Vol. 19(1), pp. 26-39, January 2024
DOI: 10.5897/ERR2023.4386
Article Number: FE84BBA71738
ISSN: 1990-3839

http://www.academicjournals.org/ERR

ACADEMIC JOURNALS
expand your knowledge

Educational Research and Reviews

Full Length Research Paper

An examination of pre-service teachers' Van Hiele levels of geometric thinking and proof perception types in terms of thinking processes

Filiz Tuba Dikkartin Övez* and Emine Özdemir

Balıkesir University, Necatibey Faculty of Education, Department of Mathematics and Science Education, Balıkesir, Türkiye.

Received 8 December, 2023; Accepted 10 January, 2024

The primary goal of advanced mathematics courses is to equip students with proof skills. That is why the competence of pre-service mathematics teachers in proof is considered an assessment of their performance. Despite the emphasis on mathematical proof in undergraduate education, students studying higher mathematics at university often face difficulties with proofs. Therefore, it is essential to examine students' perceptions and thinking processes regarding proof. Accordingly, this study aims to explore the types of proof perceptions and thinking processes of pre-service elementary mathematics teachers based on their Van Hiele geometric thinking levels. The study utilized the case study design, one of the qualitative research designs. The study group comprised 67 fourth-grade pre-service elementary mathematics teachers studying at a medium-sized education faculty in western Turkey. The study group was determined using the convenience sampling technique, one of the non-probability-based sampling methods. Van Hiele's geometric thinking level test, proof selection scale, and scale for determining proof thinking processes were employed as data collection tools. The data were analyzed descriptively. Consequently, it was found that pre-service teachers were predominantly at the fourth geometric thinking level and exhibited the B-proof perception type. Only a few students were observed to reach the level of proof in their thinking processes. According to the results obtained from the study, the geometric thinking levels and proof perception types of pre-service teachers significantly affect their thinking processes.

Key words: Geometric thinking levels, proof perception types, proof thinking processes, pre-service mathematics teachers.

INTRODUCTION

In mathematics education research, mathematical thinking is defined from two different perspectives: mathematical processes (conjecture, generalization, and proof) and the development of mathematical concepts (Isoda and

*****Corresponding author. E-mail: tdikkartin@balikesir.edu.tr.

Katagiri, 2012). The initial mathematical thinking skills of children develop through the emergence of mathematical meaning (symbol, schema, operation) in spontaneous activities within the child's own culture. Such activities can be said to develop through collaborative problem-solving processes with more experienced children (Van Oers, 2010). Objectives that students are expected to achieve at the end of problem-solving activities can be practiced by: i. Forming, testing, reviewing, reasoning, solving problems, and expressing judgments in written form, ii. Expressing solutions and judgments in writing, iii. Generalizing and predicting (making predictions), iv. Expressing them with symbols or formulas, v. Presenting them in a meaningful whole by showing or proving them with graphs or diagrams (Baykul and Aşkar, 1987). Mason et al. (1985) defined and examined the basic components of mathematical thinking in four stages: specialization, generalization, conjecture, verification, and persuasion. Tall (1991) emphasizes that mathematical thinking comprises different components such as abstraction, synthesis, generalization, modeling, problem-solving, and proof. As for proof, it plays a crucial role in the development of mathematical thinking, stabilizing mathematical practice by providing common criteria for accepting and producing new mathematical knowledge (Hemmi, 2010). Proof facilitates continuity between generations, as the deductive organization of axiomatic mathematics allows new generations to develop new problems by reapplying mathematical knowledge acquired by previous generations. At the same time, proof coordinates and disciplines mathematical reasoning, being used in reasoning and justifying conclusions (Hemmi, 2010).

As a fundamental concept in mathematics, proof plays a crucial role in developing students' analytical thinking, problem-solving skills, and mathematical reasoning skills. The development of both written and non-verbal proof skills in K-12 education is of great importance, fostering a deep understanding of mathematics, encouraging critical thinking, and empowering students to become competent problem solvers (Lamb et al., 2017). Studying formal or informal proofs allows students to break down complex mathematical problems into smaller, manageable chunks, encouraging a deeper understanding of concepts and enabling students to see the logical connections between different ideas..

In Turkey, it is observed that proof is included in the mathematics curriculum, and the concept of proof is attempted to be acquired through reasoning and association skills (MoNE, 2018). In the curricula, reasoning is defined as the process of acquiring new knowledge by using tools (symbols, definitions, relations, etc.) and thinking techniques (induction, deduction, comparison, generalization, etc.) specific to mathematics. One of the skills targeted to be developed in this context is the skill of association. The ability to associate will enable generalizations based on the relationships between mathematical objects, and with the help of these generalizations, mathematical proof processes, which are the basic characteristics of mathematics, can be completed (Yıldırım, 2014).

Another goal is reasoning skills. In the process of proof, reasoning requires being able to give examples and counter-examples that support proof claims, being aware of definitions and axioms, using different proof techniques, verifying claims by showing evidence, and making persuasive explanations (Ferrini-Mundy and Senk, 2006). To investigate the relationships between mathematical concepts and to reach generalizations is part of doing mathematics. The way of producing generalizations is specific to mathematics and is called proof (Altun, 2013).

Mathematicians decide whether a mathematical statement is true utilizing a proof. Mathematical results are considered valid only after careful proof (Ross, 1998). The criterion of truth in the eyes of the mathematician should be sought in logical proof, not in factual proof (Yıldırım, 2014). It has been revealed that students even have difficulty determining the truth of the propositions presented to them (Gibson, 1998; Goetting, 1995; Ko and Knuth, 2009; Riley, 2003). The mathematics teaching undergraduate program requires students to have the ability to defend the accuracy and validity of mathematical inferences (MoNE, 2018).

There are many theories about writing proofs and understanding proofs. One of these theories is the Van Hiele theory of geometric thinking. It examines students at five levels of geometric thinking (visualization, analysis, abstraction, deduction, and the highest level). In the first level, students verbally name shapes according to their appearance but are unable to analyze or generalize. At the analysis level, students begin to analyze shapes and features but are unable to make associations between features. To generalize their empirical results, they do not need formal proofs; they just use language such as "because," "if," "then," etc. For instance, they can physically calculate (by measuring and cutting) the sum of the interior angles of a triangle. Logical inferences cannot be understood, although definitions and axioms are meaningful for students at level 3. At this level, students can follow the proof but cannot perform the proof. They find and use strategies with reasoning to solve problems. They state their deductive arguments. However, they cannot understand the meaning of deduction in the axiomatic sense. At Level 4, they recognize and use the components of an axiomatic system but cannot compare axiomatic systems. This level of students can present the reasons for the steps of proof. They can prove and infer using induction. Level 5 students look for a domain where a mathematical theorem or principle can be applied and can generate theorems in different axiomatic systems. According to Van Hiele, to understand the nature of geometry and proof, the language used is important because each level has its language and symbols. If the level of the student is different from the level at which the teaching is done, learning and success cannot be realized (Usiskin, 1982).

Considering mathematics as a structure consisting of axioms, definitions, assumptions, theorems, and proofs (Heinze and Reiss, 2003), the importance of proof in mathematics lessons becomes better understood. Despite the significance and emphasis on mathematical proof in undergraduate education, students who pursue higher mathematics at the university level often encounter difficulties with proofs (Moore, 1994; Dreyfus, 1999; Almeida, 2000; Jones, 2000; Weber, 2004). A primary goal of advanced mathematics courses is to equip students with proof skills. Therefore, the competence of pre-service mathematics teachers in proof is considered an assessment of their performance (Weber, 2001).

Previous studies aimed at determining the reasons for students' difficulties in writing proofs have explored students' views on proof, their perception levels of proof, and their challenges in the proof process (Almeida, 2000; Raman, 2003; Solomon, 2006; Stylianides et al., 2007). For instance, in a study investigating pre-service teachers' views on proof, Almeida (2000) categorized perceptions about proofs into four types, as described below:

Type A: The student needs to work with formal proof and rejects informal proof.
Type B: The student accepts the need for formal proof but tentatively uses informal proof practices until formal proof is mastered.
Type C: The student accepts intuitive and empirical arguments as proofs. They consider formal proofs necessary for passing exams.
Type D: The student recognizes the need for formal proof but often sees it as symbol manipulation. He/she is unable to make sense of the proof.

According to the studies, pre-service teachers state that they are taught the steps of the proof process by memorizing them and that they learn proof only to pass the exam (Genç and Karataş, 2018). Besides, it is also revealed in the studies that pre-service teachers have problems in understanding the language used in the proofs made in the field courses. Furthermore, their perceptions about proof and their proof-writing skills were found to be related to Van Hiele's geometric thinking processes. Hence, proof-making and perceiving axiomatic systems, which shape the perception of proof and are important elements of it, are skills that develop at levels 3 and 4 of geometric thinking. Proof is a difficult concept to teach and learn at all levels of education. Among the difficulties that students face at the university level is the inability to complete the process despite focusing more on formal proof (Thomas and Klymchuk, 2012; Stylianides, Stylianides and Philippou, 2004; Stylianides and Stylianides, 2017). The difficulties associated with proof cause many students to have problems in moving on to more advanced studies in mathematics.

The perception of geometric thinking and proof are two fundamental elements of mathematics that play a crucial role in the development of a student's mathematical maturity. The Van Hiele theory has provided a framework for understanding the progression of levels of geometric thinking and has yielded significant insights into how students perceive mathematical proofs and interact with them (Usiskin, 1982).This relationship was also supported by the results of some research studies supporting the connection between geometric thinking levels and proof perception types. Hoffer and Moore (1983) found that students' geometric thinking levels were highly correlated with the types of proof perception. The results have clearly revealed the situation. Students at the visual and descriptive levels tended to have a more procedural and conditional perception of proof. Without fully grasping the structure of the proof, they focus on memorizing the steps and recognizing the conditions. Conversely, at the levels of experience-based inference and inference, students exhibit a strong perception of structural proof. They can develop a deeper understanding of logical relationships. In a study conducted by Herbst (2002), the relationship between students' geometric thinking levels and their perception of proof was examined. In the study, middle school students were given discovery tasks related to geometric concepts. They encourage students to actively engage with geometric problems and allow researchers to assess students' progress in both geometric thinking and proof perception. According to Herbst (2002), students' perceptions of evidence improved as they progressed in Van Hiele's geometric thinking levels.

It is considered important that the geometric thinking levels, proof perceptions, and thinking processes of pre-service teachers, who will form the mathematical understanding of future generations, play a role in the development of these skills in their students. Within this scope, it is necessary to shed light on teacher education by revealing the types of proof perceptions of pre-service teachers and examining their relationship with geometric thinking levels. Accordingly, this study aims to examine the types of proof perceptions and thinking processes of pre-service elementary mathematics teachers according to their Van Hiele geometric thinking levels.

Study problems

1. What are the types of proof perceptions of pre-service teachers according to their Van Hiele geometric thinking levels?
2. How are the proof thinking processes of pre-service teachers with different Van Hiele geometric thinking levels and proof perception types?

METHOD

Study model

The study was based on the case study, one of the qualitative research designs. A case study is defined as a study method used to answer how and why questions in current situations (Yin, 2009) or a method that allows a phenomenon or event to be examined in depth (Yıldırım and Şimşek, 2011). This study aims to describe in depth the types of proof perceptions according to Van Hiele's geometric thinking levels and the proof thinking processes of pre-service teachers with different Van Hiele geometric thinking levels and proof perception types. Since it offers the opportunity to examine complex situations with rich descriptions (Ozan-Leylum et al., 2017), the case study design was taken as a basis for the research.

Study group

The study group of the study consisted of 67 fourth-grade pre-service elementary mathematics teachers studying at a medium-sized education faculty in western Turkey. Using the convenience sampling technique, which is one of the non-probability-based sampling methods, the study group was determined. Convenience sampling is a method that aims to prevent loss of time, money, and labor. (Büyüköztürk et al., 2019). According to the gender of the pre-service elementary mathematics teachers, 42 (62.7%) were female and 25 (37.3%) were male.

Data collection tool

Van Hiele's geometric thinking level test was used to determine the geometric thinking levels of pre-service teachers. The scale was adapted into Turkish by Duatepe (2000). Within this scope, Cranbach Alpha reliability measurements were calculated as 0.82, 0.51, and 0.70 for level 1, level 2, and level 3 respectively (Duatepe, 2000). The scale has 5 questions corresponding to each level. The geometric thinking levels of Van Hiele are defined in a hierarchical structure. A level cannot be reached before the next level is passed.Hence, to pass a level, 4 or more questions in that level must be answered correctly. A person who answers at least 4 of the first 5 questions correctly is assigned to Level 1 (visual period), and a person who reaches Level 1 is assigned to Level 2 (analysis) if he/she answers at least four of the second-level questions correctly. The person who has not reached Level 1 is not assigned to the second level. If the person who reaches Level 2 answers at least four of the Level 3 questions correctly, he/she is assigned to Level 3 (inference based on experience). Similarly, if the person who reaches Level 3 answers at least four of the Level 4 questions correctly, he/she is assigned to Level 4 (inference). If the person who reaches Level 4 answers at least four of the Level 4 questions correctly, he/she is assigned to Level 5.

Proof perception types of pre-service teachers were determined using the Proof Choice Scale developed by Özdemir and DikkartınOvez (2012). The scale includes various proofs of 3 theorems. These theorems concern the sum of interior angles in a triangle, inequalities, and the sum of squares of positive integers. The proofs presented include direct proofs, proofs by inference, induction, proofs using geometric modeling as well as erroneous proofs. Pre-service teachers are asked to choose the most convincing, unconvincing, and incorrect ones from the given proofs and write brief explanations for their choices. The scale was developed based on the choice scale developed by Almedia (2000) to determine perceptions of proof. It was aimed to reveal the types of proof perception of pre-service teachers according to their evaluation of the proofs given.

In the study, a scale for determining proof-thinking processes consisting of open-ended items was developed to examine pre-service teachers' proof-thinking processes. In the scale, theorems requiring students to prove basic perpendicularity, three perpendiculars and Pythagorean theorems were selected. While determining the problems, geometry textbooks and geometry questions with validity and reliability provided in the exams of higher education institutions were scanned. Six problems involving geometry topics were selected. The items in the scale were evaluated in terms of measuring the targeted construct and their suitability for language and purpose by three mathematics education experts, and necessary adjustments were made to the scale. The pre-test scale consisting of three items was applied to 15 pre-service teachers outside the study group to determine its comprehensibility and application time. The pre-service teachers were asked to explain in writing in detail which theorems they used in solving the problems and how they reached their results. As a result of this application, the sections that were not understood by the pre-service teachers were corrected and the scale was finalized in line with the expert opinion. The open-ended items in the scale for determining proof thinking processes are as follows:

1. The face ABC of a tetrahedron ABCD is an equilateral triangle with side length a, and the face BDC is a triangle with angle D perpendicular to it. Since the separation AD is perpendicular to the plane BDC, calculate the volume of this tetrahedron in terms of a.
2. Prove the Pythagorean theorem in at least 2 different ways.
3. Prove the surface area of the sphere using the surface area of the cone.

Data analyses

Pre-service teachers' responses to the proof choice scale were categorized according to the types of proof perception. The data obtained from the proof choice scale were coded into proof perception types by two researchers, and the agreement between the researchers' coding was found to be 95%. Identified proof perception types were classified according to Van Hiele's geometric thinking levels and descriptive analysis was performed by calculating percentage and frequency values.In the study, secondly, the thinking processes of pre-service teachers with different Van Hiele geometric thinking levels and proof perception types were examined. In this direction, the thinking processes in the solutions of pre-service teachers were examined using the characterization developed by Gutierrez and Jaime (1998). Gutierrez and Jaime (1998) accepted thinking processes such as recognition, definition, classification, and proof as a combination of two or more levels of geometric thinking and made a classification accordingly. The relevant classification is presented in Figure 1. The data were analyzed descriptively and direct quotations were included, taking into account the categorization defined and the responses given.

FINDINGS AND INTERPRETATIONS

Findings on proof perception types according to Van Hiele geometric thinking levels In the study, firstly, the types of proof perceptions of pre-service elementary mathematics teachers according to their Van Hiele geometric thinking levels were examined.

The descriptive findings obtained according to the data

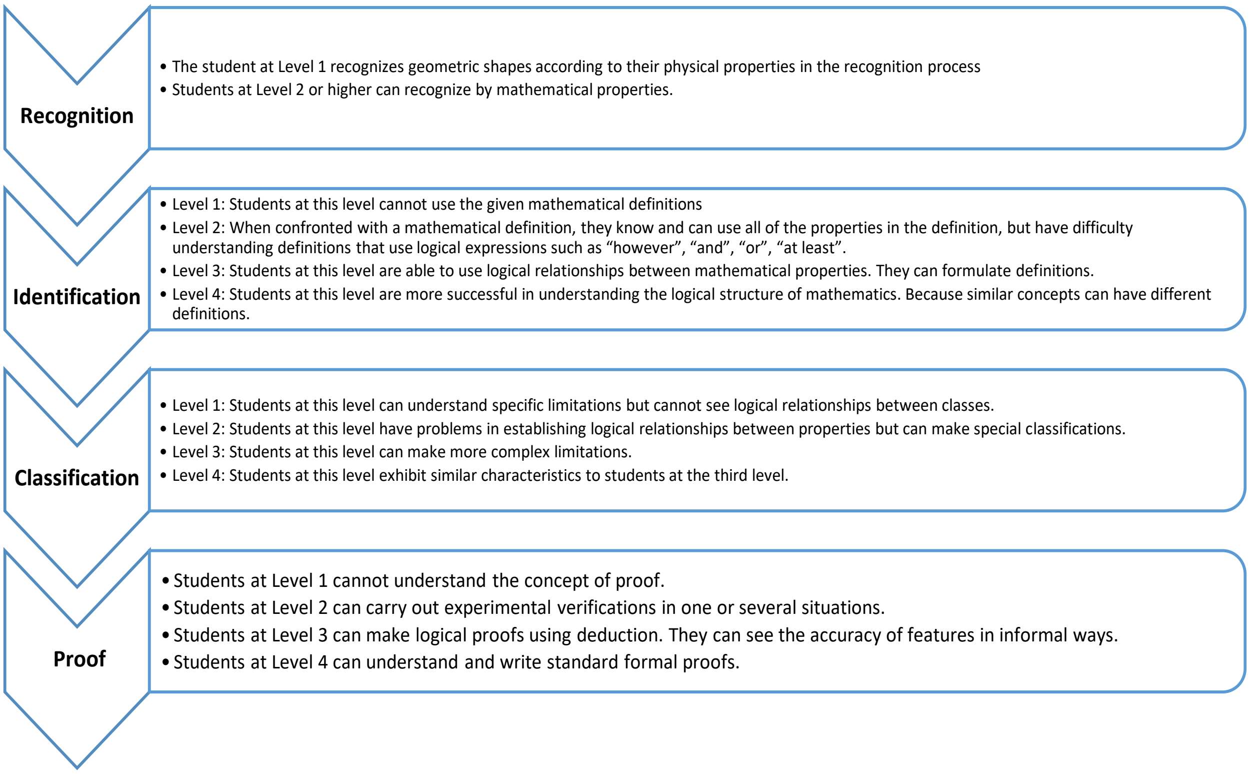

Figure 1. Gutierrez and Jaime (1998) categorization.

Table 1. Descriptive findings on geometric thinking levels according to proof perception types.

Types of proof perception	Van Hiele's geometric thinking levels									
	Level 2		Level 3		Level 4		Level 5		Total	
	f	%	f	%	f	%	f	%	f	%
Type A	0	0	0	0	3	4.4	5	7.5	8	12.0
Type B	0	0	0	0	32	47.8	0	0	32	47.8
Type C	5	7.5	7	10.4	0	0	0	0	12	17.9
Type D	3	4.5	12	17.9	0	0	0	0	15	22.3
Total	8	12.0	19	28.3	35	52.2	5	7.5	67	100

obtained from the scale for determining the types of proof perception and the scale for determining the level of Van Hiele's geometric thinking are given in Table 1.

According to the data obtained from the van hiele geometric thinking level determination scale it was determined that 12% of the pre-service teachers were at Level 2 (Analysis), 28.3% were at Level 3 (Inference based on experience), 52.2% were at Level 4, and 7.5% were at Level 5.When Table 1 is analyzedproof perception types of pre-service elementary mathematics teachers were examined, it was found that 12.0% of them were in type A, 47.8% in type B, 17.9% in type C and 22.3% in type D. It is seen that 7.5% of the pre-service teachers in type A are at level 5 of Van Hiele geometric thinking levels. Students at the 5th level of geometric thinking are known to be able to comprehend the meaning and importance of a proof based on axioms, theorems, and definitions and to be able to prove other theorems deductively by using previously proven theorems and axioms (Van Hiele, 1986; Olkun and Toluk, 2003). When the answers given by this type of pre-service teacher in the proof selection scale were analyzed, it was observed that they mostly preferred inductive and deductive proof techniques. Similarly, pre-service teachers in type A and level 5 identified proofs using different axiomatic systems as the most convincing. For a pre-service teacher at level 5 of the Van Hiele geometric thinking levels, this is expected because a pre-service teacher at this level can analyze and compare different

Table 2. Descriptive findings on the thinking processes of pre-service teachers at different geometric thinking levels and proof perception types.

Van Hiele's geometric thinking levels	Types of proof perception	Thinking processes			
		Recognition	Identification	Classification	Proof
		f	f_1	f_1	f_1
Level 5	Type A	5	5	5	5
Level 4	Type A	5	5	5	5
	Type B	32	32	32	0
Level 3	Type C	7	7	3	0
	Type D	12	12	0	0
Level 2	Type C	5	5	0	0

axiomatic systems.It is seen that 37.5 % of the pre-service teachers in type A and all of the pre-service teachers in type B are at level 4 of Van Hiele levels. According to Van Hiele's theory, pre-service teachers at level 4 can recognize geometric shapes according to their mathematical properties and use logical relationships between mathematical properties.

Within this scope, when the answers given to the scale for determining the types of proof perception were analyzed, it was revealed that the pre-service teachers with type A and B and at level 4 preferred the proofs in which deduction was used and showed the truth of the theorems in more informal ways.

It is seen that 58.33% of the pre-service teachers in type C and 80% of the pre-service teachers in type D are at level 3 of Van Hiele levels. According to Van Hiele's theory, pre-service teachers at level 3 can recognize geometric shapes according to their mathematical properties, and when they encounter a mathematical definition, they know all the properties in the definition and can use these properties. In contrast, since pre-service teachers in types C and D had difficulty understanding proofs using logical expressions and had problems establishing logical relationships (Almedia, 2000), they found the proofs that were limited to the experimental verification of one or a few situations to be the most convincing in the proof selection scale. Similarly, the pre-service teachers in types C and D, who were determined to be at Van Hiele level 2, were successful in recognizing, identifying, comparing, and explaining geometric shapes in their answers, but they did not understand the correct proofs and could not identify them most convincingly.

The findings revealed that the pre-service teachers showed the characteristics of their Van Hiele geometric thinking levels and responded to the proof selection scale accordingly. Therefore, Van Hiele's geometric thinking levels can be said to affect the answers given to the proof selection scale and to be effective in the formation of various types of proof perceptions. These findings are in line with the findings of Gutierrez and Jaime (1998) and Aydin, Halat (2009).

Findings on the proof thinking processes of pre-service teachers with different Van Hiele geometric thinking levels and proof perception types

In the study, secondly, the thinking processes of pre-service teachers with different Van Hiele geometric thinking levels and proof perception types were examined. The solutions developed by pre-service elementary mathematics teachers for the given geometry problems were examined. The thinking processes of pre-service teachers were classified according to the characterization developed and used by Gutierrez and Jaime (1998) in their study. Table 2 presents the descriptive findings related to pre-service teachers' thinking processes in problem-solving according to this categorization. The f values in the table give the amount of observation of the thinking processes in the solutions posed to the pre-service teachers.

Findings related to the recognition process

Pre-service teachers are expected to recognize geometric shapes according to their mathematical properties in the *recognition* process, which is the first of the thinking processes. Based on the findings, it was seen that all pre-service teachers who participated in the study successfully completed the recognition process. In this process, the pre-service teachers:

1. draw the tetrahedron correctly according to the mathematical properties.
2. were able to draw a right triangle with side lengths a,b,c,

and the squares that accept these sides as sides to construct Pythagoras' theorem at levels 4 and 5 drew squares that accepted the sides of the right triangle as sides, as well as various geometric shapes (trapezoid, unit circle, square, congruent right triangles) according to their mathematical properties.
3. were able to draw the height of the cone and the radius of the base, as well as a sphere with radius R.Besides, pre-service teachers at level 5 drew the hemisphere with the largest volume that can be drawn inside a cone with radius r according to its mathematical properties in problem 3, which requires higher level thinking.

Findings related to the identification process

Pre-service teachers are expected to know all of the features included in the definitions and to be able to use these definitions in the *identification* process, which is the second of the thinking processes. All of the pre-service teachers at Levels 4 and 5 of the Van Hiele thinking levels and all of the pre-service teachers in the A and B proof perception types were able to complete this process. 3 or 4 of the pre-service teachers at the level of 3rd geometric thinking and C proof perception type and 2 or 3 of the pre-service teachers at the level of 2nd geometric thinking and C proof perception type were able to complete this process. None of the pre-service teachers in the 2nd and 3rd geometric thinking levels and none of the pre-service teachers in the D proof perception type were successful in this process.

Level 1: Students at this level cannot use the given mathematical definitions
Level 2: When confronted with a mathematical definition, they know and can use all of the properties in the definition, but have difficulty understanding definitions that use logical expressions such as "however", "and", "or", "at least".
Level 3: Students at this level can use logical relationships between mathematical properties. They can formulate definitions.
Level 4: Students at this level are more successful in understanding the logical structure of mathematics. Because similar concepts can have different definitions.

It was seen that pre-service teachers at level 2 of Van Hiele's geometric thinking levels were able to express the properties in the definition of tetrahedron, right triangle, cone, and sphere. However, these pre-service teachers had difficulty in understanding logical expressions and therefore they were not successful in using the theorems related to the problems. Pre-service teachers at the 3rd level of geometric thinking and C proof perception type were able to use logical relationships between the properties of geometric shapes and formulate definitions by utilizing these properties. For example, they were able to write various equations depending on variables using the side and length properties of the triangles forming the given tetrahedron. Similarly, pre-service teachers at the 4th level of geometric thinking were able to see the relationship between the height of one face of the tetrahedron and the height of the object.

Findings related to the classification process

The third of the thinking processes, *classification*, requires pre-service teachers to be able to establish correct logical relationships and make more complex classifications.As in the identification process, similar to the definition process, all of the pre-service teachers in the 4th and 5th geometric thinking levels and in the A and B proof perception types were able to complete this process successfully. Pre-service teachers at the 3rd geometric thinking level and C proof perception type, who were successful in the identification process, also completed the classification process.It was found that while the pre-service teachers in the D proof perception type were not successful in the classification process, only some of the pre-service teachers in the C proof perception type at level 3 completed this process. Pre-service teachers at the 2nd level of geometric thinking, with proof types C and D, could not reach the classification process because they could not establish the necessary logical relationships. The response of the pre-service teacher at Level 3 and C proof perception type is given in Figure 2.

Figure 2 shows that the pre-service teacher with Level 2 and proof perception type C applied the Pythagorean relation and verified the equality by working on the right triangle 3-4-5, which is a special triangle.Pre-service teachers who presented proofs in this way made inferences informally.The students at Level 2 are known to be unable to formulate formal definitions, to explain subclasses by selecting only some of them from a list of properties, and to provide examples for special cases without the need for formal proof to generalize their empirical results. Similarly, some pre-service teachers in Level 3 and C proof perception type were observed to be able to reach the classification level by establishing the necessary relationships and even tried informal ways to verify the equations they found. The proof of the Pythagorean relation in this case involves an informal proof instead of formal proof methods. The pre-service teacher decided that the claim was true by evaluating a special right triangle and was convinced that the result obtained by writing the numbers 3-4-5 in the Pythagorean relation, which contains an algebraic expression, was proof. This finding bears resemblance to Harel and Sowder's (1998;2007) observation of students in the inductive proof scheme, making decisions about the validity of a claim by evaluating the results of one or

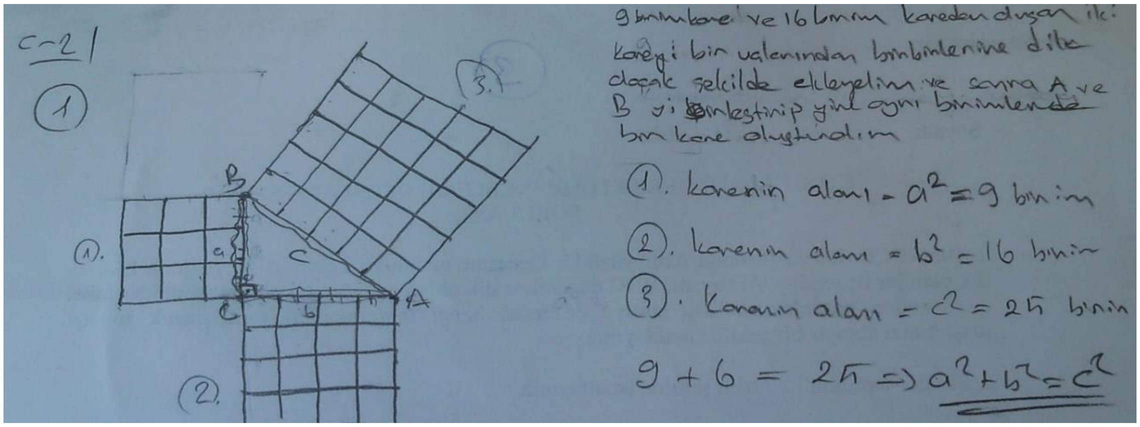

Figure 2. Solution of the pre-service teacher at level 2 and C proof type. Let's add 2 squares consisting of 9 unit squares and 16 unit squares perpendicular to each other at one end, then draw another square that accepts the line segment AB as an edge, the area of the first square becomes 9 unit squares, the area of the second square becomes 16 unit squares, the area of the third square becomes 25 unit squares. this proves the Pythagorean relation.

several examples or being convinced based on results obtained by substituting a few specific numbers into an algebraic expression.

On the contrary, pre-service teachers in level 3 and D proof perception type could not write the necessary equations and the relations between these equations due to their confusion about the symbols. Consequently, they could not reach the definition and classification processes in any of the questions. In the 4th and 5th geometric thinking levels, the pre-service teachers in the A and B proof perception types were able to go through the classification process similarly to the definition process. For example:

1.He/she was able to see the so-called super triad in a right triangle from the three perpendicular theorems, i.e. the height of the right triangle at the base and the height of the tetrahedron (object) according to the fundamental perpendicularity theorem.
2. By drawing four congruent right triangles inside the square, he/she determined the length of one side of the inner square as the hypotenuse. He/she wrote the side lengths of a right triangle within the unit circle using trigonometric expressions. They found the relationship between the height and base lengths of a right trapezoid and the side lengths of a right triangle.
3. They found relationships between lengths using similarity.

Findings related to the proof process

The fourth of the thinking processes, the proof process, requires pre-service teachers to make standard formal or higher-level proofs. All of the pre-service teachers at the 4th and 5th levels of Van Hiele's geometric thinking and in the A proof perception type were successful in this process. 25 of the pre-service teachers at the 4th geometric thinking level and B proof perception type were able to reach correct proofs in question 2.All pre-service teachers were able to prove the Pythagorean theorem in at least one way, but this was not observed for questions 1 and 3. Studies in the literature show that success in standard formal proofs is higher (Senk, 1989). The problem involving the Pythagorean theorem is an example of such a situation. Because the foundations of the relevant subject are given from middle school onwards. On the other hand, in question 3, which requires using the properties and relations of Euclidean geometry and transferring these properties to other environments and spaces, the pre-service teachers failed to do so.

According to the Van Hiele geometric thinking levels and proof perceptions of the pre-service teachers, the response of the pre-service teachers at Level 5, who completed the proof process, to the first problem is given in Figure 3, and to the third problem is given in Figure 4. When the answer of the pre-service teacher, who is at the 5th level of Van Hiele geometric thinking and has a proof perception type of A, presented in Figure 3 is analyzed, it was determined that she made a correct formal proof by using the direct proof method by stating her proof with her justifications. In this process, it is seen that he utilized the basic perpendicularity theorem, the three perpendiculars theorem, and the Pythagorean theorem and reached his conclusions by using sequential and logical justifications.

Figure 4 shows that the pre-service teacher obtained the surface area formula of the sphere with the help of the limit based on the maximum volume hemisphere drawn inside

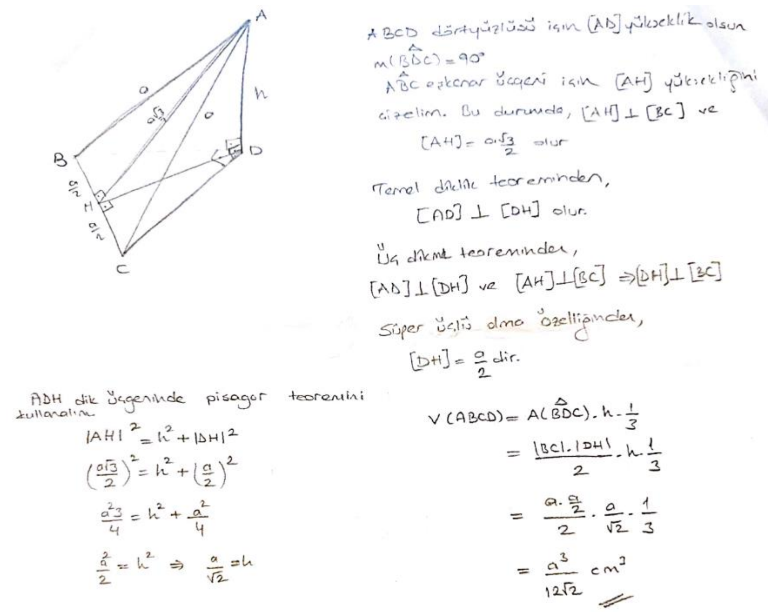

Figure 3. The solution of a pre-service teacher at level 5 and A proof perception type for problem 1.

the cone that she claimed to exist. Some similar proofs were made by pre-service teachers who used pyramids in which the base of the circle is assumed to cover the entire surface area of the sphere. Considering the characteristics of Van Hiele's geometric thinking level, a person at Level 5 is expected to be able to analyze different axiomatic systems, write differences and similarities, and make inferences by proposing theorems. Besides, Level 5 students can use and interpret the properties, relations, and theorems of Euclidean geometry by transferring them to other media or spaces. When evaluated in this context, it is seen that students in the A proof perception type and at Level 5 were able to perform the formal proof correctly and conclude the proof in the desired direction by employing their thinking processes in this direction.

Figure 5 presents the responses of the pre-service teachers at level 4 and in the A proof perception type who completed the proof process from the thinking processes regarding the second problem. Figure 5 shows that the pre-service teacher in level 4 and A proof perception type proved the Pythagorean relation similar to Euclid's method (Baki, 2014). This proof is one of the first known proofs in the history of mathematics. The proof using transformations is presented in four stages. The pre-service teacher consecutively matched the area of the square numbered 1 with the area of two parallel sides and placed the parallel side obtained in the square numbered 3 after rotation. Similarly, it performed the same operations for square 2. Using the area relation, he proved the correctness of the Pythagorean relation as a result of this process. When the characteristics of Level 4, which is the inference level among the Van Hiele geometric thinking levels, are examined, it is expected that pre-service teachers at this level will be able to recognize the components of the axiomatic system and use them flexibly, make inferences by proof, understand the importance of definitions and axioms, determine the necessary and sufficient conditions, and explain the reasons for the steps in the proof. It was also observed that the pre-service teacher did proof in accordance with the characteristics of type A among the types of perception of proof and could work with formal proof. In this respect, it was determined that the pre-service teacher was at the level of proof in terms of the thinking process and could understand and write standard formal proofs. Figure 6 shows the proof process completed by Pre-service Teacher 20, who was at level 4 and type A and the proof she presented for the Pythagorean relation.

Figure 6 shows that the pre-service teacher at level 4 and type A responded at the level of proof thinking and proved the Pythagorean relation similar to the method in Chinese sources (Baki, 2014).This proof is based on the square formed by using four equilateral right triangles and the area calculations made on it.

Figure 7 shows the proof of Pythagorean relation presented by Preservice Teacher 15, who is at Level 3 and type C. Figure 7 shows that the pre-service teacher tried to make the proof by using experimental arguments but was not successful. Drawing isosceles triangles on the sides of the triangle in the center, placing beads inside the

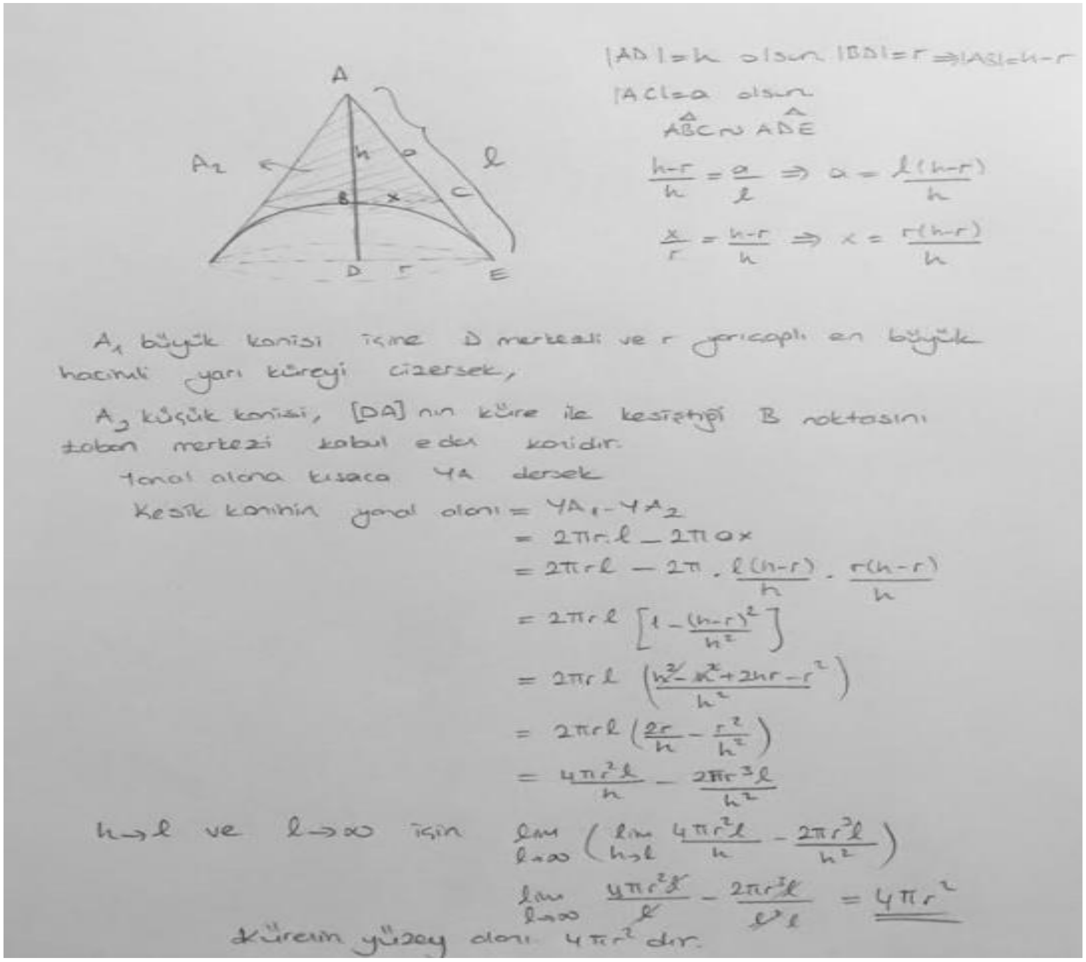

Figure 4. The solution of a pre-service teacher at level 5 and A proof perception type for problem 3.Let us draw the hemisphere of the largest volume with center D and radius r inside the large cone. The small cone A2 is the cone that takes the point B where it intersects the sphere as its base center. If we call the lateral area YA for short, the lateral area of the truncated cone is 4πr^2

isosceles triangles at the top, and pouring these beads into the largest isosceles triangle, he/she thinks that he/she proved the Pythagorean relation by using the area relation in the triangle. Considering that the pre-service teacher thinks that he/she has made proof based on the argument he/she has put forward when the thinking processes are analyzed, it shows that he/she is still at level 1 of the identification process and that he/she has misunderstood the two dimensions. Students at this stage cannot use the given mathematical definitions Likewise, this pre-service teacher, who was at Level 3 in terms of Van Hiele geometric thinking levels, could not establish the necessary logical relationships. When Yıldırım's (2014) statement that "the criterion of truth in the eyes of the mathematician should be sought in logical proof, not in factual proof" is taken into consideration, the solution made here cannot be accepted as proof.

CONCLUSION AND SUGGESTIONS

This study was conducted to examine in depth the types of proof perceptions and thinking processes of pre-service elementary mathematics teachers according to their Van Hiele geometric thinking levels.Accordingly, the geometric thinking levels and proof perception types of pre-service teachers were first determined, and then the relationship between proof perception types and Van Hiele's geometric thinking levels was revealed. Pre-service teachers' thinking processes while doing proofs were analyzed in comparison with their geometry thinking levels and proof perception types. The study concludes:

12% of the pre-service teachers were at Level 2 (Analysis), 28.3% were at Level 3 (Inference based on experience), 52.2% were at Level 4, and 7.5% were at Level 5; 12%,

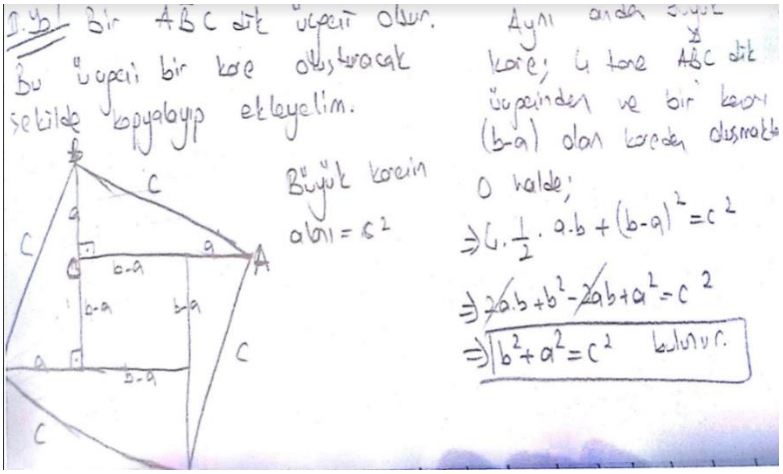

Figure 6. The solution of a pre-service teacher at level 5 and A proof perception type for problem 2.

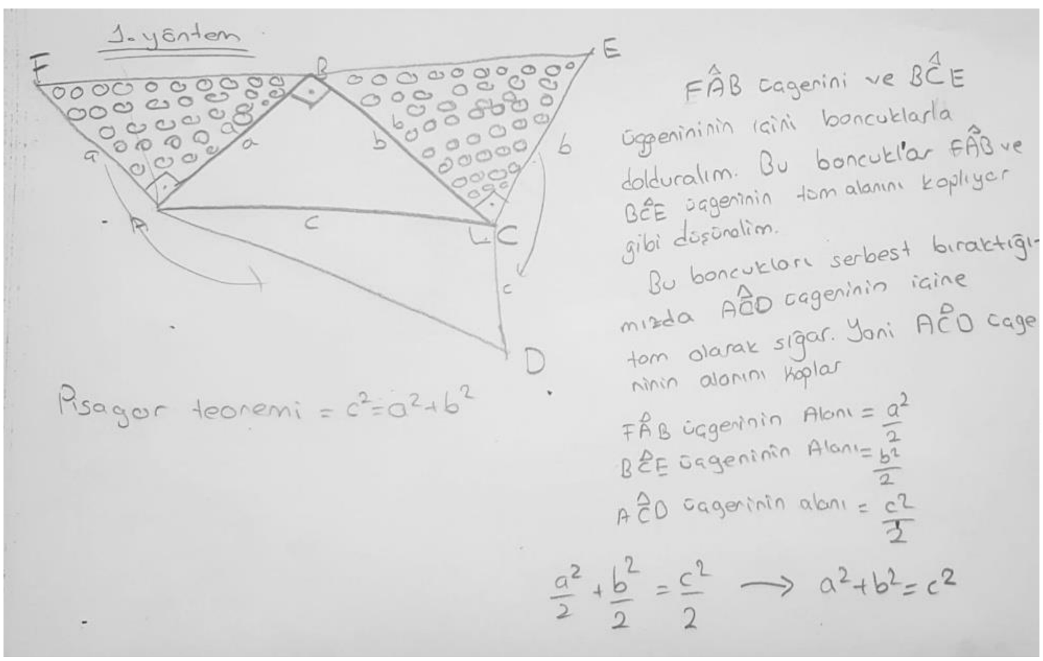

Figure 7. The solution of a pre-service teacher with Level 3 and C proof perception type. Preservice Teacher 15: Let's fill triangle FAB and triangle BCE with beads. let's pretend that these beads cover the whole area of triangle FAB and triangle ACD. when we release these beads, they fill triangle ACD completely. So it covers the area of triangle ACD.

47.8%, 17.9% and 22.3%of the pre-service teachers were in the A, B, C, and D proof perception types, respectively; 7.5% of the pre-service teachers reached the 5th level of Van Hiele geometric thinking, and these people had a perception of proof in Type A (Student needs to work with formal proof and rejects informal proof) category and successfully completed their thinking processes; 47.8% of thepre-service teachers reached the 4th level of geometric thinking, almost all of them had Type B (Student accepts the necessity of formal proof, but uses informal proof practices temporarily until he/she masters formal proof) and the others had Type A perception of proof; Pre-service teachers with Type C (Student accepts intuitive and empirical arguments as proofs, considers formal proofs

necessary for passing exams) and Type D (Student accepts the need for formal proof, but usually sees it as symbol manipulation and does not see the proof as meaningful) are at the second and third level of geometric thinking; Among the pre-service teachers at the 5th level of geometric thinking, those with the type A perception type of proof were able to reach the level of proof from the thinking processes and completed all levels;
Among the 4th-level geometric thinking level pre-service teachers, those in the type A and type B categories were able to reach the level of proof from the thinking processes; Among the thinking processes, those in the 5th level, type A; 4th level, type A and type B; and 3rd level, type C categories were able to reach the classification level; Among the 2nd and 3rd thinking levels, only those in the type C category were able to reach the definition level, but they could not move to the next level;

Type D pre-service teachers at the 2nd and 3rd levels remained at the recognition level, which is the first of the thinking levels and could not make any progress.

According to the results obtained from the study, the geometric thinking levels and proof perception types of pre-service teachers affect their thinking processes. It is thought that determining the current status of pre-service teachers' geometric thinking levels and proof perception types in the teaching practices to be carried out in this direction and conducting proof studies accordingly will have positive effects on pre-service teachers' perceptions towards proof and proof-making. In schools, an inductive assessment approach is applied, which originates from the way mathematics is taught, affects students' approaches to proof, and causes them to emphasize the final process of proof. This is why most students prefer informal proof and find formal proof difficult because of the technical difficulties involved, they rely more on their intuition and experience. Some students who accept the necessity of formal proof see it as symbolic manipulation.In other words, according to students, proof becomes a procedure, they see formal proof as unnecessary and cannot do it. Within this scope, teaching practices should be organized to improve the geometric thinking levels of pre-service teachers to improve their proof skills and thinking processes. In this way, pre-service teachers can understand the proofs given or the language of the proof.

For pre-service teachers to be effective educators, it is important to understand their geometric thinking levels, the types of proof perception and the thinking processes they follow when writing proofs. This is because pre-service teachers' geometric thinking levels significantly affect their ability to teach geometry effectively (González et al., 2022). Van Hiele's theory provides a framework for understanding the stages of geometric thinking development. To prepare future educators, the main goal should be to enable them to reach the level of inference from geometric thinking levels. They can then understand and construct formal and informal proofs and build a solid foundation for teaching mathematics topics. Pre-service teachers' perception types of proofs are also thought to directly affect their ability to use and teach proofs and their results effectively by relating them to daily life. Accordingly, future educators should develop a structured perception of evidence, as this can enable them to see evidence as coherent and logical arguments rather than a series of memorized steps. Developing teachers and pre-service teachers' perception types of proofs may enable them to effectively convey logical reasoning and the structure of proofs to their students in the mathematics teaching process.

Pre-service teachers should have strong proof-writing skills in addition to their geometric thinking levels and proof perception types. Effective proof-writing is essential for the clear communication of mathematical ideas and logical reasoning. Teachers need to be able to generate the evidence themselves to teach their students how to carry out this process. In this direction, undergraduate programs can offer in-depth geometry and geometry teaching courses to help pre-service teachers enhance their geometric thinking levels, proof perception types, and proof-writing skills.In these courses, comprehensive practices that focus not only on the content but also on the logical structure of geometric proofs can be carried out to help them reach high levels of geometric thinking and develop a perception of structural evidence in the pre-service period. Furthermore, participating in problem-solving activities that require making predictions and constructing proofs can improve both the geometric thinking and proof-writing skills of pre-service teachers. Pre-service teachers can be encouraged to explore and verify mathematical ideas to develop a deeper understanding of the subject matter. Studies that allow pre-service teachers to evaluate themselves by acting as a mirror to their thinking processes toward proof can be conducted. These assessments can allow them to evaluate not only the correctness of a proof but also the clarity and logical structure of the argument.In future studies, various studies can be conducted using these strategies to improve the types of proofs, proof perceptions, and geometric thinking levels of pre-service teachers. This can be done by enabling them to discover the connections between geometry and other mathematical fields. Different theorems and proofs from the theorems used in this study can be used to extend the data and contribute to experimental studies.

It is recommended to organize a learning environment to improve Van Hiele's geometric thinking levels of pre-service teachers, to develop informal proofs and intuitions and to encourage pre-service teachers to make formal proofs, to conduct studies to develop positive attitudes towards proof for pre-service teachers who see proofs as unnecessary or as a pile of procedures in line with their perceptions of proof and to extend the study to secondary mathematics pre-service teachers.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

ACKNOWLEDGEMENTS

The authors are grateful for the financial support given by Balıkesir University Scientific Research Projects.

REFERENCES

Almeida D (2000). A survey of mathematics undergraduates' interaction with proof: some implications form mathematics education. International Journal of Mathematical Education in Scienceand Technology 31(6):869-890.

Altun M (2013). Mathematics teaching in secondary schools (6th, 7th, 8th grades). Bursa: Aktüel Alfa Akademi Publications

Aydın N, Halat E(2009). The Impacts of Undergraduate Mathematıcs Courses on College Students' GeometrıcReasonıng Stages. The Montana Mathematics Enthusiast 4(2):151-164.

Baykul Y, Aşkar P (1987). Problem and problem solving, Mathematics teaching. Open Education Faculty Publications.

Büyüköztürk Ş, Kılıç Çakmak E, Akgün ÖE, Karadeniz Ş, Demirel F (2019). Scientific research methods in education. Pegem Academy.

Dreyfus T (1999). Why Johnny can't prove. Educational Studies in Mathematics 38:85-109.

Duatepe A (2000). An İnvestigation On the Relationship Between Van Hiele Geometric Level of Thinking And Demographic Variables For Preservice Elementary School Teachers. Master's Thesis, Middle East Technical University.

Ferrini-Mundy J, Senk S (2006). Knowledge of algebra for teaching: Framework, item development and pilot results. Research symposium at the research presession of NCTM annual meeting. St. Louis, MO.

Genç M, Karataş İ (2018). Integrating History of Mathematics into Mathematics Teaching: Al-Khwarizmi's Completing The Square Method. KastamonuEducation Journal 26(1):219-230.

Gibson D (1998). Students' use of diagrams to develop proofs in an introductory analysis course. InA. H. Schoenfeld J, Kaput, E. Dubinsky (Eds.), Research in collegiate mathematics education. Ill (pp. 284-307). Providence, RI: American Mathematical Society.

Goetting M (1995). The college students' understanding of mathematical proof. Unpublished Doctoral Dissertation. Available from ProQuest Dissertations and Theses database. https://www.proquest.com/openview/b8aae178c0ebeb0c9db8911a5d5daac9/1?pq-origsite=gscholarandcbl=18750anddiss=y

González A, Manero V, Arnal-Bailera A, Puertas ML (2022). Proof levels of graph theory students under the lens of the Van Hiele model. International Journal of Mathematical Education in Science and Technology 1-19.

Gutierrez A, Jaime A (1998). On the assessment of the Van Hiele levels of reasoning. Focus on Learning Problems in Mathematics 20(2/3):27-46.

Harel G, Sowder L (1998). Students' proof schemes: Results from exploratory studies. American Mathematical Society 7:234-283.

Harel G, Sowder L (2007). Toward comprehensive perspectives on the learning and teaching of proof. InF. Lester (Ed.),Second handbook of research on mathematics teaching and learning. Reston, VA: National Council of Teachers of Mathematics, pp. 805-842.

Heinze A, Reiss KM (2003). Reasoning and proof: Methodological knowledge as a component of proofcompetence. In: M. A. Mariotti (Ed.), European Research in Mathematics Education III: Procedings of theThird Conference of the European Society for Research in Mathematics Education (pp. 1-10). Bellaria, Italy: University of Pisa and ERME.

Hemmi K (2010). Three styles characterising mathematicians' pedagogical perspectives on proof. Educational Studies in Mathematics 75:271-291.

Herbst PG (2002). Engaging students in proving: A double bind on the teacher. Journal for research in mathematics education 33(3):176-203.

Isoda M, Katagiri S (2012). Mathematics thinking: How to develop it in the classroom. Singapore: World Scientific Publishing.

Jones K (2000). The Student Experience of Mathematical Proof at University Level. International Journal of Mathematical Education in Scienceand Technology 31(1):53-60.

Ko YY, Knuth E (2009). Undergraduate mathematics majors' writing performance producing proof sand counter examples about continuous functions. The Journal of Mathematical Behavior 28(1):68-77.

Lamb S, Maire Q, Doecke E (2017). Key skills for the 21st Century: an evidence-based review. Report prepared for the State of New South Wales (Department of Education) Sydney. Retrieved from http://vuir.vu.edu.au/35865/1/Key-Skills-forthe-21st-Century-Analytical-Report.pdf

Mason J, Burton L, Stacey K (1985). Thinking mathematically. Revised Edition. England: Addison-Wesley Publishers, Wokingham.

Ministry of Education [MoNE] (2018). Teaching mathematics (1, 2, 3, 4, 5, 6, 7 and 8th grades) program. Ankara: Ministry of National Education, Board of Education.

Moore RC (1994). Making the transition to formal proof. Educational Studies in Mathematics 27:249-266.

Olkun S, Toluk Z (2003) Activity-based mathematics teaching in primary education. Ankara: Anı Publishing.

Ozan-Leylum Ş, Odabaşı HF, Kabakçı YI (2017). The Importance of Case Study Research in Educational Settings, Journal of Qualitative Research in Education 5(3):369-385.

Özdemir E, OvezDikkartın FT (2012). A research on proof perceptions and attitudes towards proof and proving: some implications for elementary mathematics prospective teachers. Procedia-Social and Behavioral Sciences 46:2121-2125.

Riley KJ (2003). An investigate of prospective secondary mathematics teachers' conceptions of proof and refutations. Unpublished Doctoral Dissertation, Available from ProQuest Dissertation and Theses database.https://www.proquest.com/openview/538f1016fc75f665d3e3bfefa9535238/1?pq-origsite=gscholarandcbl=18750anddiss=y

Ross KA (1998). Doing and proving: The place of algorithms and proofs in school mathematics. The American Mathematical Monthly 105(3):252-255.

Senk SL (1989). Van Hiele levels and achievement in writing geometry proofs. Journal for Research in Mathematics Education 20(3):309-321.

Stylianides GJ, Stylianides AJ, Philippou GN (2004). Undergraduate students' understanding of the contraposition equivalence rule in symbolic and verbal contexts. Educational Studies in Mathematics 55:133-162.

Stylianides GJ, Stylianides AJ,Philippou GN (2007). Preservice teachers' knowledge of proof by mathematical induction. Journal of Mathematics Teacher Education 10:145-166.

Tall D (1991). Advanced mathematical thinking. USA: Kluwer Academic Publishers.

Thomas M, Klymchuk S (2012). The school–tertiary interface in mathematics: Teaching style and assessment practice. Mathematical Education Research Journal 24:283-300.

Van Oers B (2010). Emergent mathematical thinking in the context of play. Educational Studies In Mathematics 74:23-37.

Usiskin Z(1982). Van Hiele Levels and achievement in secondary school geometry. CDASSG Project,. University of Chicago.

Van Hiele PM (1986). Structure and insight: a theory of mathematics education. Academic Pres. Inc. Orlando, Florida.

Weber K (2001). Student Difficulty in Constructing Proofs: The Need for Strategic Knowledge. Educational Studies in Mathematics 48:101-119.

Weber K (2004). A Framework for Describing the Processes that Undergraduates Use to Construct Proofs. In: M. J.Hoinesand AB. Fuglestad (Eds.) International Group for the Psychology of Mathematics Education 28th, Bergen Norway 14-18.

Yıldırım C (2014). Mathematical thinking Istanbul: Remzi Publications.

Yıldırım A, Şimşek H (2011). Qualitative Research Methods In Social

Sciences (8th Edition). Ankara: Seçkin Publishing.
Yin RK (2014). Case study research: design and methods (5th edition). SAGE Publications.