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Full Length Research Paper

The construction of deductive warrant derived from inductive warrant in preservice-teacher mathematical argumentations

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This study discusses the construction of deductive warrant derived from inductive warrant in mathematical argumentations expressed by pre-service teacher. In completing a mathematics task, a problem solver needs argumentation to determine, reveal, and support a reasonable solution. A mathematical argumentation can be analyzed by Toulmin scheme consisting of data, claim, warrant, backing, refutation and qualifier. This study focuses on *warrant* because it is one quality determinant of an argumentation. Inglis, Mejia-Ramos, and Simpson (2007) mentioned types of warrant in mathematical argumentation, including inductive, structural-intuitive, and deductive. This study aims to describe the construction of deductive warrant in mathematical argumentation. Students must use deductive warrant in mathematical argumentation, and that the truth of the conclusion obtained is absolute and there is no rebuttal. This study uses qualitative approach with written works, think aloud, and interview to collect data. The subjects are asked to investigate the truth of the mathematical statement. The result shows that the subjects, in constructing a deductive warrant, initially used inductive warrant. The subjects use an inductive warrant to reduce the uncertainty of the result, so the needs of using deductive warrant exist to reveal a certain conclusion.

Key words: Construction, warrant, deductive, inductive, mathematical, argumentation.

INTRODUCTION

Argumentation is a person's competence to link data to make a claim (Jimenez et al., 2000). It is a kind of informal reasoning, which is central to intellectual competence in problem-solving, judging and decision making, generating idea and faith building (Kuhn and Udell, 2003). When students are able to do argumentation, they can leave their hesitancy in completing a task, they can be more acquitted to select an idea as well. They can even suggest a reasonable answer for the task.

Although competence to argument is necessary in completing a task, many students failed to do that. Cerbin (1988) mentioned that students are not well-versed in constructing a convince argument. Kuhn (1992)

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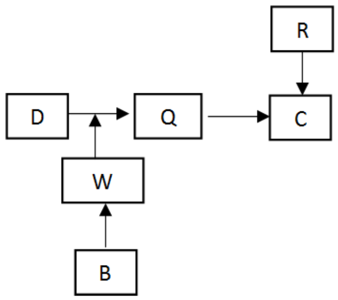


Figure 1. Toulmin scheme for general argument.

suggested that argumentative skill in reasoning did not evenly occur in every school. He also showed that capability to make a reasoned assessment must become a part of capability to think well. Brem and Rips (2000) and Klaczynski (2000) mentioned some difficulties and limitations by the subjects in their youth and young adults in the term of constructing and improving arguments. Kuhn (1991) investigated 160 students in various ages and found that only few of them consistently improved qualified arguments. Thus, a study of mathematical argumentation is further necessary to conduct as an evaluation toward a process of students' thinking and a model of mathematical argumentation which is likely to exist.

A person's argumentation needs to be analyzed using a richer format so he/she does not merely distinguish between premises and conclusion (Toulmin, 2003). Hence, he proposed a layout of argument, known as Toulmin Scheme. It consists of data (D), claim (C), warrant (W), backing (B), rebuttal (R), and qualifier (Q). Data refers to facts for supporting claim. Claim refers to a proposition supported by the data. Warrant is a guarantee for data in supporting the claim. The warrant is supported by backing, and backing provides further evidence including legal basis as the foundation of warrant. Refutation refers to an exceptional condition for an argument, and qualifier enables it to reveal the strength of the data toward the claim by the warrant (Figure 1).

In analyzing a mathematical argumentation, Toulmin scheme can be use. In mathematics education, Krummheuer (1995) started the trend of using Toulmin scheme to analyze the mathematical argumentation. However, Krummheuer (1995), Whitesnake and Knipping (2002) and Conner (2007) focused on the mathematical argumentation in the discussion. Before someone builds a valid argument with others, it is better they build a valid argument individually, so that the individual will be ready to discuss. An argument can exist in a dialogue or non-dialogue (Walton, 1990). Critically, discussion is one instance that arguments exist in a dialogue that every participant attempts to bring out a correct view through their arguments addressed to other participants. Planning or solving a problem is one instance of argument in non-dialogue. While planning or solving a problem, an interactive reasoning can naturally take place, that the same person, in turns, plays roles as an initiator and respondent. Self-approach and self-debate will exist as well. Therefore, the focus of this research is the analysis of mathematical argumentation individually.

This study focuses on the components of warrant used when mathematical argumentation occurs. Toulmin (2003) also mentioned the importance of recognizing and understanding mathematical argumentation especially in the re-construction of the components of warrant. He stated that there is a chance to conduct a research on how a person builds a warrant in the field of mathematics by attempting to demonstrate variability or field-dependence. The warrant of an argument may depend on a particular limitation, of which to be concerned so that the truth of the argument will never be conflicted.

Weber and Alcock (2005) suggested that at least, there are three important elements as a core of an argument: data, conclusion, and warrant. Whenever a person brings out an argument, he/she is attempting to persuade audience with particular statement called conclusion. Supporting the conclusion, a presenter usually keep on providing evidence or data. An explanation presented by the presenter on how the provided data may support the conclusion is called warrant. In this phase, the audience may accept the data but not for the explanation stating that data determine the conclusion. In other words, the authority of warrant can be resisted. If this happens, the presenter needs to present additional supports for correcting the warrant. Thus, the warrant is the core of a valid argument. This study focused on the warrant of Freeman and Inglis (2005) and Ramos and Simpson (2007).

Freeman (2005) classified warrants into priori, empirical, institutional and evaluative. Inglis et al. (2007) classified three kinds of warrant: inductive, structural-intuitive, and deductive. The researchers refer to the classification of the types of warrant (Inglis et al., 2007) because the category has similarities with the proof-schemes outlined by Harel and Sowder (1998). Harel (2001) and Tall (2004) stated that it is a must to use a deductive warrant in undergraduate level. Therefore, this research focuses on how undergraduate students construct such a deductive warrant.

The researchers selected undergraduate students of Mathematics education program as the subject of their study; due to the fact that they are pre- service mathematics teachers for the future who will affect the development of students' thinking process in mathematical argumentation. Students must be able to

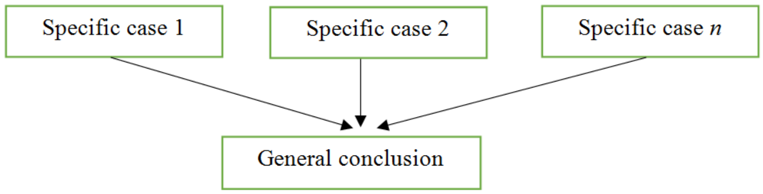


Figure 2. A Process of inductive warrant thinking.

make an argument for establishing the validity of allegation (NCTM, 2000). The students' argument depends on the construction of theorem culture during the class, the characteristics of the given task, and the kind of certain reasoning teacher emphasizes (Boero et al., 1999).

Thus, what teachers do can encourage their students to explain, note, and correct their intention during a class discussion. Whitenack and Knipping (2002) and Yackel (2002) found that what teacher did would encourage his/her students to explain, note, and correct their intention during the class discussion in order to improve their argument. Following Conner (2007), a concept of evidence and proof by mathematics teachers affects their practical teaching in facilitating students to argue.

This study aims to describe the construction of deductive warrant in mathematical argumentation done by pre-service teacher in completing certain mathematics tasks. The research questions of this study are as follows:

1. How is the structure of students' thinking in mathematical argumentation based on the components of Toulmin scheme?
2. How is the construction of deductive warrant derived from inductive warrant in mathematical argumentation based on Toulmin scheme?

REVIEW OF LITERATURE

Kinds of warrant in mathematical argumentation

Inglis et al. (2007) classified three kinds of warrant: inductive, structural-intuitive and deductive.

Inductive warrant

Inductive *warrant* occurs when students ensure themselves and persuade some one else about the truth of an allegation by evaluating that allegation in one or more specific cases in order to reduce uncertainty of a conclusion made (Inglis et al., 2007). However, Soejadi (2000) addressed inductive warrant as a mind-set which begun with specific things, and then gradually leads to a general conclusion or common trait. Solso et al. (2008) suggested that in inductive reasoning, a conclusion is often expressed both explicitly and implicitly in the context of a statement of possibility. Following those experts, it can be concluded that inductive is a process of thinking which conclusion is made from specific cases to become a common trait, based on an observation toward that cases. A process of formulating a conclusion through inductive can be seen in Figure 2.

Structural-intuitive warrant

The very first big study of intuition in mathematics education is conducted by Fischbein (1987). Fischbein (1987) suggested that intuition is often described as a kind of spontaneous thinking directly accepted through individual's belief when the information is related to their previous experience. It may be not identically be similar to someone else's belief, but it can be understood by anyone else. Kustos (2010) addressed that intuition in problem-solving can exist through 4 components:

1. Understanding a problem based on instinct, a response in thinking of a problem being faced.
2. Solving a problem based on intervention, relating the problem with learned knowledge.
3. Solving a problem needs a prior perception, and then runs that perception, until the truth of the answer becomes individual belief.
4. Global is that a problem solving conducted in whole package.

Structural-intuitive warrant occurs when students use observation or experiment, some kinds of mental structures whether visual or the otherwise, which leads them to a conclusion (Inglis et al., 2007). Following Fischbein (1987), Inglis et al. (2007) and Kustos (2010), the characteristics of intuitive and analytical/formal thinking can be concluded as indicated in Table 1. Here, structural-intuitive warrant occurs when students use a direct cognition beginning with perception, and then

Table 1. Characteristics of intuitive and analytical thinking.

Intuitive thinking	Analytical thinking
Immediate or spontaneous answer based on direct cognition	The answer is based on systematical and logical thinking
A direct cognition which begins with perception and then run the perception made	The cognition is based on some rules, procedures of completion, applying a formula, definition, and theorem
Resulting some creative ideas	Verify and formulate more precise ideas as the final step of a creative process

running the perception made, which results to a spontaneous or immediate answer based on that direct cognition.

Deductive warrant

Deductive warrant is a correction of formal mathematics which is used to guarantee the conclusion of related argument. Such correction can be derived from various ways: deduction of axioms, manipulation of algebra, or the usage of *counterexamples*, in which all will be classified as deductive warrant (Inglis et al., 2007). Deductive mind-set is simply defined as a way of thinking derived from common traits that are applied or led to specific things (Soejadi, 2000). Harel and Sowder (1998) and Harel (2001) suggested that deductive scheme is the most sophisticated scheme. People with this scheme will use the deduction of axioms to create a truth. However, in this study, deductive warrant refers to a theory from Inglis et al. (2007) being a foundation derived from a justification process of formal mathematics which aims to guarantee the conclusion made. Such justification of formal mathematics comes from a deduction of axioms, manipulation of algebra or the usage of *counterexamples*.

The construction of deductive warrant

The construction of deductive warrant from inductive one as stated in this study refers to the usage of inductive warrant in constructing a deductive warrant. Such construction occurs when students use inductive warrant in their mathematical argumentation. Hence, it will reveal a conclusion, since the qualifier of that conclusion is still probable, thus, the needs of using deductive warrant in that mathematical argumentation will exist in order to eliminate the uncertainty within the conclusion made.

Similarly, Inglis et al. (2007) did not deny that different kinds of warrant may probably exist, like combination of inductive and structural-intuitive. However, a combination found in this study is between inductive and deductive, thus, students construct deductive warrant beginning with inductive.

Many researches have examined argumentation on mathematics field (Krummheuer, 1995; Yackel, 2001; Kuhn and Udell, 2003; Verheij, 2005; Cross, 2009; Bizup, 2009; Tristanti et al., 2015; Tristanti et al., 2016). However, all have not discussed the process of constructing deductive warrant which begun with inductive in mathematical argumentation, yet.

Following the previous studies dealing with Toulmin scheme (Inglis et al., 2007) stating that particular parts of the scheme (backing and refutation as the most frequent ones) are not explicitly verbalized by a presenter due to the fact that the researchers do not only report the data from written answers, but they also explain the attitude and words expressed by the subject of their studies, although the subjects did not directly bring those out. Hence, the researchers need to see three important points: the written works, think-aloud subjects in mathematical argumentation when solving particular problems, and interview conducted toward the subjects.

METHODOLOGY

Subjects

The subject of this study is undergraduate students of STKIP PGRI Jombang of The Mathematics education program in the 6th semester. They are pre-service teacher in junior high and high school. They are selected due to the fact that they, in that semester, have learned relational concept. So that they can have the ability to solve a given problem. Selection of the subjects in this study were students who had argued with warrants deductive through inductive.

Procedures

In the first phase, the subjects are asked to maximally express what they think of a given mathematics task (*think aloud*) during the process of problem-solving. Secondly, the researchers conduct a task-base interview. At a glance, this interview is conducted to find out what subjects are thinking when they conclude and do an act. The question can be in the form of "How do you think of this? Or "what are you thinking right now? Some questions are also delivered to see the reasons behind the way they are thinking.

Instrument

This study applies two kinds of instruments, main and supporting instruments. The researchers place themselves as the main

Some information is given as follows.

Definition 1. Cartesian product A and B are set of all ordered counterpart (a, b) with $a \in$ A and $b \in$ B, in the form of $A \times B = \{(a, b) \mid a \in A$ and $b \in B$.

Definition 2. R is binary relation to set S, if R refers to a non-empty set of S x S.

Definition 3. Given $a, b \in$ S and R is binary relation to set S. a relates to b, notated as a R b if $(a, b) \in$ R.

Definition 4. Given R is a binary relation to set S. S is partially ordered set of R if it meets some criteria:

1. Reflective: if a R a for every $a \in$ S,
2. Transitive: if a R b and b R c, a R c for every $a, b, c \in$ S,
3. Antisymmetric: if a R b and b R a, $a = b$ for every $a, b \in$ S.

Definition 5. Given S is a partially ordered set of binary relation R to S, and A is subset of S. Then, A refers to *chain*, if each of two different elements of a, b in A fits whether a R b or b R a.

Definition 6. S is partially ordered set of binary relation R to S, and B is a subset of S. B refers to antichain, if each of two different elements of a, b in B fits both a R b and b R a.

Investigate the truth of the following mathematical statements.

4. If $\mathbb{Z}$ is a set of integers and P is binary relation to $\mathbb{Z}$ notated as $P = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a - b = 7k$, for k is whole number$\}$, then, binary relation P on set $\mathbb{Z}$ is antisymmetric.
5. If P is not chain, P is antichain.

Figure 3. Problem-solving mathematics task.

instrument. Furthermore, they use other three kinds of instruments as the supporting ones, including mathematics task, interview manual, and video recording. Mathematics task is applied to figure out which kind of warrant is in mathematical argumentation as shown in Figure 3.

The researchers adapt the task based on the research instruments applied by Inglis et al. (2007). According to the instruments, some information is given, and then subjects need to investigate certain statements. Each of them is asked to investigate the two statements at different time. While completing the task, they are asked to verbally express what they are thinking during the process as much as possible. The researchers take a video recording to tape every activity done by the subjects during the process.

Data analysis

Data from *think aloud* and interview are scripted and analyzed. The data collection process can be seen in Figure 4. The researchers conduct a three-phase analysis of qualitative data by Miles and Huberman (1992), and a six-phase analysis and interpretation of qualitative data by Creswell (2012). Those analysis phases are:

1. Data transcription
2. Data reduction
3. Data coding
4. Check the validity of the data or the triangulation of data
5. Data analyzing
6. Finding interpreting
6. Conclusion making.

RESULTS

The researchers involve 50 undergraduate students of Mathematics education major in STKIP PGRI Jombang. This study was conducted in February 25 to March 22,

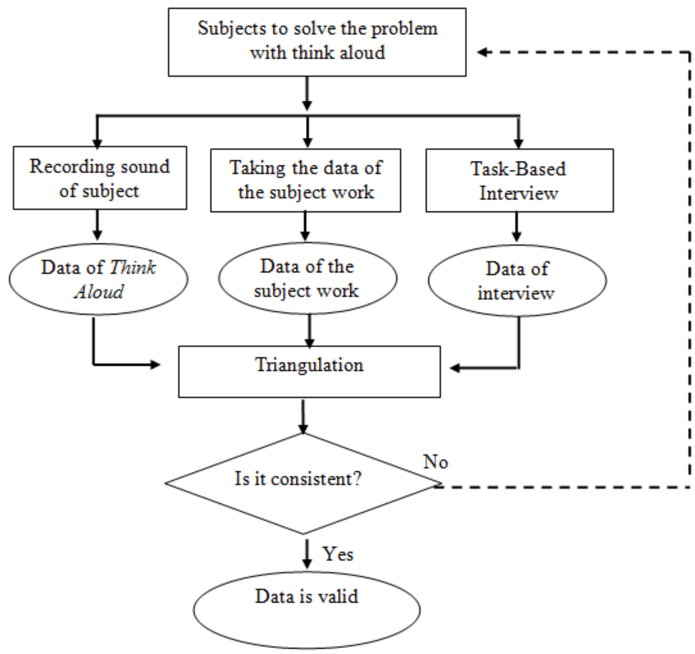


Figure 4. Data collection techniques.

Table 2. Kinds of warrant used by students.

Kinds of warrant	The number of students using warrant for statement 1	The number of students using warrant for statement 2
Inductive	5	6
Structural-intuitive	3	3
Deductive	5	7
Inductive and structural-intuitive	4	3
Structural intuitive and deductive	16	18
Inductive and deductive	17	13

2016. Table 1 presents the kinds of warrant the students used, as follows.

Based on Table 2, it shows that, in mathematical argumentation of statement 1, 5 students use inductive warrant, other 3 use structural-intuitive, other 5 used deductive, other 4 use inductive and structural-intuitive, other 16 uses structural intuitive and deductive, and other 17 use inductive and deductive. Whereas, in statement 2, 6 students use inductive warrant, other 3 uses structural-intuitive, other 7 use deductive, other 3 use inductive and structural-intuitive, other 18 use structural intuitive and deductive, and other 13 use inductive and deductive.

More than 60% of the students construct deductive warrant from non-deductive ones. Inductive and structural-inductive are non-deductive warrant. Thus, the researchers are interested to describe a process of constructing deductive warrant beginning with non-deductive ones. The researchers will describe the process from inductive one in mathematical argumentation expressed by the subjects, since this construction may whether strengthen or oppose the conclusion made by inductive warrant.

A construction of deductive warrant from inductive one for task 1, begins with subject ND expresses given information, $\mathbb{Z}$ refers to integers, and gives models of $\mathbb{Z}$ element: -2, -1, 0, 1, 2. Other elements of $\mathbb{Z}$ is coded with ". . .". Next, subject ND defines binary relation P, P = $\{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a - b = 7k$, k is element of whole number$\}$. The intention of subject ND expressing such information is because it will become the data to conclude that whether or not binary relation P on set $\mathbb{Z}$ is antisymmetric. It is seen based on the data of *think aloud* by subject ND during the task completion, interview transcription, and his written work.

Researcher: "After reading the task, why did you write this? (*showing subject ND's work*)
Subject ND: "This is information given within the task"
Researcher: "There is much information within this task, why did you take this information to be noted?"
Subject ND: "I will use this information to see whether or not binary relation P on set $\mathbb{Z}$ is antisymmetric" (data)

After subject ND shows the data used and the conclusion made, the subject determines element models of P. Doing so, given the value of $a = 5$ and $b = -2$. Subject ND randomly determines the value of $a = 5$ from $\mathbb{Z}$. The reason why he determines $b = -2$ since $b \in \mathbb{Z}$ and the result of $a - b$ is $7k$. Subject ND seeks certain integers as b which fits $5 - b = 7k$, k is a whole number. With $k = 1$, the subject determines $b = -2$. Thus, subject ND determines (5, -2) as the element model of P. Next, he examines whether or not (5, -2) is antisymmetric. He use the term "antisymmetric" referring to "if $a \neq b$, so a P b or b P a" in examining (5, -2). Subject ND states that $5 \neq -2$, 5 P -2 and -2 P 5. a P b refers to a and b has such a binary relation P that fits $a - b = 7k$, with k as whole number. 5 P -2 indicates that 5 and -2 has such a binary relation P that fits $5 - (-2) = 7k$, with $k = 1$. -2 P 5 indicates that -2 and 5 has no such a binary relation P because $-2 - 5 \neq 7k$. Subject ND suggests that (5, -2) meets an antisymmetric characteristic due to $5 \neq -2$ and fulfills whether 5 P -2 or -2 P 5. He also suggests that binary relation P is antisymmetric because $(5, 2) \in$ P is antisymmetric. This is based on *think aloud* data by Subject ND while completing the task and also based on his written work.

Inductive warrant

Subject ND: "Both a R b and b R a are antisymmetric, so $a = b$, with a, b as elements of $\mathbb{Z}$. The P equals to (*keeping silent for a while, and then going back on his reading*) 5 and -2 since $a - b = 7k$, $5 - (-2) = 7k$, $7 = 7k$, so $k = 1$, 1 is an element of whole number, so (5, -2) is element of P. Next, about antisymmetric (*keeping silent for awhile*), it is if a R b and b R a so it is whether $a = b$ or $a \neq b$, it is also whether a R b or b R a. $5 \neq -2$, yes, of course, 5 R -2 is yes, -2 R 3 is no, so binary relation P on set $\mathbb{Z}$ is antisymmetric."

Based on the *think aloud* data and the subject's written work as shown in Figure 6, it seems that Subject ND determines the element model of P, evaluates them with a definition of antisymmetric, and conclude that binary relation P is antisymmetric. He uses a specific model to make such conclusion. Thus, he uses inductive warrant in his mathematical argumentation to guarantee the truth of his conclusion. He determines another element model of P, (14, 7), with $a = 14$ and $b = 7$. Subject ND randomly determines the value of $a = 14$ from $\mathbb{Z}$ as well. His intention determining $b = 7$ is because $b \in \mathbb{Z}$ and the result of $a - b$ is $7k$. He seeks integers as b that fits $14 - b = 7k$, k is a whole number. With $k = 1$, Subjek ND takes $b = 7$. Next, he examines whether or not (14, 7) is antisymmetric. In doing so, he uses the same strategy as when he examined (5, -2) with a definition of antisymmetric. He also mentions that binary relation P is antisymmetric since $(14, 7) \in$ P is antisymmetric. This is based on what is stated in *think aloud* data by Subjek ND while completing the task and also based on what his written work.

Subject ND: "then, the P equals to (*keeping silent for awhile*) 14 (*keeping silent for awhile*) 7, $a - b = 7k$, and $14 - 7 = 7k$, $7 = 7k$, so $k = 1$, 1 is whole number. Next, about antisymmetric (*going back on his reading*), this $a \neq b$ so if $a \neq b$ then it is whether a R b or b R a. It is not 14 = 7, but 14 R 7, 7 R 14 is a no, because it meets $a \neq b$ so it is whether a R b or b R a, thus, P is antisymmetric. After all, binary relation P on set $\mathbb{Z}$ is antisymmetric."

Inductive warrant

Based on the *think aloud* data and the written works on Figures 6 and 7, it seems that subject ND determines an element model of P with $a \neq b$ and then evaluates the model through a definition of antisymmetric, "If $a \neq b$ so it is whether a relates to b (a R b) or b relates to a (b R a)". Next, he determines an element model of P with $a = b$, (3, 3). He randomly determines such model from set $\mathbb{Z}$. Then, he examines whether or not the model (3, 3) meets the definition of binary relation P. Thus, he substitutes $a = b = 3$ into $a - b = 7k$ and takes (3, 3) as the element of P. He examines whether or not the element (3, 3) meets the nature of antisymmetric. He uses the same strategy as when he examined (5, -2) and (14, 7) with a definition of antisymmetric. At this time, however, he does not such definition stating "If $a \neq b$ so it is whether a R b or b R a so P is antisymmetric". He uses a definition of antisymmetric stating "If a R b and b R a so it is $a = b$". Subject ND also notes that $(3, 3) \in$ P is antisymmetric. This is based on what is stated on *think aloud* data expressed by subject ND when completing the task and also based on his written work.

Subject ND: "Hemp…, given the P equals to (3, 3), $a - b = 7k$, and $3 - 3 = 7k$, $0 = 7k$, then it will be $k = 0$, $k = 0$ is whole number, it is done (*keeping silent for a while*). Yeach,… then both a R b and b R a are antisymmetric and $a = b$, since from its beginning, the a is 3, both 3 R 3 and 3 R 3 are right, thus, 3 = 3 is right, it is antisymmetric."

Subject ND makes another element model of P with $a = b$, (10, 10). He uses the same strategy as when he examined whether or not (3, 3) meets the definition of binary relation P and the definition of antisymmetric as

Data used by Subject ND to make a conclusion

Subject ND : "(reading the task) ℤ is a set of integers of {. . ., -2, -1, 0, 1, 2, . . .} (*keeping silent for a while and then going back on his reading*) P is binary relation of ℤ, noted as binary relation due to a non-empty space of ℤ × ℤ. Definition of binary relation P = {(a, b) ∈ ℤ × ℤ | $a - b = 7k$, k is the element of whole number }. Is binary relation P on set ℤ antisymmetric? Antisymmetric is both a R b and b R a, so $a = b$, with a, b as element of ℤ".

The conclusion that will be gained by subject ND

Figure 5. Subject ND's work.

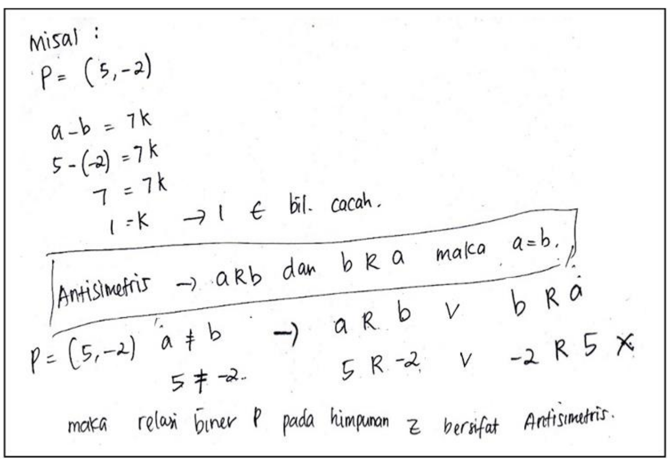


Figure 6. Subject ND's work.

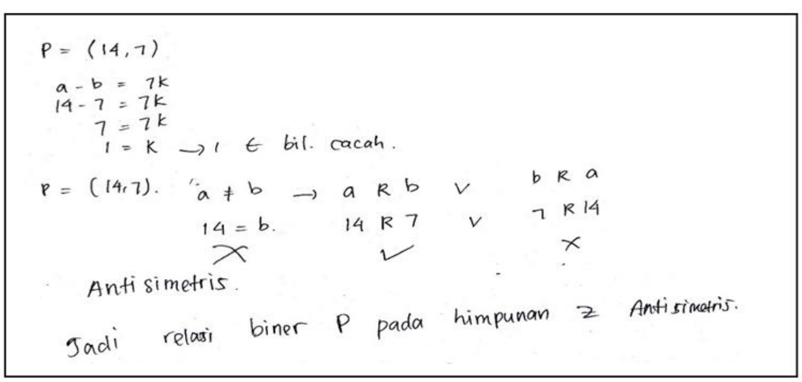


Figure 7. Subject ND's work

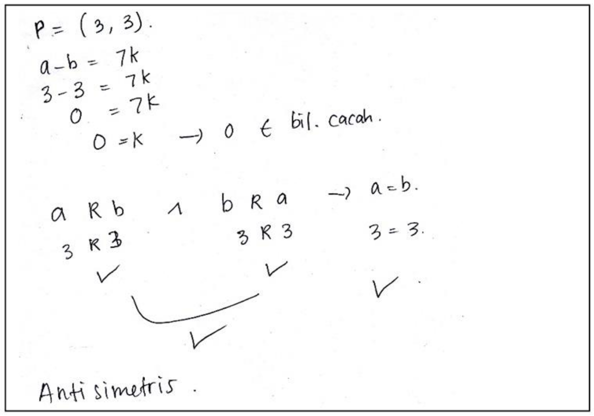


Figure 8. Subject ND's work.

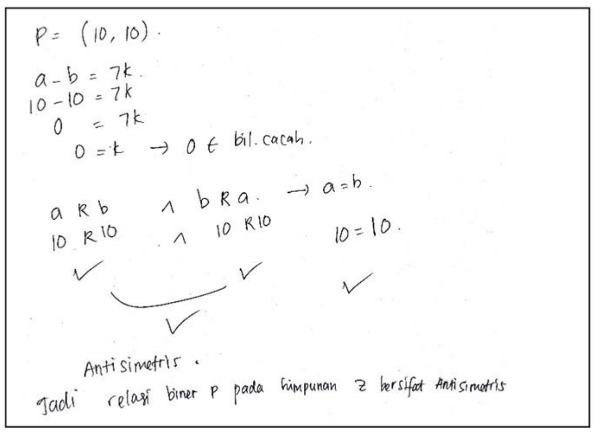


Figure 9. Subject ND's work.

as well. He suggests that (10, 10) ∈ P is antisymmetric. He also states that binary relation P on set Z is antisymmetric based on an observation conducted on both (3, 3) and (10, 10). This is in accordance to *think aloud* data done by subject ND while completing the task and also based on his written work.

Subject ND: "Hemp,… if another given P is (10, 10), it will be $a - b = 7k$, $10 - 10 = 7k$, $0 = 7k$, hence, $k = 0$, 0 is whole number. It is right, isn't it? Thus, it will become antisymmetric is a R b and b R a, so, $a = b$. 10 R 10, it is right, and 0 R 10 is also right, so 10 = 10 is right, thus, it is antisymmetric. As (3, 3) and (10, 10) are both antisymmetric, the binary relation P on set ℤ is antisymmetric."

Based on *think aloud* data and subject's written work as shown in Figures 5, 6, 7, 8, and 9, it indicates that in giving a guarantee for the truth of his conclusion, Subject ND uses some specific element models of P including (5, -2), (14, 7), (3, 3) and (10, 10). He uses similar strategy to examine whether or not each of his models is antisymmetric. Thus, he uses inductive warrant in his mathematical argumentation. He makes an instance (a, b) ∈ P by considering 2 states: $a \neq b$ and $a = b$.

Nevertheless, after ensuring his conclusion with inductive warrant, subject ND starts constructing a deductive warrant. This is because some elements of ℤ that meet binary relation P and identifying some elements model of P with a definition of anti-symmetrical nature. *Qualifier* of a conclusion in its mathematical argumentation is *probable* since the conclusion is made based on some specific data.

Subject ND needs to construct deductive *warrant* in his mathematical argumentation to easily make general conclusion without any requisite or disclaimer. This is based on *think aloud* data done by the subject and his written work while completing the task. The interview transcription conducted by the researchers is included as well.

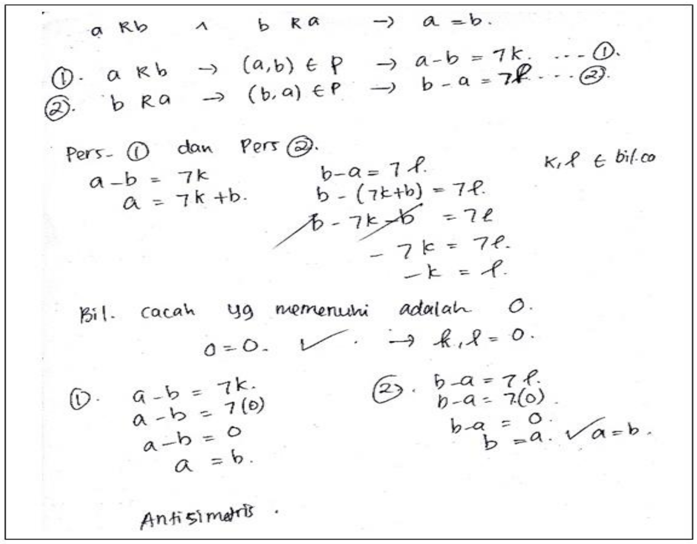
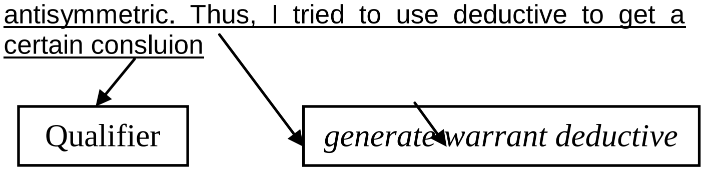


Figure 10. Subject ND’s work.

Subject ND: “(*keeping silent for a while, and going back on his reading again)* hemp,… I try to use deductive to make the conclusion generally accepted.” (*constructing deductive warrant*)

Furthermore, the results of interviews with subject ND are as follows:

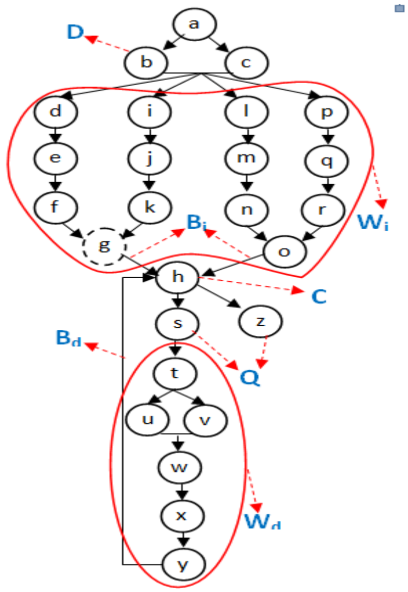
Researcher : “Why did you use this deductive as your way?”
Subject ND : “Here, for inductive, I just took some pairs of ℤ to be the elements of P, then, I identified those elements of P with a definition of antisymmetric. Hence, binary relation P on set <u>ℤ is likely to be antisymmetric. Thus, I tried to use deductive to get a certain consluion</u>



Subject ND starts writing down a definition of antisymmetric by using “a R $b \wedge b$ R $a \Rightarrow a = b$” as the symbol. He relates the definition of antisymmetric and binary relation P, a R b indicating that $(a, b) \in$ P with $a - b = 7k$, which k is whole number and b R a indicating that $(b, a) \in$ P with $b - a = 7l$, which l is whole number. He calls $a - b = 7k$ as equation 1 and $b - a = 7l$ as equation 2. Then, he keeps substituting both equations until he finds $-k = l$. He states $k = l = 0$ for the whole numbers within $-k = l$ is 0. He substitutes k, $l = 0$ into both equations that $a = b$ to be reached. He concludes that binary relation P on set ℤ is antisymmetric. This is based on *think aloud* data done by the subject and his written work while completing the task.

Subject ND: “If a R b and b R a so it is $a = b$. Hence, the first a R b has a and b in set ℤ with (a, b) as the elements of P. This is in accordance to the definition stated in the task which $a - b = 7k$ to be equation 1. The second b R b indicating that (b, a) is an element of P so that $b - a = 7k$, this becomes equation 2. Based on both equations, Hemp… $a - b = 7k$ so that $a = 7k + b$, equation 1 can be substituted into equation 2, $b - a = 7k$, wait a minute… (*keeping silent for a while*) $a - b = 7k$ so that $b - a = 7l$. Thus, it will be $b - a = 7l$, a is substituted into $7k + b$, $b - (7k + b) = 7l$, $b - 7k - b = 7l$, b minus b = 0 becomes $-7k = 7l$, so, it will be $-k = l$. Hemp,… (*keeping silent for a while and then going back on his reading)* how comes….it is not allowed to be distributed into equation 1, k is whole number. It means that the whole number that fits $-k = l$ is 0, indicating that 0 = 0. And, (*keeping silent*) let’s distribute it into equation 1 and 2, respectively, for equation 1, $a - b = 7k$, $a - b$ = 7 times 0, $a - b = 0$, then it will be $a = b$. For equation 2, $b - a = 7l$, $b - a$ = 7 times 0, $b - a$ =0 then it will be $b = a$ or $a = b$. And yeach..it is antisymmetric” (*describing deductive warrant in detail*).

Based on *think aloud* data and the subject’s written work on Figure 10, it seems that when giving a guarantee for the truth of his conclusion, subject ND uses a part of the definition of antisymmetric and manipulates the algebra. *Qualifier* of the conclusion in his mathematical argumentation is *certain* since the uncertainty of that conclusion has been removed. The structure of subject ND’s thinking can be seen on Figure 11, whereas the scheme of subject ND’s mathematical argumentation is presented in Figure 12.



a	Read the task given	r	Decide (10, 10) $\in$ P
b	Determine data used	s	Provide a *qualifier* to the conclusion, *probable*
c	Determine conclusion target		
d	Make such a prospective element model of P with $a \neq b$, which is (5, -2)	t	Write down a definition of antisymmetric
e	Identify (5, -2) with the definition of binary relation P, $a - b = 7k$, with k as whole number.	u	Relate aRb within the definition of antisymmetric to the definition of binary relation P, aRb indicating that $(a, b) \in$ P with $a - b = 7k$
f	Determine (5, -2) $\in$ P		
g	Identify (5, -2) $\in$ P and (14, 7) $\in$ P with the definition of antisymmetric, "if $a \neq b$ and bRa then $a R b$"	v	Relate bRa within the definition of antisymmetric to the definition of binary relation P, bRa indicating that $(b, a) \in$ P with $b - a = 7l$
h	Conclude that binary relation P on set $\mathbb{Z}$ is antisymmetric	w	Substitute $a - b = 7k$ into $b - a = 7l$
		x	Determine $-k = l = 0$
i	Make such a prospective element model of P with $a \neq b$, (14, 7)	y	Substitute $k, l = 0$ to $a - b = 7k$ and $b - a = 7l$
j	Identify (14, 7) with the definition of binary relation P, $a - b = 7k$, with k as whole number	z	Provide a *qualifier* to the conclusion, *certain*
		B_i	*Backing* to inductive *warrant*
k	Determine (14, 7) $\in$ P	B_d	*Backing* to deductive *warrant*
l	Makes such a prospective element model of P with $a = b$, which is (3, 3)	C	Conclusion
		D	Data
m	Identify (3, 3) by the definition of binary relation P, $a - b = 7k$, with k as whole number	W_i	*Warrant* inductive
		W_d	*Warrant* deductive
		→	The order of thinking when completing the task
n	Determine (3, 3) $\in$ P		
o	Identify (3, 3) $\in$ P with the definition of antisymmetric, "if aRb and bRa then $a = b$"	- →	The components of a scheme of mathematical argumentation is based on Toulmin scheme
p	Make such a prospective element model of P with $a = b$, (10, 10)	◯	Activity right thinking
		◌	activity wrong thinking
q	Identify (10, 10) with the definition of binary relation P, $a - b = 7k$, with k as whole number	◯ (red)	A set of thinking activities on the components of *warrant*

Figure 11. The structure of undergraduate student's thinking in mathematical argumentation based on the components of Toulmin scheme.

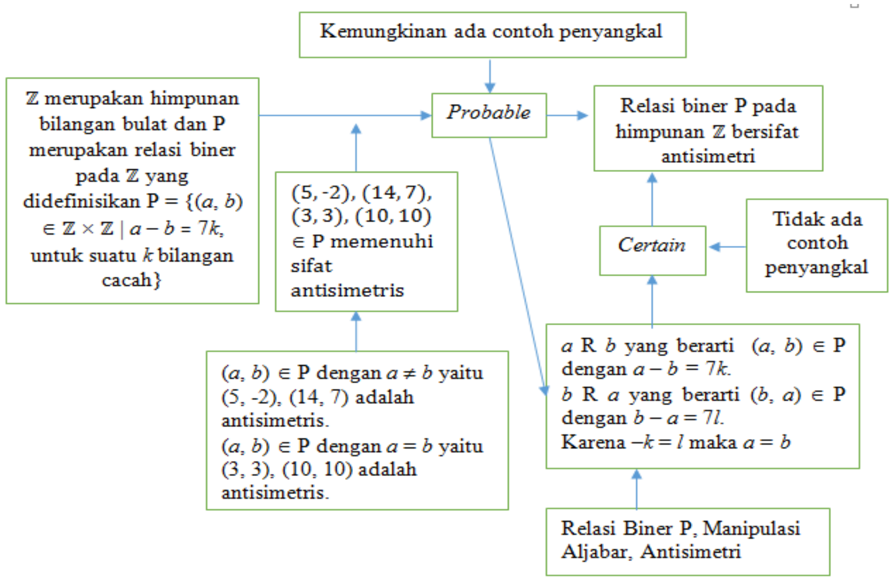


Figure 12. The Scheme of Subject ND's Mathematical Argumentation based on Toulmin Scheme.

DISCUSSION

Based on the data analysis of *think aloud*, interview and written work done by the subject, it shows that in constructing a deductive warrant in mathematical argumentation, the subject needs to use inductive warrant. The subject does not directly use some abstract objects; he uses some concrete objects as the bridge to understand those abstract ones. This is in line with Martin and Harel (1989) and Fischbein and Kedem (1982) that neither inductive nor deductive argument can stand by its own as mathematical evidence. This indicates that inductive framework is constructed at the beginning of deductive framework.

Construction deductive warrant occurs when a student uses inductive warrant in mathematical argument resulting in a conclusion, because the conclusion obtained qualifier is still probable, then there is a need to use deductive warrant in mathematical argument that the uncertainty of conclusions can be removed. A process of constructing deductive warrant begins when *qualifier* of a conclusion derived from inductive warrant is *probable*. Thus, the conclusion is questionable that a disclaimer may exist. Therefore, it is necessary to have deductive warrant so that any hesitancy can be vanished.

Soejadi (2007) suggested that a nature/theorem/principle is initially found through inductive ways, that should be strengthened by deductive ones. A conclusion derived from inductive warrant may be different from that with deductive one. Thus, a deductive warrant may whether strengthen or even weaken the conclusion derived from inductive warrant. However, it is common that undergraduate students should have changed their inductive mind-set into deductive, since a deductive scheme is the only warrant accepted for the truth of mathematical argumentation (Inglis, Ramos & Simpson, 2007; Harel, 2001; Harel and Sowder, 1998; Tall, 2004).

The study data indicate students do not always reach the conclusion through mathematical argumentation with deductive warrant. However, students are required to reach the conclusion through mathematical argumentation with deductive warrant. Therefore, educators must help students to construct a valid mathematical argumentation. Valid mathematical argumentation is argument with warrant deductive. In constructing mathematical argument with deductive warrants can be through inductive warrant because mathematics in finding theorem or definition of indirect deductively. Educators in helping students to construct mathematical argumentation can use Toulmin scheme, educators can indicate the possibility of rebuttal if using non-deductive warrant. Educators can also indicate qualifier on the conclusions reached through non-deductive warrant.

Conclusion

Students do not directly use deductive warrant in expressing their mathematical argumentations. They initially need an inductive warrant as a bridge to construct such a deductive one. Based on the data analysis, undergraduate students construct a deductive warrant due to the lack of a conclusion derived from inductive warrant. Thus, a deductive warrant is needed to strengthen the conclusion. A deductive warrant will change the *qualifier* of a conclusion from *probable* to *certain*. Inductive warrant plays an important role in mathematical argumentation in the term of generating a deductive warrant. Although a deductive warrant is needed in mathematical argumentation, it still requires an inductive mind-set that includes presentation of its nature, theorem, and definition with some models in the first phase, consisting of specific things that lead to a general conclusion.

Conflict of Interests

The authors have not declared any conflict of interests.

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