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Using menelaus' theorem and dynamic mathematics software to convey the meanings of indeterminate forms to students

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This study investigates the effectiveness of a teaching activity that aimed to convey the meaning of indeterminate forms to a group of undergraduate students who were enrolled in an elementary mathematics education programme. The study reports the implementation sequence of the activity and students' experiences in the classroom. To assess the effectiveness of the adopted approach, an individual homework assignment, in which students were asked to generate examples for indeterminate forms as geometrical constructions, was applied. The findings revealed that the students had various misunderstandings about the meanings of indeterminate forms prior to the study. Moreover, the use of the term "indeterminate" to designate these limit forms was found to be inappropriate because of its connotation. In light of the results, some suggestions are made to improve the teaching of indeterminate forms.

Key words: Calculus teaching, indeterminate forms, limit concept, concept of infinity.

INTRODUCTION

Indeterminate forms are special cases of limits that occur when it is not possible to determine the limit value of an expression solely by recognizing the limiting behavior of its subexpressions. Possibly the most important indeterminate forms are the types $\frac{0}{0}$ and $\frac{\infty}{\infty}$, because to calculate the limit of the others, one should transform them to these types in order to apply L'Hopital's theorem. Instruction on types of indeterminate forms and how to use L'Hopital's theorem to compute their limit value is covered in analysis courses at the university level. On several occasions during teaching, it has been seen that most of the students who were able to use L'Hopital's theorem to calculate the limits of indeterminate forms were unable to correctly grasp the meaning of what they stand for. Namely, most students cannot interpret $\frac{\infty}{\infty}$ as the ratio of two quantities that increase without bound, or $\frac{0}{0}$ as the ratio of two quantities both approaching zero.

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Instead, they tend to regard these symbols as a division operation on two numbers, such as $\frac{3}{5}$ or $\frac{0}{-1}$. It was concluded based on the fact that in almost every teaching session involving this topic, students ask questions such as:

"*We know that any number multiplied by zero is zero; if this is so, why then do we regard 0.∞ as indeterminate? Should not the result be zero?*" or "*Division by zero is meaningless, so why are we searching for the limit value of a meaningless expression?*" or "*Is not the division of zero by a number zero? So why do we call it indeterminate*?"

Although no study have specifically focused on students' understanding of indeterminate forms, or one that reports similar observations, there is an extensive body of research about the concepts of limit, infinity, and division by zero; which together constitute indeterminate forms. In relation to the limit concept, one major difficulty reported in the literature is students' belief that the limit of a function at a point is the same as the value of the function at that point or the point where the limit to be taken must belong to the domain of the function (Przenioslo, 2004; Bezuidenhout, 2001; Elia et al., 2009).

Another common difficulty that students face is the belief that a function can never reach its limit value. In other words, they see the limit as an endless process that approaches a certain value, but never reaches it (Cornu, 1991; Szydlik, 2000). One possible reason for this inaccurate reasoning about limit seems to be lack of comprehension of the distinction between potential and actual infinities. The research on this distinction has shown that the concept of infinity is mainly seen as a process, rather than an object; and consequently, this influences students' ability to cope with problems that deal with actual infinity (Monaghan, 2001; Tirosh 1991). An additional factor that might impede the comprehension of the indeterminate form $\frac{0}{0}$ is the difficulties that students experience in the case of division by zero.

For instance, Ball (1990) interviewed 19 (10 elementary and 9 secondary) prospective mathematics teachers about a fictitious student's question asking for the quotient of 7 divided by 0. The findings showed that only four students could give a satisfactory explanation as to why this operation is undefined. Seven other students explained the division by zero in terms of a correct rule, but they could not justify their answers; and five other students' responses were based on the incorrect rule "anything divided by zero is zero". In another study that focused on 35 in-service mathematics teachers, Quinn et al. (2008) used the same question that was applied in Ball's study. The findings revealed that only thirty-one percent of the teachers could give the correct answer and understood the underlying concept. Tsamir and Tirosh (2002) suggested that one reason for students' difficulty in understanding why dividing by zero is undefined is the intuitive belief that every mathematical operation should result in a number.

In sum, teaching experiences indicate that students cannot grasp the meaning of indeterminate forms, and the earlier mentioned studies offer some explanations for this situation. To address this issue, the study designed a teaching activity that would help students in comprehending the meaning of indeterminate forms. The activity relies on the understanding that teaching should lead from concrete to abstract; therefore, it aims first to embody the indeterminate forms in geometrical context and to have students experiment on geometrical constructs; and then to have students link their actions to the algebraic context. This study reports the implementation of the teaching activity and students' classroom experiences during the activity. Specifically, the study seeks to answer the following question:

Is the designed teaching activity effective in conveying the meaning of indeterminate forms to students?

At this point, to avoid any misinterpretation of the aim of the present study, the study wish to clarify the preference for the word "meaning" instead of "understanding" in the research question. In this sense, philosophers have differing interpretations of the link between understanding and meaning. While some expound meaning in terms of understanding, others explain understanding in terms of meaning (Sierpinska, 1994). For the researcher, the understanding of indeterminate forms requires, at least, the knowledge to answer why L'Hopital's theorem works. However, the intention in designing the activity was not to present any formal knowledge about indeterminate forms to students, but rather to help students to comprehend the mathematical process that is represented by indeterminate forms. Thus, the term "meaning" was considered to be more appropriate to reflect the aim of the study.

Menelaus' theorem as a window on the meanings of indeterminate forms

Menelaus' theorem, named for Menelaus of Alexandria, is a theorem in plane geometry (Figure 1). The theorem states:

Let ABC be a triangle, and F, E are two points on]AC[and]AB[respectively. If D is the intersection point of FE and CB, then $\frac{|DB|}{|DC|}\cdot\frac{|CF|}{|FA|}\cdot\frac{|AE|}{|EB|}=1$ ……….. (I)

Although the theorem seems to be static on paper, it has a dynamic nature. This dynamism results from the

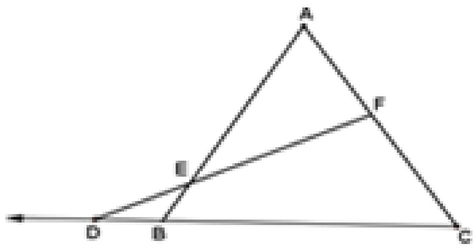


Figure 1. The depiction of Menelaus' theorem.

arbitrariness of the positions of the points F and E. As a consequence of this arbitrariness, the theorem includes some of the indeterminate forms in its structure. First, imagine that, holding point D as fixed and keeping points D, E, and F collinear, point F gets closer and closer to point A. Then the ratio $\frac{|CF|}{|FA|}$ increases without bound, and the ratio $\frac{|AE|}{|EB|}$ decreases continuously to zero. Therefore, as F approaches A, the limit of $\frac{|CF|}{|FA|}\cdot\frac{|AE|}{|EB|}$ yields the indeterminate form $\infty.0$. Because the equality (I) holds irrespective of the positions of F and E, that limit is equal to the ratio $\frac{|DC|}{|DB|}$. In a similar manner, as point F approaches point A, the limit of $\frac{\frac{|CF|}{|FA|}}{\frac{|EB|}{|AE|}}$ yields the indeterminate form $\frac{\infty}{\infty}$. However, for the same reason, the limit is equal to the ratio $\frac{|DC|}{|DB|}$. Lastly, the indeterminate form $\frac{0}{0}$ comes from the ratio $\frac{|AE|}{|FA|}$ as point F approaches point A; as a result of equation (I), the limit of this ratio is equal to $\frac{|DC|.|AB|}{|DB|.|CA|}$. As a result of possessing these indeterminate forms in its structure, the researcher considered Menelaus' theorem to be an appropriate context to concretely represent these abstract concepts in order to help students grasp their meaning.

METHODOLOGY

Participants and setting

The research sample for this study consisted of 40 (12 male and 28 female) pre-service elementary mathematics teachers. The participants were in their second year at the university and had taken a Euclidean Geometry course in which Menelaus' theorem had been introduced, as well as a Calculus-1 course in which indeterminate forms and L' Hopital's theorem had been introduced, prior to the onset of the study. The structure of the Calculus-1 course was traditional, and the examples given for indeterminate forms were all in the form of operations on algebraically represented functions (i.e. $\lim_{x\to 0}\frac{x}{sinx}$); no examples in the form of geometrical constructions were given. Moreover, the Turkish calculus textbooks recommended to the students by their instructor for the Calculus-1 course did not include examples of indeterminate forms in the form of geometrical constructions; nor had the researcher encountered any Calculus textbooks written in Turkish that included examples in the form of geometrical constructions up to the time of the study.

The teaching activity reported in this study was administered in an elective course titled Graphical Calculus that was offered in the third semester of the teacher education program. The general aim of this course, of which the researcher was the instructor, was to introduce students to the geometrical representations of basic calculus concepts. The teaching session took place in a computer laboratory equipped with a projector that was connected to the instructor's terminal. The students formed pairs, and each pair had a computer on which the supplementary GeoGebra file was installed for the completion of the student worksheet (see Appendix 1). The instruction took place during two consecutive sessions. The first session lasted 80 min, followed by a 20 min break; and the second session lasted 40 min. All of the students had enough technical knowledge about the software to complete the worksheet.

Data collection tools

The data collection tools used in this study consisted of an open ended test, the researcher's classroom observation notes, and a homework assignment in which the students were asked to generate examples for indeterminate forms using geometrical constructions. Before the classroom intervention, in order to reveal the students' existing knowledge about the indeterminate forms $\frac{\infty}{\infty}$, $\frac{0}{0}$ and $\infty.0$, an open-ended test was administered. The questions on the test asked students to explain why these limit forms are called indeterminate.

While students were completing the worksheet in the classroom, the researcher was available for assistance and guidance if needed. During this time, the researcher took notes on important issues discussed between the partners, as well as between the pairs and the researcher himself. These field notes were used as a data source. After the instruction took place, the researcher assigned an individual homework task in which students were asked to generate at least one example for each type of indeterminate as a geometrical construction. The aim of this homework assignment was to assess whether the students had correctly grasped the meanings of these indeterminate forms. As a hint, the researcher recommended that students re-think the geometry theorems that they had covered in their Euclidean Geometry course in terms of whether they possess any of these indeterminate forms in their structures, as with Menelaus' theorem. The students were given one week to finish and return the assignment. The data presented in this paper were translated from Turkish into English while trying to maintain the essence of the meaning.

Classroom activities and implementation sequence

The teaching activities took place in two back-to-back sessions. The first session lasted approximately 80 min, and after a 20 min break, the second session began and lasted approximately 40 min. The

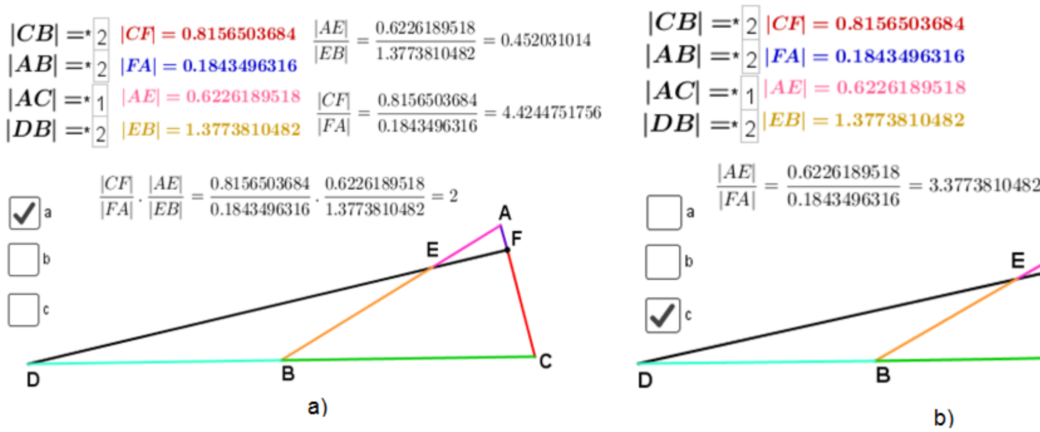


Figure 2. Screenshots for parts A and C in step 1.

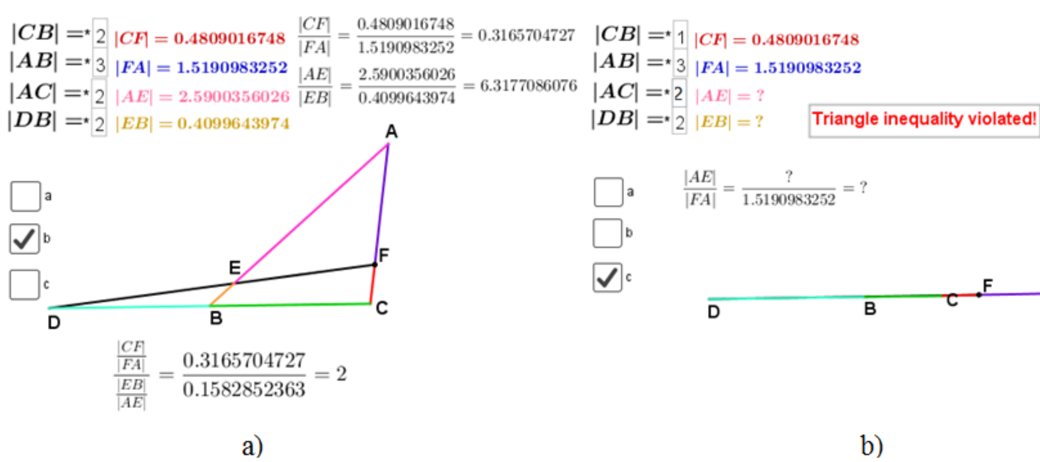


Figure 3. Screenshots for different assigned values for the lengths of the construction.

first session was devoted to the implementation of the student worksheet. In the second session, the researcher gave a presentation in which he investigated the limits in the student worksheet in the Cartesian coordinate plane by focusing on a specific case. The aim of the second session was to get students to link their actions in Euclidean geometry to the algebraic context. The implementation of these sessions is detailed below.

Implementation of the first session

In the first session of the teaching activity, the students were asked to complete the worksheet, which consisted of three phases: exploration, experimentation and generalization. The first three questions constituted the exploration phase, and in this phase, students did not make use of the software. The questions in this phase asked students to predict the indeterminate forms that would emerge as a result of point F approaching point A. The aim of these questions was to embody these rather abstract indeterminate forms into geometrical entities in order for students to gain comprehension of their meaning.

The next three steps following the first three questions on the worksheet constituted the experimentation phase. After students had completed the first three questions, they opened the GeoGebra file developed by the researcher. In the first step, the students were asked to determine the results of the limits by making experimental observations with the software. At this point, students were able to check the predictions that they had made in the exploration phase. The students had the opportunity to change the position of point F and consequently observe the results of the calculations on the screen simultaneously. In Figure 2, screenshots of the software with the default lengths for parts A and C of step 1 are shown.

In steps 2 and 3, the students were asked to assign new integer values for the indicated lengths of the construction and re-investigate the associated limits. In Figure 3, screenshots for different assigned values are shown. Once a new value for a length using the input bars at the top left corner of the screen was assigned, the construction was changed accordingly (Figure 3a). If

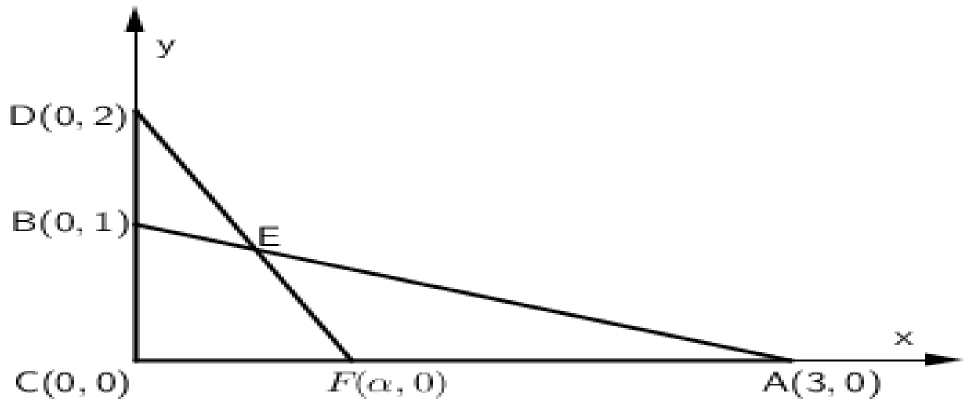


Figure 4. The construction for the figure in the worksheet.

the assigned value for a length did not suffice for triangle inequality, a warning message indicating that the triangle inequality was violated appeared on the screen (Figure 3b).

In step 2, assigning new values to $|AB|$ and $|AC|$ affects the result of the limit in part C; but it does not affect the results of the limits in parts A and B, because of the equation (I). However, in step 3, assigning new values to $|CB|$ and $|DB|$ does affect the results of all limits. Following several experimentations in steps 2 and 3, students were expected, at minimum, to notice that the results of the limits in parts A and B were equal to the same value, which is the ratio $\frac{|DC|}{|DB|}$, and also to comprehend that the results of the limits might be a consequence of Menelaus' theorem.

The final step on the worksheet constituted the generalization phase. In this step, students were asked to find expressions for the results of the associated limits when the lengths in the construction were given as variables. After all of the groups completed the worksheet, a class discussion about the results of the limits in step 4 and the relation between these results and Menelaus' theorem took place.

Implementation of the second session

In the second session of the teaching activity, the instructor gave a presentation that aimed to help students link the actions they had performed in the previous session in Euclidean geometry to the algebraic context. To this end, the researcher focused on a specific case of the construction and investigated the limits given on the student worksheet in the Cartesian coordinate plane. During the presentation, the students were not simply passive listeners; the instructor also asked questions as appropriate to ensure their active participation.

However, as they do not pertain directly to the study itself, these questions are not included here. In the course of the presentation, in order to make the calculations simpler and hence to make the content easier for the students to comprehend, the instructor chose a right triangle whose perpendicular sides lay on the coordinate axes in constructing the figure on the worksheet. The content of the presentation was as follows. The figure investigated in the worksheet was constructed in the Cartesian coordinate system as seen in Figure 4 (Let the arbitrary point F be on]CA[and its abscissa be represented by α).

First of all, let's determine the coordinates of point E in terms of α. Point E lies at the intersection of AB and DF; therefore, we need to find the equations of these lines and then specify their intersection. By using the two-point form of a line, we get:

$\overleftrightarrow{AB}: \frac{y-0}{x-3} = \frac{1-0}{0-3} \rightarrow y = \frac{-x}{3} + 1$ and $\overleftrightarrow{DF}: \frac{y-0}{x-\alpha} = \frac{2-0}{0-\alpha} \rightarrow y = \frac{-2}{\alpha}x + 2$

If we find the intersection of these lines using their equations, we get:

$$\overleftrightarrow{AB} \cap \overleftrightarrow{DF}: \frac{-x}{3} + 1 = \frac{-2}{\alpha}x + 2 \rightarrow x = \frac{3\alpha}{6-\alpha}$$

This gives us the abscissa of point E in terms of α. To get the ordinate of point E, we can use the equation of $\overleftrightarrow{AB}$; thus, we get:

$y = \frac{\frac{-3\alpha}{6-\alpha}}{3} + 1 = \frac{6-2\alpha}{6-\alpha}$ hence $E(\frac{3\alpha}{6-\alpha}, \frac{6-2\alpha}{6-\alpha})$.

Now that we have specified the coordinates of E, we can find the expressions for the lengths of the segments in terms of α. By using the formula for the distance between two points, we get:

$$|CF| = \alpha, |FA| = 3-\alpha, |AE| = \frac{2\sqrt{10}(3-\alpha)}{6-\alpha}, |EB| = \frac{\sqrt{10}\alpha}{6-\alpha}$$

Now let's transform the limits that we dealt with on the worksheet by using the variable α in algebraic form.

Part a)

$$\lim_{F \to A} \overset{\infty.0}{\frac{|CF|}{|FA|} \cdot \frac{|AE|}{|EB|}} = \lim_{\alpha \to 3^-} \frac{\alpha}{3-\alpha} \cdot \frac{\frac{2\sqrt{10}(3-\alpha)}{6-\alpha}}{\frac{\sqrt{10}\alpha}{6-\alpha}} = \lim_{\alpha \to 3^-} \frac{2\sqrt{10}}{\sqrt{10}} = 2 = \frac{|DC|}{|DB|}$$

Part b)

$$\lim_{F \to A} \overset{\frac{\infty}{\infty}}{\frac{\frac{|CF|}{|FA|}}{\frac{|EB|}{|AE|}}} = \lim_{\alpha \to 3^-} \frac{\frac{\alpha}{3-\alpha}}{\frac{\frac{\sqrt{10}\alpha}{6-\alpha}}{\frac{2\sqrt{10}(3-\alpha)}{6-\alpha}}} = \lim_{\alpha \to 3^-} \frac{\frac{\alpha}{3-\alpha}}{\frac{\sqrt{10}\alpha}{6-\alpha} \cdot \frac{6-\alpha}{2\sqrt{10}(3-\alpha)}} = 2 = \frac{|DC|}{|DB|}$$

Part c)

$$\lim_{F \to A} \frac{\overset{\frac{0}{0}}{|AE|}}{|FA|} = \lim_{\alpha \to 3^-} \frac{\frac{2\sqrt{10}\,(3-\alpha)}{6-\alpha}}{3-\alpha} = \lim_{\alpha \to 3^-} \frac{2\sqrt{10}}{6-\alpha} = \frac{2\sqrt{10}}{3} = \frac{|DC|}{|DB|} \cdot \frac{|AB|}{|CA|}$$

FINDINGS

In this study, the data were gathered from three sources: an open-ended test implemented before the instruction, the researcher's observation notes, and student-generated examples. The findings that emerged from these sources are presented in separate sections below.

Findings from the test

Before the classroom intervention to reveal the meaning that students attached to the indeterminate forms $\frac{\infty}{\infty}$, $\frac{0}{0}$ and $\infty.0$, an open-ended test was administered. The students were asked to explain why each of these limit forms are called indeterminate. After analyzing the students' written responses, it was seen that the students' explanations as to why these limit forms are called indeterminate had similarities and differences. Based on these similarities and differences, the researcher assigned categories for the students' explanations. In Table 1, the categories, an excerpt of students' explanations for each category, and the number of students that fell in each category are presented.

The analysis of the responses showed that the students could not interpret the symbol "∞" as describing the process in which a quantity is increasing without bound. Instead, as seen from Table 1, students interpreted the symbol "∞" as an object to which they attached different meanings, and as a result, they used fallacious rationales in explaining why this limit form is called indeterminate.

The findings also indicate that the students interpreted the number zero in the indeterminate form $\frac{0}{0}$ as an object; and hence, almost all of the students who provided an explanation proposed the impossibility of division by zero as a rationale to explain why it is called indeterminate. The students could not interpret the number zero as a process in which a quantity decreases to zero.

According to the data in Table 1, the frequency of students who did not provide an explanation for the $0.\infty$ type was the highest among the others. In line with the previous cases, the students interpreted the symbol "∞" and the number zero as objects. Therefore, as shown in the last excerpt, the students expressed that the number zero might not absorb the number infinity because of its incomprehensibly large value.

In summary, the findings that emerged from the students' responses did not contradict the researcher's expectations, because they are consistent with his previous experiences as outlined in the introduction.

Classroom observations during the teaching sessions

While the students were making their experimental observations, using the software to complete the experimentation phase of the worksheet, some of the groups had difficulties in answering the questions in Step 1 and asked for help from the researcher. The dialogues occurring during the interactions between these groups and the researcher revealed two difficulties regarding the limit concept.

The first difficulty is related to the limit given in part C of step 1. Some of the groups could not arrive at a solution for that limit and asked for help. The result of that limit is $\frac{|DC|}{|DB|} \cdot \frac{|AB|}{|CA|}$, which is equal to 4, according to the default lengths. The dialogue between these students and the researcher showed that, although these students moved point F very close to point A by using the zoom feature and observed the value of $\frac{|AE|}{|FA|}$ becoming very close to 4 (Figure 5), they still denied that the limit exits. To justify their conclusion, the students proposed that one could always make point F closer to point A, and hence make the value of the quotient larger; therefore, this quotient would never reach a constant value. Whereupon the researcher attempted to help the students realise that the value of the quotient could be made as close to 4 as desired; but the students seemed to be reluctant to agree that this limit exists. However, after the content of the second session was presented, these students, without being asked, told the researcher that they had changed their decision about the existence of the limit and realised their fallacious reasoning.

While the researcher was monitoring the students' activities in the classroom, he noticed that some groups concluded that the limits in parts A and B of step 1 did not exist. The result of these limits is $\frac{|DC|}{|DB|}$, which is equal to 2, according to the default values. Moreover, because the ratio $\frac{|DC|}{|DB|}$ is independent of the position of point F, the numerical values of the expressions in these limits remain fixed (as can be seen in Figure 2a and Figure 3a) as point F moves. The researcher asked these students to explain the reasoning for their decisions. According to their understanding, these limits did not exist because the values of the expressions in these limits remain unchanged; and therefore, they do not approach a value.

Table 1. The categories that emerged from students' responses and the frequencies of students in each category.

Type	Categories	Student excerpts	f(40)
$\frac{\infty}{\infty}$	Interpreting the symbol "∞" as an incomprehensibly large number	If we divide the number infinity into the number infinity, it does not correspond to a real number. That is why it is called indeterminate	8
	Interpreting the symbol "∞" as a collection of very large numbers	In this case, the functions in the numerator and the denominator can take on an infinite number of high values. Therefore, we cannot calculate the result of this division.	6
	Interpreting the symbol "∞" as an unknown quantity	Because we do not know what infinity is, we cannot make calculations with it; hence this quotient called indeterminate	12
	No explanation		14
$\frac{0}{0}$	Division by zero is undefined or meaningless	The denominator of a fraction cannot be zero; therefore, this quotient is called indeterminate.	21
	The numerical value of 0/0 is unknown	This is called indeterminate because we do not know the numerical value of 0/0	3
	No explanation		16
0.∞	∞ is inconceivable	Because we do not know the number by which we multiply zero; in other words, because we do not know what infinity is, this is called indeterminate.	8
	Interpreting the symbol "∞" as an incomprehensible large number	Normally, zero is the absorbing element for multiplication; but in this case, because we do not know how big infinity is, we cannot say the result is zero. Therefore, it is indeterminate	10
	No explanation		22

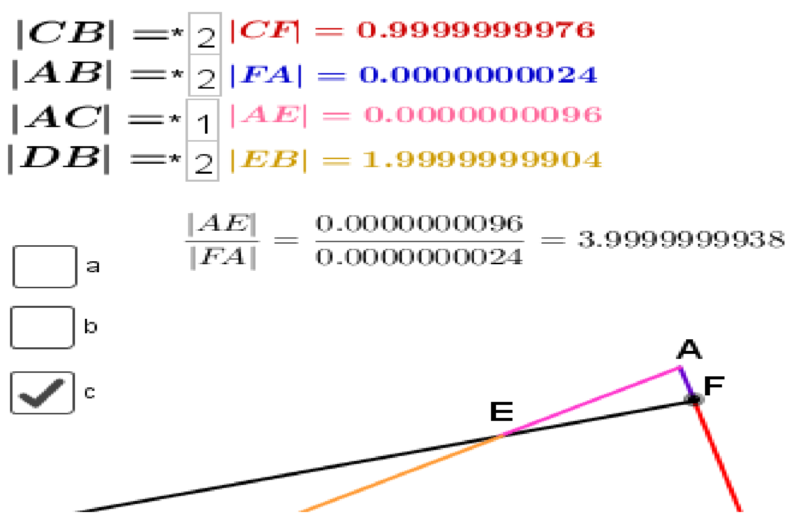


Figure 5. The screenshot of the software when point F is close to point A.

These students believed that in order for an expression in a limit to approach a value, it must change. Although they had encountered the limit of a constant function in their Calculus-1 course, they could not transfer this knowledge to the present context. However, after the researcher prompted these students to recall the theorem about the limit of a constant function, they were able to overcome this misconception.

At the end of the instruction in the second session, one student asked a pedagogically valuable question out loud in the classroom. This student complained about the term “indeterminate” as used to designate these limit forms, and the other students backed him up in this idea. Because this issue may be context-bound, it would be beneficial to clarify the meaning of the term in Turkish that corresponds to “indeterminate.” Namely, the Turkish word used to designate these limit forms is “belirsiz,” which has the same meaning as the word “indeterminate”; and, as in English, it is the antonym of the word “belirli”, which has the same meaning as the word “determinate”.

The student asked, “If we can determine the value of these limit forms, why then do we call them indeterminate”? This question indicates that the term used to designate these limit forms does not convey the essential meaning of the forms to students.

Selected examples of indeterminate forms from students

To evaluate the effectiveness of the activity designed for teaching students the meaning of indeterminate forms, the researcher used an individual homework assignment in which students were asked to generate examples of indeterminate forms in as geometrical constructions. The students were asked to turn in the assignment to the instructor in person, so that he would have the opportunity to ask them to explain their examples and thereby ensure that they had correctly grasped the meaning of a particular indeterminate form.

Moreover, if a student turned in an example that had been previously submitted by another student, the instructor asked the second student to change his/her example. By this means, the instructor aimed to prevent students from copying each other’s homework. In total, sixteen students provided an admissible example with their homework assignments; however, among these, two turned in examples that had been previously been submitted by other students. These students were asked to change their examples and re-submit their homework, but they failed to do so. All 14 of the students who turned in admissible examples were able to explain them and to justify the results of the limits that they had constructed. Furthermore, although they had not been asked to do so, ten students had also constructed their examples using the software and had checked their results experimentally before submitting the assignment.

Four of these students’ examples can be seen in Table 2. The first and second examples in the table belong to two of the students who had experienced the first and second difficulties related to the limit concept that are accounted respectively in the section on classroom observations.

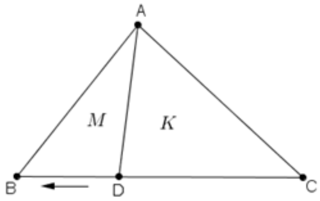
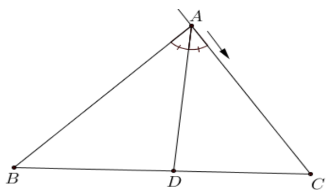
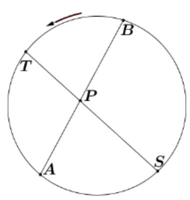
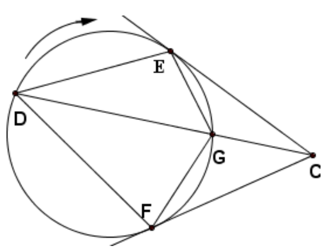
Conclusion

The present study addresses the question “is the designed teaching activity effective in conveying the meaning of indeterminate forms to students”? The researcher’s informal interviews with the students about the effectiveness of the activity and his observations with respect to their success in interpreting the limit forms in subsequent lessons indicate a positive answer to this question. However, the proportion of the students who actually completed the homework assignment fell short of the researcher’s expectations. In total, fourteen out of 40 students provided an example and showed confidence in explaining the meaning of the limit forms that they had formed. Two reasons for this shortfall were identified. First, eight of the 40 students did not attend the second session of the teaching activity, and none of these students completed the homework assignment. Second, the assignment was not graded, and it did not contribute to the final grade in the course. Having discussed this with the students, the researcher realized that this adversely affected students’ motivation to complete the assignment.

The responses to the open-ended test that was administered prior to the teaching activity revealed that none of the students could correctly explain the meanings of the indeterminate forms, although they had encountered this topic in their calculus courses. It was found that the students interpreted the symbols 0 and ∞ as objects, rather than as representing two processes. Consequently, they suggested the impossibility of division by zero and the impossibility of determining the value of infinity as rationales for referring to these limit forms as indeterminate. One possible cause for students’ understanding of infinity as an object representing a tremendously large number may be statements such as “x approaches positive/negative infinity” that are used in discussing limits involving infinity in the classroom. In this respect, due to the meaning of the verb “approach”, this statement seems to suggest that there is a number called infinity and that the variable x gets closer to it. To alleviate this misconception, the researcher recommends that it would be more appropriate to say “as x increases/ decreases without bound”, instead. However, further investigation is needed to clarify this issue.

In addition to this misunderstanding, the students’ responses to the worksheet revealed two difficulties in grasping the limit concept. First, some of the students were unable to comprehend the distinction between *potential* and *actual* infinities. These students conceived of the limit as an endless process that approaches a certain value, but never reaches it; this misconception has also been reported by researchers such as Cornu

Table 2. Selected examples from the students for indeterminate forms.

Theorem	0/0	∞/∞	0.∞
 ABC is a triangle; if M and K are the areas of ABD and ADC respectively, then $\frac{M}{K}=\frac{\lvert BD\rvert}{\lvert CD\rvert}$ (I)	As D approaches B along [BC], $\lim_{D\to B}\frac{M}{\lvert BD\rvert}$ yields the indeterminate form 0/0. However, $\lim_{D\to B}\frac{M}{\lvert BD\rvert}=\lim_{D\to B}\frac{K}{\lvert CD\rvert}=h$ h is the height of the altitude to the base BC.	As D approaches B along [BC], $\lim_{D\to B}\frac{\frac{\lvert CD\rvert}{\lvert BD\rvert}}{\frac{K}{M}}$ yields the indeterminate form ∞/∞. Because of the equation (I), the limit is equal to 1.	As D approaches B along [BC], $\lim_{D\to B}\frac{M}{K}\cdot\frac{\lvert CD\rvert}{\lvert BD\rvert}$ yields the indeterminate form 0.∞. Because of the equation (I), $\lim_{D\to B}\frac{M}{K}\cdot\frac{\lvert CD\rvert}{\lvert BD\rvert}=1$
 ABC is a triangle and [AD] is angle bisector. $\frac{\lvert CA\rvert}{\lvert CD\rvert}=\frac{\lvert AB\rvert}{\lvert BD\rvert}$ (II)	As A approaches C along [CA, $\lim_{A\to C}\frac{\lvert CA\rvert}{\lvert CD\rvert}$ yields 0/0 indeterminate form. However, as both $\lvert AB\rvert$ and $\lvert BD\rvert$ approach to $\lvert BC\rvert$ $\lim_{A\to C}\frac{\lvert CA\rvert}{\lvert CD\rvert}=\lim_{A\to C}\frac{\lvert AB\rvert}{\lvert BD\rvert}=1$	As A approaches C along [CA, $\lim_{A\to C}\frac{\frac{\lvert AB\rvert}{\lvert CA\rvert}}{\frac{\lvert BD\rvert}{\lvert CD\rvert}}$ yields ∞/∞ indeterminate form. Because of the equation (II), the limit is equal to 1.	As A approaches C along [CA, $\lim_{A\to C}\frac{\lvert BD\rvert}{\lvert CD\rvert}\cdot\lvert CA\rvert$ yields 0.∞ indeterminate form. Because of the equation (II), $\lim_{A\to C}\frac{\lvert BD\rvert}{\lvert CD\rvert}\cdot\lvert CA\rvert=\lim_{A\to C}\lvert AB\rvert=\lvert BC\rvert$
 The points B, T, A, and S are on the circle. P is the intersection of the chords. $\frac{\lvert PT\rvert}{\lvert PB\rvert}=\frac{\lvert PA\rvert}{\lvert PS\rvert}$ (III)	As B approaches T along the circle, $\lim_{B\to T}\frac{\lvert PT\rvert}{\lvert PB\rvert}$ yields the indeterminate form 0/0. Because of the equation (III), $\lim_{B\to T}\frac{\lvert PT\rvert}{\lvert PB\rvert}=\lim_{B\to T}\frac{\lvert PA\rvert}{\lvert PS\rvert}=\frac{\lvert AT\rvert}{\lvert ST\rvert}$	As B approaches T along the circle $\lim_{B\to T}\frac{\frac{\lvert PS\rvert}{\lvert PB\rvert}}{\frac{\lvert PA\rvert}{\lvert PT\rvert}}$ yields the indeterminate form ∞/∞. Because of the equation (III), the limit is equal to 1	As B approaches T along the circle, $\lim_{B\to T}\frac{\lvert PT\rvert}{\lvert PA\rvert}\cdot\frac{\lvert PS\rvert}{\lvert PB\rvert}$ yields 0.∞ indeterminate form. Because of the equation (III), the limit is equal to 1
 D, G, and C are collinear. E and F are tangency points. $\lvert DE\rvert.\lvert FG\rvert=\lvert EG\rvert.\lvert DF\rvert$ (IV)	As D approaches E along the circle, $\lim_{D\to E}\frac{\lvert DE\rvert}{\lvert EG\rvert}$ yields indeterminate form 0/0. However, as both $\lvert DF\rvert$ and $\lvert FG\rvert$ approach $\lvert EF\rvert$, $\lim_{D\to E}\frac{\lvert DE\rvert}{\lvert EG\rvert}=\lim_{D\to E}\frac{\lvert DF\rvert}{\lvert FG\rvert}=1$	As D approaches E along the circle, $\lim_{D\to E}\frac{\frac{\lvert DF\rvert}{\lvert DE\rvert}}{\frac{\lvert FG\rvert}{\lvert GE\rvert}}$ yields the indeterminate form ∞/∞. Because of the equation (IV), the limit is equal to 1	As D approaches E along the circle, $\lim_{D\to E}\frac{\lvert EG\rvert}{\lvert FG\rvert}\cdot\frac{\lvert DF\rvert}{\lvert DE\rvert}$ yields the indeterminate form 0.∞. Because of the equation (IV), the limit is equal to 1

(1991) and Szydlik (2000). Second, as a result of the meaning of the verb “approach”, some of the students believed that an expression that evaluates to a constant value does not have a limit. In this case, the researcher has not come across any reports in the literature with similar findings. On the other hand, while the researcher did not aim to remedy these misconceptions in designing the teaching activity, it was found that the teaching sessions did aid students in both realizing and overcoming their misconceptions. Thus, it may be argued that providing students first with the opportunity to investigate these limits in geometrical context by allowing them to experiment on geometrical entities, and then to establish a link to the algebraic context, was the main reason for this positive outcome.

Another important finding of this study is that the term “indeterminate” as used to designate these limit forms does not truly represent their meaning. As the students expressed in their complaints at the end of the teaching session, this terminology caused them to erroneously believe that the limit forms cannot have a limit value. Therefore, the researcher contends that using the term “ambiguous”, rather than “indeterminate”, to designate these limit forms would more accurately convey their meaning.

The teaching activity discussed in the study requires both a substantial investment of time in the classroom and students who have a certain degree of technical knowledge of the software used in implementing it. For these reasons, it may not be suitable for introducing students to these limit forms in calculus courses. However, the researcher recommends that beginning this topic with a discussing of a geometry theorem that possesses ambiguous forms and then translating them into algebraic expressions and re-investigating them might help students to gain a better understanding of their meanings.

Conflict of Interests

The author has not declared any conflicts of interest.

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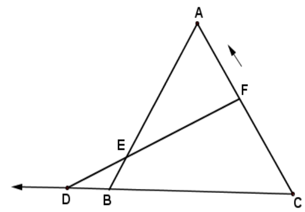
Appendix 1

Student worksheet



In the figure to the right, ABC is a triangle, E $\in]AB[$, $F \in]AC[$, D is a fixed point on the extension of [CB], and points D, E, F are collinear.

Answer the questions below concerning the given information.

Q1. Which indeterminate form does the product $\frac{|CF|}{|FA|}\cdot\frac{|AE|}{|EB|}$ yield as point F approaches closer and closer to point A? Answer: .....

Q2. Which indeterminate form does the ratio $\frac{\frac{|CF|}{|FA|}}{\frac{|EB|}{|AE|}}$ yield as point F approaches closer and closer to point A? Answer: ......

Q3. Which indeterminate form does the ratio $\frac{|AE|}{|FA|}$ yield as point F approaches closer and closer to point A? Answer: ......

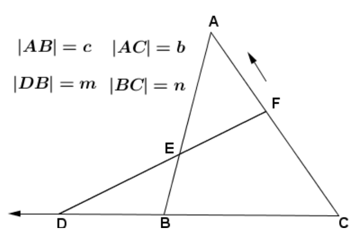
Open the GeoGebra file named Work. You will find the above figure constructed on the opening window. By clicking on the input boxes at the upper left corner of the graphical window, you can assign new values for the given parts of the construction (leave them in default until you move to step 2). By clicking on the question buttons, you will make visible the associated values on the screen. Point F lies on [AC] and can be dragged; as you drag it, you can monitor the values of the resulting segments on the screen. Try to answer the questions below.

Step 1. Drag point F as close to point A as you can (use the magnification tool) and monitor the values of the product and the ratios in the above questions. Based on your experimentations, try to determine the results of the limits below.

a) $\lim_{F \to A} \frac{|CF|}{|FA|}\cdot\frac{|AE|}{|EB|} =$ b) $\lim_{F \to A} \frac{\frac{|CF|}{|FA|}}{\frac{|EB|}{|AE|}} =$ c) $\lim_{F \to A} \frac{|AE|}{|FA|} =$

Step 2. Assign new integer values to $|AB|$ and $|AC|$ and then reinvestigate the limits above.

Step 3. Assign new integer values to $|CB|$ and $|DB|$ and then reinvestigate the limits above.



Step 4. Let the lengths of the segments in the construction be given as variables, as seen in the figure to the right. Based on your experiments so far, and considering Menelaus' theorem, can you determine the results of the above limits in terms of these variables?