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Learning difficulties experienced by students and their misconceptions of the inverse function concept

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The aim of this study is to determine students' learning difficulties and misconceptions related to the "inverse function". The study group was composed of 137 first-grade students enrolled in the elementary mathematics teaching program of an Eastern Anatolia University in Turkey during the fall term of the academic year 2010–2011. The case study method was employed in the study. Three different question forms with one open-ended question are used as data collection instruments. Also, students were interviewed to conduct a more detailed analysis of their responses to the open-ended questions, and semi-structured interviews were used as a data source in this study. As a result, it was concluded that the students experienced difficulties in demonstrating surjection of a function when finding the inverse of a given function and had misconceptions like "If a function's inverse is not available, the inverse image of any element in the range is not either" and "The expressions of inverse function, inverse of a function, and inverse relation all have the same meaning." Some specific teaching strategies are recommended based on the conclusions to prevent misconceptions and address learning difficulties specific to the topic of inverse functions.

Key words: Inverse function, learning difficulties, misconceptions, surjective function, inverse relation.

INTRODUCTION

The concept of a function is one of the basic and central concepts in mathematics. It emerges from the urge of humans to uncover patterns among quantities, which is as ancient as mathematics (Gagatsis and Shiakalli, 2004; Iliada et al., 2007). The concept of a function is fundamental for undergraduate mathematics and is essential in related areas of the sciences. A clear understanding of the function concept is also crucial for any student to better understand calculus - a critical direction for the rise of future scientists, engineers, and mathematicians (Carlson and Oehrtman, 2005). This concept taught as part of mathematics curricula in Turkey as it does in other countries (NCTM, 2000; NCTM, 2009). In Turkey's high school and higher educational mathematics curricula, the concepts of set, ordered pair, and relation are taught in this order according to Bourbaki's definition, which is based on the set concept. The function concept, however, is introduced as a particular type of relation (Jones, 2006). Therefore, the function concept is explained visually by using the set mapping diagram, and it is presented with four different representations (set mapping diagram, ordered pairs' set, equations, and graphics), following a conceptual definition (Davis, 2005; Akkoç, 2006; Jones, 2006).

Students have been introduced to function and relation concepts early in life as in certain kinds of mapping like child-mother or citizen-ID number relations, which is the base of the function concept, before they start school. Elementary school students experience this concept intuitively while performing addition and multiplication operations which can be identified as functions in natural numbers' set. For example, the multiplication symbol (x)

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in the equation 3x6=18 converts the operand 3 and the operand 6 of natural numbers' set to another element, 18. Similarly, finding symmetry in a geometric shape acts as a function (Bayazıt, 2010; Bayazıt and Aksoy, 2010). Such practices of the function concept lay a foundation for learning the formal definition of a function introduced in the secondary school mathematics curriculum.

A relation is a generalization of arithmetic relations, such as = and <, that occur in statements, such as "5 < 6" and "2 + 2 = 4". An inverse relation, on the other hand, is the relation when you switch the elements in the relation. An example of an inverse relation would be the relation for "child of" and the relation for "parent of". A relation defines a link among sets.

Functions are relations only when every input has a different or separate output and named according to their properties. Two of them are injective and surjective functions. A function is injective (one-to-one) if there is a relation between a set of inputs (domain) and a set of acceptable outputs (range) with the property that each input is related to exactly one other output. A function is surjective (onto) when each and every element of the domain has a link to each element in the range and no element stays unconnected. The function concept is related to many other concepts like set, ordered pair, Cartesian multiplication, and relation. Each of these concepts has some specific features, and they are presented in the forms of algebraic and graphical multi-representations. Thus, learning the concept of a function becomes more difficult (Jones, 2006). Literature also states that students experience difficulties in dealing with the concept of a function when using textual, algebraic and graphical representations involving daily life situations (Baker et al., 2000; İşleyen, 2005; Montiel et al., 2009; Bayazıt and Aksoy, 2010). Researchers also report some difficulties in student learning in understanding the roles of dependent and independent variables in algebraic presentations (Zaskis et al., 2003; Li, 2006; Polat and Şahiner, 2007) and have many misconceptions.

An inverse function is a function that undoes the other function. Square and square root functions are examples of an inverse function within the domain of nonnegative numbers. When a function has an inverse function, it is called invertible. Not every function is invertible, but every function has an inverse relation, even if not an inverse function. The inverse function concept is one of the important topics in secondary and high educational mathematics subject programs, and includes two more concepts, injection and surjection, in addition to the conditions of being a function, and a different presentation like f^{-1} is used. Thus, learning these concepts becomes harder. According to previus studies, students experience learning difficulties with the concept of an inverse function or have misconceptions (Carlson et al., 2005; Ural, 2006). The essential reason for such difficulties and misconceptions is the fact that the concept of an inverse function is generally taught based on memorization and routine rules (Wilson et al., 2011). Such type of teaching approach prevents students from performing the operations in a correct way, understanding how they are used, and interpreting them (Baki and Kartal, 2004; NCTM, 2009).

Wilson et al. (2011) reported that there are two types of misconceptions related to the inverse function. The first one is to write the inverse of the $y = f(x)$ function as $y = f^{-1}(x)$. This is a common conceptual mistake made in finding inverse of a function. The second emerges in obtaining the graph of the inverse function by finding its symmetry according to the line of $y = x$ after the diagram of a given function is drawn-in other words, presenting the function and the inverse of a function on the same coordinate plane. The basic reason for this and similar mistakes is the replacement operation taking place between x and y variables since this replacement operation changes the meaning of the variables (Bayazıt, 2010; Carlson and Oehrtman, 2005).

It is possible to see some expressions like "the inverse of $y = f(x)$ function is $f^{-1}(x)$" and "presenting a function and inverse of a function on the same coordinate plane" as stated in the secondary school mathematics curriculum (MEB, 2011). Similar expressions may be seen in textbooks at the secondary and higher educational levels. Using such expressions in teaching inverse functions or selecting teaching strategies not complying with the subject may cause the formation of misconceptions. These are important obstacles in meaningful learning and play a role in failing to eliminate mistakes (Ann and Miroslav, 2009). It is believed that determining students' misconceptions and learning difficulties with regard to inverse function will make a significant contribution to the field, foster understanding, and improve student performance in finding inverse function as well as other related subjects.

Objective

The objective of the study was to determine students' learning difficulties and misconceptions related to the inverse function. Therefore, answers for the following questions were investigated:

1. What are the common mistakes made by students, and what are learning difficulties experienced by them in finding the inverse function of a given function as well as the reasons behind these issues?
2. Which types of misconceptions do students have in regard to the concepts of inverse function, inverse of a function, inverse relation, and inverse image of an element?
3. What are the common reasons why students might have misconceptions about inverse function, inverse of a function, inverse relation, and inverse image of an

element?

METHOD

Qualitative methods were employed in this study. The case study design was used because it was believed that it complied with the nature of the study. According to Yıldırım and Şimşek (2011), the case study is used to obtain detailed information about the subject under study and to deeply understand the case from various perspectives.

The group under study

The group under study was composed of 137 first grade students enrolled in the elementary mathematics teaching program of an Eastern Anatolia University in Turkey during the fall term of the academic year 2010–2011. These students are separated into three different groups. The first group involved 43 students, the second group made up 52 students, and third one was consisted of 42 students. All the students enrolled in general mathematics course offered by the same lecturer, and they were taught the concepts of relation and function along with some other topics during this course.

Data collection

Three different question forms for the three groups were used as data collection instruments. Each form had one open-ended question related to functions. Each group of students is asked to type their solutions in a different form. Also, students were interviewed to collect detailed information about their responses and these semi-structured interviews were used as a data source in this study.

The researcher, in his previous experiences, observed a common student tendency of confusing the inverse function with the inverse image of an element and to use all three of the concepts (inverse function, inverse of a function, and inverse relation) as if they were the same concept. Questions in forms were formulated based on these personal experiences. Which question form would be used for which group was determined randomly. The group to which the first question form would be distributed was named as the first group. Similarly, the rest of the groups were called the second and third groups, respectively.

Each question form included an open-ended two-step question about functions. The first form included a question about finding the inverse function and the inverse image of an element for a given function to discover whether the students know the difference between an inverse function and an inverse image of an element. The second form asks for the inverse of a given function as well as the inverse image of an element. The third question form includes a two-step question requiring finding the presence of an inverse relation and the inverse image of an element for a given function to discover whether students know the difference between an inverse relation and an inverse image of an element, similar to the other question forms. Furthermore, all three question forms questioned the presence of an inverse function, presence of inverse of a function, and presence of an inverse relation, respectively, for a given function. The aim was to collect data about whether students' answers about inverse function are similar to their answers to inverse of a function or inverse relation concepts for a given function. Questions about inverse function, inverse of a function and inverse relation appear in separate question forms and each of the question forms was distributed to separate groups because it was believed that a question might have provided a clue for the next question, and the misconceptions might not have been detectible, if any. It was also useful to use the same function in the questions for all of the groups, and this would help in ensuring equivalency among respondents without including other variables in the process.

Semi-structured interviews were carried out after the analysis of student responses in question forms. Students to be interviewed were selected as indicated in the next section. The aim of the interviews conducted was to determine whether the students consciously used expressions and operations and to analyze the written answers given to in the question forms in more detail. Interview questions were asked to the students about the reasons for their answers of the written questions. Additional questions were asked where needed to elaborate on their answers. Interviews were conducted face-to-face with every student, and the interviews were recorded by an audiorecorder with the consent of the student and were transcribed later.

Data analysis

Written answers of the students were coded with a code number. The answers were categorized into three groups. Student answers were grouped as "present" when they specifically point out a feature as injection, surjection, inverse function, inverse of a function, or inverse image of an element, depending on the question. When the student specifically states a feature as absent, this answer was classified as "absent". If there is no specific indication about a feature, then the answer was classified as "unanswered". Percentages and frequencies were then calculated for each category.

In the first group, each of the students with correct answers to both steps was interviewed. The reasons for their answers to the first and second steps of the first question were probed in the interviews and the data were analyzed descriptively. Interviews were not conducted with the second group of respondents. In the third group, each of the respondents from the "unanswered" category was interviewed. A randomly selected one-third of the third group from the "present" or "absent" categories was also interviewed. These students were asked to explain the reasons for their use of expressions and operations. The students' written answers and the data produced from the interviews were analyzed descriptively. Approximately 15 min were offered for answering the written question form, while each interview took approximately 20 min.

In the qualitative approach, validity means that the researcher investigates the research field as objectively as possible without any disambiguation. The researcher should use some additional methods to support the data and confirmatory information from participants and colleagues to achieve an integrated picture about the fact or event. It is important for a qualitative research study to report the data in detail and reveal how the conclusions are drawn. This provides validity and reliability for the study (Yıldırım and Şimşek, 2011).

In the present study, five different experts-one from measurement/evaluation, one from mathematics, and the rest mathematics education- were consulted and their views were asked to determine whether the questions in the question forms and interviews were suitable for the purposes of this study. The collected data were presented descriptively. The data were obtained through written questions and interviews. Attention was paid to getting the approval of the relevant participant during the interviews by asking questions like "Do you mean this?" to the student. Furthermore, findings and results from the research were presented to the relevant experts to confirm their approval. Necessary adjustments were made according to views of the experts, and the findings and results were finalized. On the other hand, repeatability is another significant issue for qualitative studies to ensure reliability for the research (Yıldırım and Şimşek, 2011). The measures taken for ensuring repeatability of the study were reported explicitly in detail.

Table 1. Students' answers about finding "inverse function" and "inverse image of an element" of a given function.

Question 1 (N=43)	Present		Absent		Unanswered	
	f	%	f	%	f	%
Being 1 to 1	41	95.4	1	2.3	1	2.3
Surjection	21	48.8	21	48.8	1	2.3
Inverse function	21	48.8	22	51.,2	-	-
Inverse image of an element	32	74.4	11	25.6	-	-

Since the questions for the written part of the study and all questions of the interviews are presented in the chapter of findings, they are not mentioned here again.

FINDINGS

This chapter includes written answers of the students to the open-ended questions related to the inverse function concept and the findings obtained from the semi-structured interviews conducted with the students.
The question for the first group is as follows:

Question 1**:** for the function of $f: \mathbb{R} - \left\{-\frac{1}{3}\right\} \longrightarrow \mathbb{R},\ f(x) = \frac{x+3}{3x+1}$ (a)find the inverse function, if any, and (b)find $f^{-1}(5) = ?$, if any

Frequencies and percentages for the answers provided by the students for this question are seen in Table 1.

According to Table 1, 95% of the students taking part in the first group say that the function is injective (one-to-one). It seems 2.3% of the students fail to indicate this feature. On the other hand, almost half of the students (48.8%) say that the function is surjective while in fact it is not. Some of these students say that the function is surjective without making relevant operations while the rest claim that the function is surjective according to their imperfect calculations. Surjection requires the presence of at least one x value in a domain for each y element in a codomain (range). It is seen that the students whose operations were imperfect mostly ignored the fact that y is a member of the range or the meaning of *each* in the expression "for each y element". The answer of one of these students, Ö-101, is as follows.

In Figure 1, the student's result shows there is no definite solution for $y = 1/3$. So the student ignores the condition and concludes that there is a solution for each y value, while in fact, there is not.

Table 1 shows 51.2% of the students concluded that there is no inverse function for the given function. It is interesting to see that only 48.8% of the students correctly point out that there is no surjection feature while surjection is a condition for the presence of an inverse function.

Another interesting point in Table 1 is that approximately one-fourth of the students (25.6%) believe that "if a function's inverse is not available, inverse image of any element in the range is not either." The answer of one of these students, Ö-98, is as follows.

According to Figure 2, the student correctly concludes there is no surjection; therefore, no inverse function is available. However, s/he claims there is also not a definite $f^{-1}(5)$ value. The misconception here is even if the inverse function is not available, the inverse relation may produce the result for the inverse image of an element.

According to the analysis of the written answers of the first group, it was seen that nine students (20.9%) correctly found the inverse image of an element where the inverse function is not available-in other words, image of an element in the range according to the f^{-1} relation. The answer of one of these students, Ö-132, is as follows.

The student's answer in Figure 3 indicates no misconception about the inverse function and inverse relation. S/he concludes that there is no surjection or inverse function, but they correctly state and calculate the $f^{-1}(5)$ value.

Semi-structured interviews were conducted with the students who had answered this question in written form to find whether they used their expressions and operations intentionally or unwarily. In the interviews, the students were asked why they had found "the value of $f^{-1}(5)$ while saying that inverse function is unavailable". One of these students, Ö-134 explained, "in fact, I believe that I made a mistake. Normally, it is meaningless to look for the value of $f^{-1}(5)$ where an inverse function is unavailable". Ö-130 and Ö-131 said that "they always looked for the values in inverse of a function generally by using the test technique with no respect to which function is given in high school or private courses, but they did not know the reason". Similarly, Ö-135 and Ö-136 explained the reason that "the value desired in the inverse exists in domain of the given function; in other words, it does not make the function undefined, and therefore, they found the inverse value". Ö-129, Ö-133, and Ö-137 stated that "they found the solution in the inverse formulae because this value is computable in the inverse formulae". Among these students, only Ö-132 explained the reason correctly that "the inverse of the function may not available. However, a value may exist in its inverse. If A

Table 2. Students' answers about finding "the inverse" and "inverse image of an element" of a given function.

Question 2 (N=52)	Present		Absent		Unanswered	
	f	%	F	%	f	%
Being 1 to 1	50	96.2	2	3.8	-	-
Surjection	34	67.3	17	32.7	1	1,9
Its inverse	37	71.2	15	28.8	-	-
Inverse image of an element	43	82.7	9	17.3	-	-

Table 3. Students' answers about finding "the inverse relation" and "inverse image of an element" of a given function.

Question 3 (N=42)	Present		Absent		Unanswered	
	f	%	f	%	f	%
Being 1 to 1	35	83.3	2	4.8	5	14.4
Surjection	28	66.6	9	21.4	5	11.9
Its inverse	32	76.2	9	21.4	1	2.4
Inverse image of an element	36	85.7	6	14.4	-	-

is domain and B is the range, the value of 5 exists in B, while a value exists in A corresponding to 5 value in B. This indicates presence of the inverse image. The function is not surjective, but its image may exist for a value in its inverse".

The following question was asked of the students in the second group:

Question 2: for the function of $f: \mathbb{R} - \left\{-\frac{1}{3}\right\} \longrightarrow \mathbb{R},\ f(x) = \frac{x+3}{3x+1}$,

$$\text{(a) find the inverse, if any, and (b) } f^{-1}(5) = ?$$

Frequencies and percentages for the answers provided by the students for this question are seen in Table 2.

According to Table 2, 96% of the students in the second group say that the function is injective. Similar to group 1's answers, more than half of the students (67.3%) say that the function is surjective, although it is not. Some of these students say that the function is surjective without making relevant operations, while the rest claim that the function is surjective according to their imperfect calculations. Written answers of these students were analyzed and it was found that they had made the same mistakes as the first group mentioned above about the function being surjective. As seen in Table 2, like group 1, some students (17.3%) had concluded that because the inverse function was unavailable, any image of an element was also unavailable according to the relation of f^{-1}. Furthermore, it was observed in analyzing written answers given for the second question that approximately one fifth of the students (21.1%) correctly found the inverse image of an element in the range in cases where the inverse function is not available just as it was seen in analysis of the written answers given to the first question.

The question which was asked to the students taking part in the third group is as follows:

Question 3: for the function of $f: \mathbb{R} - \left\{-\frac{1}{3}\right\} \longrightarrow \mathbb{R},\ f(x) = \frac{x+3}{3x+1}$ find, if any,

$$\text{(a) the inverse relation and (b) } f^{-1}(5) = ?$$

Frequencies and percentages for the answers provided by the students for this question can be seen in Table 3.

According to Table 3, most of the students (88.1%) had provided similar answers to the answers of those taking part in the second and third groups, although the question asks for the inverse relation of the given function and the conditions for being the inverse function. The answer of one of these students, Ö-59, is as follows:

In Figure 4, the student writes, "injection and surjection conditions must be met for an inverse relation", which is a condition for an inverse function, not for an inverse relation. Here, the student displays a clear misconception by interchanging the inverse function concept with the inverse relation concept.

From Table 3, it can be seen that 83.3% of the students in the third group claims the function is surjection. More than half of the students (66.6%) say that the function is surjective when,in fact, it is not. The students, who found that the function was not surjective (21.4%) stated that it had no inverse relation either. Some of these students (14.4%) stated that the inverse image of an element would not be available when they could not find the inverse relation. Only three students (7.14%) answered the second part of the question and found the inverse

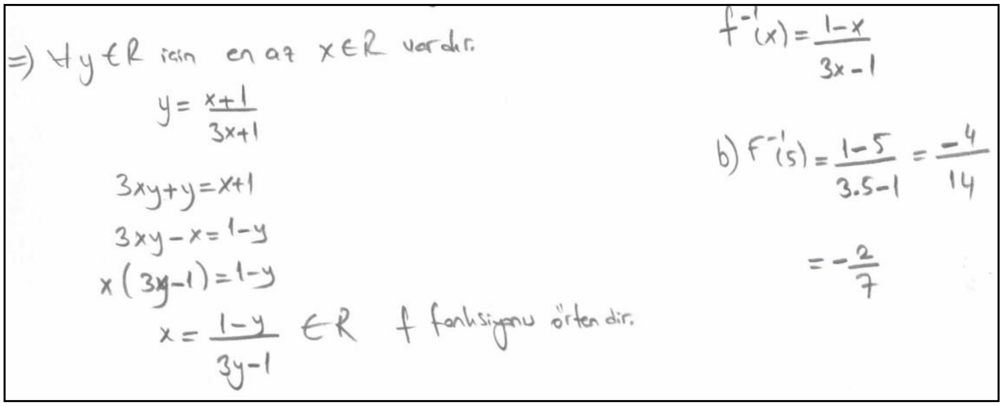

Figure 1. Answer of a student considering the surjective function.

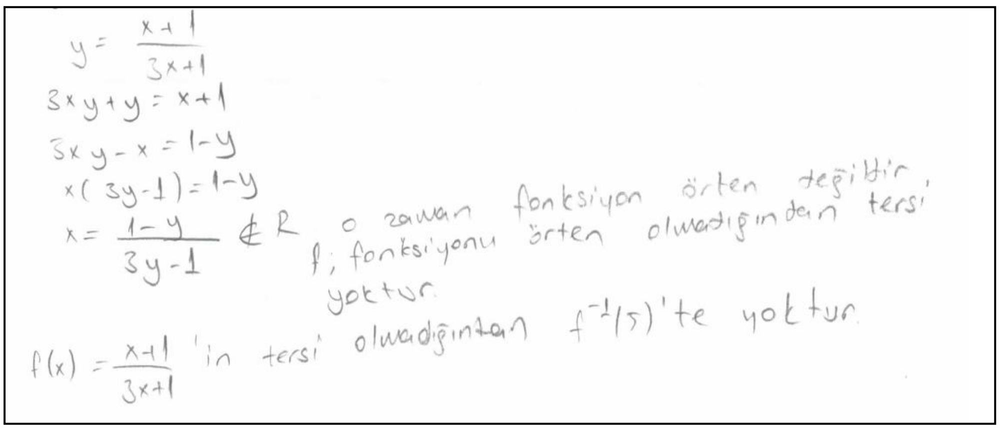

Figure 2. A student's answer saying that "if a function's inverse is not available, inverse image of any element in the range is not either" .

image of an element.

It was seen that only five students of those taking part in the third group (11.9%) had not checked the conditions for being surjection and surjection for the inverse relation, which is the correct approach. Semi-structured interviews were conducted with the students in this group like those taking part in the first group to discover whether or not they knew the real meaning behind the mathematical operations. The students were asked "why they did not question injection or surjection of the function while they were finding the inverse relation" during the interviews. The common emphasis in their reasoning is that they had memorized the formulae for finding the inverse previously, and therefore, they answered the question via the formulae without inspecting injection and surjection. Two of these students, Ö-76 and Ö-77, said that "in fact, we should check injection and surjection also for availability of the inverse relation". Ö-90 said that "to the extent I remember from my high school period, we do not need to check injection and surjection for inverse of a function. We found the inverse and value by using the formula". Ö-75 said that "as far as I know, if there is no value that makes the function undefined, the inverse is available".

Considering Tables 1 - 3 as a whole, another interesting point is the fact that students provided similar answers for three different questions. Different questions including the expressions of "inverse function", "inverse of a function" and "inverse relation" were asked to the students for a given function with the first, second, and third questions, respectively. Approximately 90% of the students in the third group to which the question of "Find the inverse relation, if any" answered this question incorrectly. Although inverse function and inverse relation are concepts required for different conditions, the students answered the question considering injection and surjection conditions for the inverse relation concept.

Semi-structured interviews were conducted with 13 randomly selected students to find the reasons for the errors made by the students who had answered the question incorrectly. The questions which were asked of the students during interviews are "If the question required the inverse function instead of the inverse

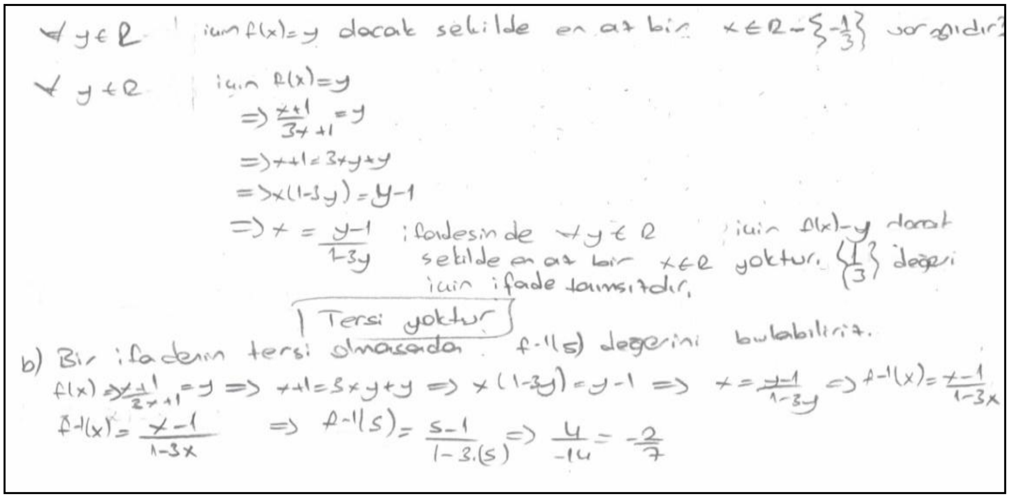

Figure 3. Answer of a student finding inverse image of an element in the range in cases that inverse function is not available.

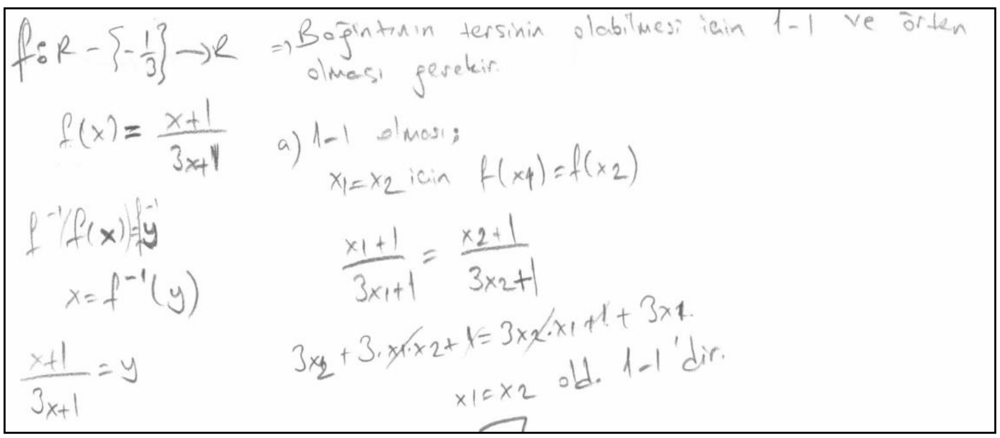

Figure 4. A student's answer examining 1 to 1 and surjection conditions for the inverse relation.

relation for the given function, would you have still made the same operations? and why?" Ten students answered the question by saying that they "would have made the same operations". In other words, "If the question required the inverse function, we would have checked conditions for being injection and surjection", while one student said that "I checked injection surjection because I perceived it as 'find the inverse function', not as the inverse relation. I would have not checked had I perceived it correctly because I don't know what an inverse relation means". Two students said that they would have not made the same operations. Ö-88, who is one of the students saying that they would have not made the same operations, said that "I would have not checked conditions for inverse of the relation, injection and surjection, but not the relation, and I would have written inverse of the function". Ö-55 stated his reason that "I was finding inverse of the function in case of inverse of a function only by replacing x and y with each other". One of ten students who stated that they would have made the same operations-in other words, they would have checked injection and surjection-did not state any reason. One student assessed the function and inverse relation concepts as the same expression. Three of these students indicated their reason as the presence of the inverse term in the question root, while five students - stated their reason had to do with the relations between the relation and the function, believing their existence, and emphasized the similarity between inverse relation and inverse function. Ö-63, who is one of these students, stated that "I would have checked injection, surjection even if it was the inverse function because it depends on variable x. A relation depends on two variables, like a pair of (x, y) while a function x depends on a single variable. Injection, surjection should be checked because there is an unknown in both of them".

Ö-56 stated that “because a relation is also a function, the inverse relation will be the inverse function. Therefore, injection, surjection should be checked for both of them”.

RESULTS AND DISCUSSION

It was concluded as a result of the study that more than half of the students (60.6%) experienced difficulties in proving surjection of a function. Approximately half of the students in the first group (48.8%), more than two-thirds of the students in the second group (67.2%), and two-thirds of the students in the third group (66.6%) made a mistake by finding the function as surjective, although it is not. Reasons for mistakes are the same for three of the groups. According to the analyses of the written answers, students especially ignored the fact that y takes place in the range and the meaning of *each* in the concept that for each y element related to the condition for surjection, the existence of at least one x value in the domain corresponding to each y value in the range is required. Similar studies (İşleyen, 2005; Şandır, 2006) reported that students experience difficulties proving the existence of the condition for surjection. Students do not experience difficulties in judging if a function is injective, while they experience difficulties in exploring if a function is surjective. The reason for this may be that evidencing injection is a process required of operation at an application level, while inspecting surjection is a process done with higher-level operations, rather than those at the synthesis level. The result also shows that students do not have sufficient prior learning experiences to learn about the inverse function concept. However, learning requires suitable preliminary knowledge in addition to other conditions. Learning becomes easier if an individual relates the new knowledge with other prior learning experiences. As a result, it is believed that eliminating the students’ weakness of judging the condition of surjection may result in more efficient learning of the inverse function concept.

Another result obtained through the study is the fact that some students (18.97%) have the misconception that the image of any element in the range according to f^{-1} is unavailable when inverse of an f function is absent. Students having misconceptions may use incorrect approaches and may produce incorrect results (Xiaobao and Yeping, 2008). It is believed that the reason for this misconception is how the inverse function is defined because the definition of the inverse of a function generally includes a phrase saying that the inverse of a function is represented by f^{-1}. Naturally, this definition leads to a misconception that f^{-1} must always represent a function. Therefore, students believe that the inverse image of an element is not available if they cannot find the inverse function. They say that “there is no inverse function. Because there is no f^{-1}, there is no value like $f^{-1}(5)$”. However, the inverse of an f relation is ( ) represented by f^{-1} also in the subject of the relation; in other words, the symbol of f^{-1} represents both the inverse relation and the inverse function. Consequently, an f^{-1} relation must be available even if the f^{-1} function is always -available for a given function, considering the fact that a function is a special relation. Furthermore, according to the definition of an inverse image of an element as follows,

$$for\ \ f: A \longrightarrow B,\ \ y = f(x)\ function, if\ b \in B, the\ inverse\ image\ of\ \ b\ element\ is\ f^{-1}(b) = \{x \in A: f(x) = b\}$$

(Kadıoğlu and Kamali, 2009).

It can be clearly seen that the image of an element in the range according to f^{-1} may be available independent from presence of the inverse function. Defining the concept of an inverse function in association with relation and inverse image of an element may help eliminating misconceptions, which is very likely to emerge, especially on this subject. It was seen from the results that some students correctly found the inverse image of an element without checking for the inverse function; however, they did this without knowing the real meaning of the operation because they memorized the operation previously. The students who answered the question correctly failed to explain their reasoning for their operation during the interviews. This may be seen as the strongest evidence of rote learning. Posing problems from some daily life events to teaching abstract concepts may help learners better conceive real meanings of the concepts and operations since the language used in expressing concepts and selecting teaching strategies not suitable for the subject may result in misconceptions (Jones, 2006).

Another significant result obtained by the study is the fact that a significant number of students have misconceptions like attributing the same meaning to different concepts like inverse function, inverse of a function, and inverse relation. In other words, students use the same operations and procedures to solve the questions related to inverse function, inverse of a function, and inverse relation for a given function. Some students attributed the same meaning to the concepts of “inverse function” and “inverse of a function”. It is believed that the reason is that both concepts are used in their secondary school curriculum (MEB, 2011) and secondary school and college textbooks (Kadıoğlu and Kamali, 2009; Komisyon, 2011) and appear to have the same meaning when teaching the concept of an inverse function. However, if the relation produced in finding the inverse of a given function ensures conditions for a function, it indicates the inverse function, but if it does not ensure them, it indicates just a relation. Thus, if the terms *inverse function* or *inverse relation* for a given function are used instead of “inverse of a function” in secondary school curricula and higher educational programs, this may help prevent potential misconceptions. Some students in the sample attribute the same meaning to the concepts of “inverse function” and “inverse relation”. A significant part of the students (88.3%) checked both the conditions of injection and surjection to answer a question requiring finding the inverse relation of a given function, although they did not need to check for the conditions of injection and surjection. On the other hand, it was understood that the students who did not check for injection and surjection conditions did not check them because they mistakenly believe that they can find the inverse formula and the value by replacing x and y variables with each other based on their knowledge acquired previously without knowing the difference between an inverse function and an inverse relation. If the relation between an inverse function and an inverse relation is highlighted in teaching the concept of an inverse function in secondary and higher educational courses, this may help eliminate the potential misconceptions in regard to this subject.

Considering the findings as a whole, it is seen that learning difficulties and misconceptions related to the inverse function generally emerge from the fact that the real meanings of x and y variables, which are used in finding an inverse function, and interchanging them are not thoroughly acknowledged by the students. Research findings supporting this result are also available in the literature (Bayazıt, 2010; Wilson et al., 2011). If the approach of determining the dependent variable and solving the problem according to this variable is preferred to the approach of the replacement of x and y with each other in the planning and practicing process for teaching activities requiring finding the inverse function, this may help prevent misconceptions, which have been found in this study. If such type of teaching approach is preferred, this may assist learning about inverse function at a higher level and may support learners’ motivation and achievement levels (Wilson et al., 2011).

Students with misconceptions can follow erroneous approaches when solving problems which in turn, produce incorrect solutions. The results of this study imply better student understanding and less misconception when the content is consistently presented in textbooks and curricula with special attention paid to notations, definitions and operations. It would also be useful when differences and similarities between concepts are clearly emphasized to construct clear distinctions. To better demonstrate abstract concepts of algebra such as inverse function, inverse relation, and inverse of a function, it can be suggested to use daily life examples for students to help them conceptualize the operational meaning of these terms. In summary, it was seen that the students in the study have certain learning difficulties and misconceptions related to the concept of inverse function. Similar studies may be conducted to find whether or not student groups in different locations and at different levels have such types of learning difficulties and misconceptions. Furthermore, experimental studies should be conducted to determine the effects of different teaching approaches on learning processes related to the inverse function, and then the teaching style for this concept may be reviewed according to the findings from future studies.

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