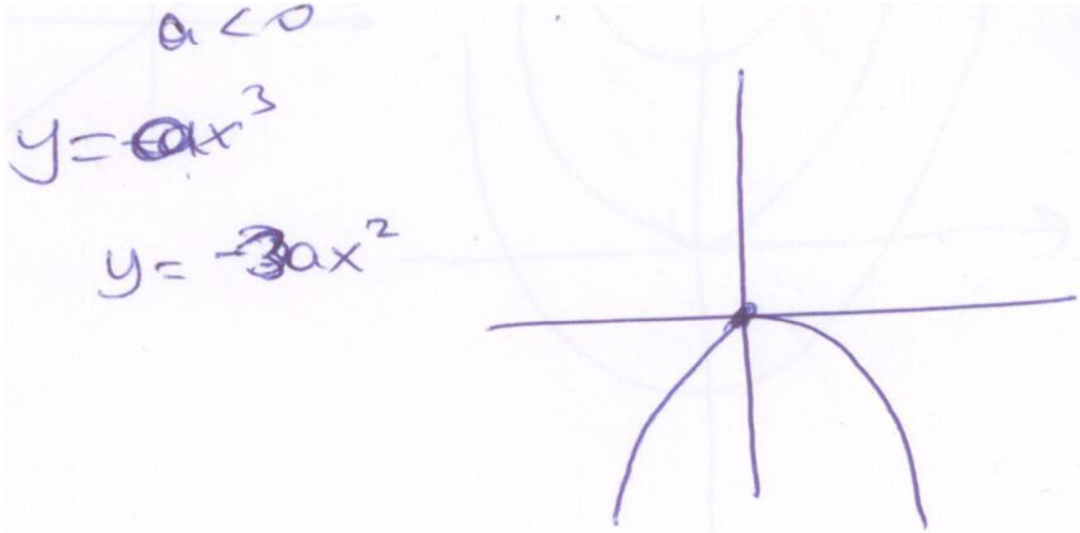

Figure 8. Alper's written explanation to Task 9.

Table 6. A summary of the interview results.

	Berk		Aysun		Alper	
	Method of solution	**Sort of conception**	**Method of solution**	**Sort of conception**	**Method of solution**	**Sort of conception**
Task 2	GR	OC	ALG	PC	ALG	PC
Task 3	GR	OC	ALG	PC	ALG	PC
Task 4	GL	OC	PO	PC	PO	PC
Task 8	GR	OC	ALG	PC	ALG	PC
Task 9	GR	OC	ALG	PC	ALG	PC

Abbreviation: GR, Graphical approach; ALG, algebraic approach; GL, global approach; PO, pointwise approach; OC, object conception; PC,process conception.

interpret the algebraic functions very few of them (Tables 1, 2 and 3) were able to shift from an algebraic to a graphical representation of the same function. For instance, 29% of them in the written exam and two students in interviews (Table 3) employed a pointwise approach – a pointwise approach is considered to be a typical indicator of a process conception (Breidenbach et al., 1992) – to sketch the graph of a piecewise function but all displayed a continuity misconception. These evidences suggest that possession of a process conception in the algebraic situation does ensure that the possessors will be able to translate this quality of understanding to the graphical context. That is, the development of a process conception is context-bounded, and the possession of it in the algebraic context may not enable one to see the connections between algebraic and graphical representations of the functions.

Connection between representations and between pieces of knowledge has been considered to be a defining aspect of conceptual knowledge (Hiebert and Lefevre, 1986; Goldin, 1998; Hiebert and Carpenter, 1992). One could develop a much better understanding of the essence as well as the different facets of the function concept providing that he/she sees the interrelation between algebraic and graphical representations of the functions. However, our results indicated that even the prospective teachers with a strong mathematics background encounter great difficulties in establishing this link; and this manifest itself, especially, in the problem situations the solution of which could be obtained by shifting from algebra to graph (Tables 1 and 2). The participants are relatively good at moving from graph to algebra, but this is evident when the graphs are familiar to them (Tables 4 and 5). Students' tendency to move from graph to algebra indicates, in fact, that they were in need of an algebraic expression to think about a function. 14% of the participants and one of the interviewees appeared to possess an object conception of function because they sketched the graphs of $y=\int f(x)d(x)$ and *f′(x)* in a global way when the graph of *f(x)* given.

Global approach (an object conception) embraces an ability to conceptualise a graph of function as a unified entity and use it in further processes as if it was a single object. It entails an ability to visualise the behaviour of a graph of function over a domain in accord with the changes in its algebraic form. It is therefore this quality of understanding is considered to be more powerful than a pointwise approach to resolve certain type of problems. On the other hand, an understanding of the connections between the representations of the functions deemed to be crucial aspect of students' knowledge about this notion. A review of literature indicates that links between these two aspects of students' knowledge have not been illustrated yet. In our study, 5 to 7 students in the questionnaire and one student in the interviews displayed a graphical approach by shifting from algebra to graph and resolved the Tasks they were given (Tables 1 and 2). 6 students in the questionnaire and one in the interviews conceptualised the graph of a function as a single entity and used it in the processes of derivative and integral (Tables 3 and 4). Cross examination of the data sets indicated that students who established graphical connections between a function and its derivative/integral and those who shifted from algebra to graph were the same students. We conclude from these cases that students' capability at shifting between algebraic and graphical representations of the functions appear to be crucial stage to develop an object conception of function in the graphical context. On the way around, our findings suggest that possession of an object conception of a graph of function could enable students to move freely between algebraic and graphical representations of this notion.

In summary, students' inclination to algebraic expressions might have relations with their background training. A follow-up study can be conducted to explore the impacts this on students' understanding of the connections between algebraic and graphical represent-tations of the functions and the relation of this to their vertical development of the concept in each of these situations. This is the issue for further research.

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