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Full Length Research Paper

New Holder-type inequalities for the Tracy-Singh and Khatri-Rao products of positive matrices

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Recently, authors established a number of inequalities involving Khatri-Rao product of two positive matrices. Here, in this paper, the results are established in three ways. First, we find new Holder-type inequalities for Tracy-Singh and Khatri-Rao products of positive semi-definite matrices. Secondly, the results are extended to provide estimates of sums of the Khatri-Rao and Tracy-Singh products of any finite number of positive semi-definite matrices. Finally, the results lead to inequalities involving the Hadamard and Kronecker products, as a special case.

Key words: Tracy-Singh products, Khatri-Rao products, positive (semi) definite matrices, Holder inequalities.

INTRODUCTION

Consider matrices $A = [a_{ij}]$, $C = [c_{ij}] \in M_{m,n}$ and $B = [b_{kl}] \in M_{p,q}$. Let A and B be partitioned as $A = [A_{ij}]$ and $B = [B_{kl}]$ $(1 \le i \le t,\ 1 \le j \le c)$, where A_{ij} is an $m_i \times n_j$ matrix and B_{kl} is a $p_k \times q_l$ matrix ($m = \sum_{i=1}^{t} m_i$, $n = \sum_{j=1}^{c} n_j$, $p = \sum_{i=1}^{t} p_i$, $q = \sum_{j=1}^{c} q_j$). Let $A \otimes B$, $A \circ C$, $A \Theta B$ and $A * B$ be the Kronecker, Hadamard, Tracy-Singh and Khatri-Rao products, respectively. The definitions of the mentioned four matrix products are given by Al-Zhour and Kilicman (2007); Al-Zhour and Kilicman (2006a); Al-Zhour and Kilicman (2006b); Cao et al. (2002); Liu (2002); Mond and Pecaric (2000) and Visick (2000).

$$A \otimes B = \left(a_{ij} B\right)_{ij} \ ; \ A \circ C = \left(a_{ij} c_{ij}\right)_{ij} \ ; \tag{1}$$

$$A * B = \left(A_{ij} \otimes B_{ij}\right)_{ij} \ ; \ A \Theta B = \left(A_{ij} \Theta B\right)_{ij} = \left(\left(A_{ij} \otimes B_{kl}\right)_{kl}\right)_{ij}. \tag{2}$$

Additionally, the Khatri-Rao product can be viewed as a generalized Hadamard product and the Tracy-Singh product as a generalized Kronecker product, that is, for a non-partitioned matrix A and B, their $A \Theta B$ is $A \otimes B$ and $A * B$ is $A \circ B$. For any compatibly partitioned matrices A, B, C, D, we shall make frequent use the following properties of the Tracy-Singh product (Al-Zhour and Kilicman, 2007; Al-Zhour and Kilicman, 2006a and b; Cao et al., 2002; Liu, 2002):

$$(A \Theta B)(C \Theta D) = (AC) \Theta (BD); \tag{3}$$

$$(A \Theta B)^* = A^* \Theta B^*. \tag{4}$$

The Hermitian matrix A is called positive semi-definite (Written $A \ge 0$) if $\langle Ax, x \rangle \ge 0$ for any vector x and positive definite (Written $A > 0$) if $\langle Ax, x \rangle > 0$ for any non-zero vector x. For Hermitian matrices A and B, the relations $A \ge B$ mean that $A - B \ge 0$ is a positive semi-definite. Given a positive semi-definite matrix A and $k \ge 1$ be a given integer, then there exists a unique positive semi-definite matrix B such that $A = B^k$, written as $B = A^{1/k}$. Denote H_n^+ be the set of all positive definite

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$n \times n$ matrices. If $A \in H_n^+$, then the spectral decomposition of A assures that there exists a unitary matrix U such that (Zhang, 1999):

$$A = U^* D\, U = U^* diag(\lambda_i) U \;,\; U^* U = I_m \;.$$

Here, $D = diag(\lambda_i) = diag(\lambda_1, \cdots, \lambda_m)$ is the diagonal matrix with diagonal entries λ_i (λ_i are the positive eigen values of A). For any real number r, A^r is defined by:

$$A^r = U^* D^r U = U^* diag(\lambda_i^r) U \;. \tag{5}$$

Now we present the following basic results (Al-Zhour and Kilicman, 2007; Al-Zhour and Kilicman, 2006b; Cao et al., 2002; Rao and Rao 1998).

Lemma 1

Let $A_i \in M_{m(i),n(i)}$ $(1 \le i \le k, k \ge 2)$ be compatibly partitioned matrices ($m = \prod_{i=1}^{k} m(i)$, $n = \prod_{i=1}^{k} n(i)$, $r = \sum_{j=1}^{t}\prod_{i=1}^{k} m_j(i)$, $s = \sum_{j=1}^{c}\prod_{i=1}^{k} n_j(i)$, $m(i) = \sum_{j=1}^{t} m_j(i)$, $n(i) = \sum_{j=1}^{c} n_j(i)$). Then there exists two real matrices Z_1 of order $m \times r$ and Z_2 of order $n \times s$ such that $Z_1^T Z_1 = I_1$, $Z_2^T Z_2 = I_2$ and:

$$\prod_{i=1}^{k} {}_* A_i = Z_1^T \left(\prod_{i=1}^{k} \Theta A_i \right) Z_2 \,. \tag{6}$$

Here, I_1 and I_2 are identity matrices of order $r \times r$ and $s \times s$, respectively.

Lemma 2

Let a_i and b_i $(1 \le i \le k)$ be positive scalars. If $1 \le p, q < \infty$ satisfy $(1/p) + (1/q) = 1$. Then the scalar Holder inequality is given by:

$$\sum_{i=1}^{k} a_i b_i \le \left(\sum_{i=1}^{k} a_i^p \right)^{1/p} \left(\sum_{i=1}^{k} b_i^q \right)^{1/q} . \tag{7}$$

Corollary 1

Let a_i and b_i $(1 \le i \le k)$ be positive scalars. If $0 < p < \infty$ and $0 < q < 1$ satisfy $(1/q) - (1/p) = 1$. Then:

$$\sum_{i=1}^{k} a_i b_i \ge \left(\sum_{i=1}^{k} a_i^{-p} \right)^{-1/p} \left(\sum_{i=1}^{k} b_i^q \right)^{1/q} \tag{8}$$

Proof: The condition $(1/q) - (1/p) = 1$ can be rewritten as $\frac{1}{(p/q)} + \frac{1}{(1/q)} = 1$.

Since $1 \le \frac{p}{q} < \infty$ and $1 \le \frac{1}{q} < \infty$, then by Lemma 2 we have:

$$\sum_{i=1}^{k} (b_i)^q = \sum_{i=1}^{k} a_i^{-q} (a_i b_i)^q \le \left(\sum_{i=1}^{k} \left((a_i)^{-q} \right)^{p/q} \right)^{q/p} \left(\sum_{i=1}^{k} \left((a_i b_i)^q \right)^{1/q} \right)$$

$$= \left(\sum_{i=1}^{k} (a_i)^{-p} \right)^{q/p} \left(\sum_{i=1}^{k} (a_i b_i) \right)^q .$$

hence

$$\sum_{i=1}^{k} a_i b_i \ge \left(\sum_{i=1}^{k} a_i^{-p} \right)^{-1/p} \left(\sum_{i=1}^{k} b_i^q \right)^{1/q} .$$

This completes the proof of Corollary 1. ϒ

The problem may occur that we cannot find Holder-type inequalities for usual product of positive semi definite matrices, but here, we can find new Holder-type inequalities for the Tracy-Singh, Khatri-Rao, Kronecker and Hadamard products of positive matrices which are very important for applications to establish new inequalities involving these products. Since it is sometimes difficult to compute, for example, ranks, determinants, eigen values, norms of large matrices, it is of great importance to provide estimates of sums of these products of any finite number of matrices by applying Holder-type inequalities of positive matrices.

MAIN RESULTS

Based on the aforementioned basic results, and the general connection between the Khatri-Rao and Tracy-Singh products in Lemma 1, we derive some inequalities with respect to the Tracy-Singh and Khatri-Rao products and extend these results to any finite number of matrices. These results lead to inequalities involving Kronecker and Hadamard products, as a special case.

Theorem 1

Let $A_i \in H_n^+$ and $B_i \in H_m^+$ be partitioned matrices ($1 \le i \le k$). If $1 \le p, q < \infty$ satisfy $(1/p)+(1/q)=1$. Then:

$$\sum_{i=1}^{k} A_i \Theta B_i \le \left(\sum_{i=1}^{k} A_i^p\right)^{1/p} \Theta \left(\sum_{i=1}^{k} B_i^q\right)^{1/q}. \qquad (9)$$

Proof: By assumption there exist a unitary matrix $U \in M_n$ and a unitary matrix $V \in M_m$ such that $A_i = U^* D_i U$ with $D_i = diag(d_{i1}, \cdots, d_{in})$ and $B_i = V^* T_i V$ with $T_i = diag(t_{i1}, \cdots, t_{im})$, where d_{ij}, t_{ij} are nonnegative real numbers for all i and j. It follows that:

$$A_i \Theta B_i = (U^* D_i U)\Theta (V^* T_i V) = (U^* \Theta V^*)(D_i \Theta T_i)(U\Theta V)$$
$$= (U\Theta V)^* diag(d_{i1}t_{i1}, \cdots, d_{i1}t_{im}, \cdots, d_{in}t_{i1}, \cdots, d_{in}t_{im})(U\Theta V)$$

So, by using Lemma 2, we have:

$$\sum_{i=1}^{k} A_i \Theta B_i =$$
$$(U\Theta V)^* diag\left(\sum_{i=1}^{k} d_{i1}t_{i1}, \cdots, \sum_{i=1}^{k} d_{i1}t_{im}, \cdots, \sum_{i=1}^{k} d_{in}t_{i1}, \cdots, \sum_{i=1}^{k} d_{in}t_{im}\right)(U\Theta V)$$

$$\le (U\Theta V)^* diag\left[\left(\sum_{i=1}^{k} d_{i1}^p\right)^{1/p}\left(\sum_{i=1}^{k} t_{i1}^q\right)^{1/q}, \cdots, \left(\sum_{i=1}^{k} d_{i1}^p\right)^{1/p}\left(\sum_{i=1}^{k} t_{im}^q\right)^{1/q}\right.$$
$$, \cdots, \left(\sum_{i=1}^{k} d_{in}^p\right)^{1/p}$$
$$\left.\left(\sum_{i=1}^{k} t_{i1}^q\right)^{1/q}, \cdots, \left(\sum_{i=1}^{k} d_{in}^p\right)^{1/p}\left(\sum_{i=1}^{k} t_{im}^q\right)^{1/q}\right](U\Theta V)$$
$$= (U\Theta V)^* \left\{ diag\left[\left(\sum_{i=1}^{k} d_{i1}^p\right)^{1/p}, \cdots, \left(\sum_{i=1}^{k} d_{in}^p\right)^{1/p}\right]\right.$$
$$\left.\Theta diag\left[\left(\sum_{i=1}^{k} t_{i1}^q\right)^{1/q}, \cdots, \left(\sum_{i=1}^{k} t_{im}^q\right)^{1/q}\right]\right\}(U\Theta V)$$
$$= (U^* \Theta V^*) \left\{\left(\sum_{i=1}^{k} D_i^p\right)^{1/p} \Theta \left(\sum_{i=1}^{k} T_i^q\right)^{1/q}\right\}(U\Theta V)$$
$$= \left\{\left(U^* \left(\sum_{i=1}^{k} D_i^p\right)^{1/p} U\right)\Theta\left(V^*\left(\sum_{i=1}^{k} T_i^q\right)^{1/q} V\right)\right\}$$

$$= \left(\sum_{i=1}^{k} A_i^p\right)^{1/p} \Theta \left(\sum_{i=1}^{k} B_i^q\right)^{1/q}. \Upsilon$$

Corollary 2

Let $A_i \in H_n^+$ and $B_i \in H_m^+$ be partitioned matrices ($1 \le i \le k$). If $1 \le p, q < \infty$ satisfy $(1/p)+(1/q)=1$. Then:

$$\sum_{i=1}^{k} A_i * B_i \le \left(\sum_{i=1}^{k} A_i^p\right)^{1/p} * \left(\sum_{i=1}^{k} B_i^q\right)^{1/q}. \qquad (10)$$

Proof: Follows immediately by applying Lemma 1 and Theorem 1.. Υ

Corollary 3

Let $A_i^{(j)} \in H_{n^{(j)}}^+$ $(1 \le i \le k)$ be partitioned $n^{(j)} \times n^{(j)}$ matrices, $(1 \le j \le r)$. Let $1 \le \left\{p^{(j)}\right\}_{j=1}^{r} < \infty$ satisfy $\sum_{j=1}^{r}(1/p^{(j)}) = 1$. Then:

$$\sum_{i=1}^{k}\left(\prod_{j=1}^{r} \Theta A_i^{(j)}\right) \le \prod_{j=1}^{r} \Theta\left(\sum_{i=1}^{k} (A_i^{(j)})^{p^{(j)}}\right)^{1/p^{(j)}}. \qquad (11)$$

Proof: Using Theorem 1, the corollary follows by induction on k. Υ

Corollary 4

Let $A_i^{(j)} \in H_{n^{(j)}}^+$ $(1 \le i \le k)$ be commutative partitioned $n^{(j)} \times n^{(j)}$ matrices, $(1 \le j \le r)$. Let $1 \le \left\{p^{(j)}\right\}_{j=1}^{r} < \infty$ satisfy $\sum_{j=1}^{r}(1/p^{(j)}) = 1$. Then:

$$\sum_{i=1}^{k}\left(\prod_{j=1}^{r} * A_i^{(j)}\right) \le \prod_{j=1}^{r} *\left(\sum_{i=1}^{k} (A_i^{(j)})^{p^{(j)}}\right)^{1/p^{(j)}}. \qquad (12)$$

Proof: Using Corollary 3 and Lemma 1, the corollary follows by induction on k .Υ

We give an example using products of three matrices ($r = 3$). Let $A_i^{(j)}$ $(1 \le i \le k)$ be $n \times n$ positive definite partitioned matrices, $(1 \le j \le 3)$. Let $1 \le \left\{p^{(j)}\right\}_{j=1}^{3} < \infty$ satisfy $(1/p^{(1)}) + .(1/p^{(2)}) + (1/p^{(3)}) = 1$. Then:

(i)
$$\sum_{i=1}^{k} A_i^{(1)} \Theta A_i^{(2)} \Theta A_i^{(3)} \le \left(\sum_{i=1}^{k} A_i^{p^{(1)}}\right)^{1/p^{(1)}} \Theta \left(\sum_{i=1}^{k} A_i^{p^{(2)}}\right)^{1/p^{(2)}} \Theta \left(\sum_{i=1}^{k} A_i^{p^{(3)}}\right)^{1/p^{(3)}} . \quad (13)$$

(ii)
$$\sum_{i=1}^{k} A_i^{(1)} * A_i^{(2)} * A_i^{(3)} \le \left(\sum_{i=1}^{k} A_i^{p^{(1)}}\right)^{1/p^{(1)}} * \left(\sum_{i=1}^{k} A_i^{p^{(2)}}\right)^{1/p^{(2)}} * \left(\sum_{i=1}^{k} A_i^{p^{(3)}}\right)^{1/p^{(3)}} . \quad (14)$$

Theorem 2

Let $A_i \in H_n^+$ and $B_i \in H_m^+$ be partitioned matrices ($1 \le i \le k$). If $0 < p < \infty$ and $0 < q < 1$ satisfy $(1/q) - (1/p) = 1$. Then:

$$\sum_{i=1}^{k} A_i \Theta B_i \ge \left(\sum_{i=1}^{k} A_i^{-p}\right)^{-1/p} \Theta \left(\sum_{i=1}^{k} B_i^{q}\right)^{1/q} \quad (15)$$

Proof: By assumption there exist a unitary matrix $U \in M_n$ and a unitary matrix $V \in M_m$ such that $A_i = U^* D_i U$ with $D_i = diag(d_{i1}, \cdots, d_{in})$ and $B_i = V^* T_i V$ with $T_i = diag(t_{i1}, \cdots, t_{im})$, where d_{ij}, t_{ij} are nonnegative real numbers for all i and j. It follows that:

$$A_i \Theta B_i = \left(U^* D_i U\right) \Theta \left(V^* T_i U\right) = \left(U^* \Theta V^*\right)\left(D_i \Theta T_i\right)\left(U \Theta V\right)$$
$$= \left(U \Theta V\right)^* diag(d_{i1}t_{i1}, \cdots, d_{i1}t_{im}, \cdots, d_{in}t_{i1}, \cdots, d_{in}t_{im})\left(U \Theta V\right)$$

So, by using Corollary 1, we have:

$$\sum_{i=1}^{k} A_i \Theta B_i =$$
$$\left(U\Theta V\right)^* diag\left(\sum_{i=1}^{k} d_{i1}t_{i1}, \cdots, \sum_{i=1}^{k} d_{i1}t_{im}, \cdots, \sum_{i=1}^{k} d_{in}t_{i1}, \cdots, \sum_{i=1}^{k} d_{in}t_{im}\right)\left(U\Theta V\right)$$
$$\ge \left(U\Theta V\right)^* diag\left[\left(\sum_{i=1}^{k} d_{i1}^{-p}\right)^{-1/p}\left(\sum_{i=1}^{k} t_{i1}^{q}\right)^{1/q}, \cdots, \left(\sum_{i=1}^{k} d_{i1}^{-p}\right)^{-1/p}\left(\sum_{i=1}^{k} t_{im}^{q}\right)^{1/q}\right.$$
$$, \cdots, \left(\sum_{i=1}^{k} d_{in}^{-p}\right)^{-1/p}\left(\sum_{i=1}^{k} t_{i1}^{q}\right)^{1/q}$$
$$\left., \cdots, \left(\sum_{i=1}^{k} d_{in}^{-p}\right)^{-1/p}\left(\sum_{i=1}^{k} t_{im}^{q}\right)^{1/q}\right]\left(U\Theta V\right)$$
$$= \left(U\Theta V\right)^* \left\{diag\left[\left(\sum_{i=1}^{k} d_{i1}^{-p}\right)^{-1/p}, \cdots, \left(\sum_{i=1}^{k} d_{in}^{-p}\right)^{-1/p}\right]\right.$$
$$\left.\Theta diag\left[\left(\sum_{i=1}^{k} t_{i1}^{q}\right)^{1/q}, \cdots, \left(\sum_{i=1}^{k} t_{im}^{q}\right)^{1/q}\right]\right\}\left(U\Theta V\right)$$
$$= \left(U^* \Theta V^*\right)\left\{\left(\sum_{i=1}^{k} D_i^{-p}\right)^{-1/p} \Theta \left(\sum_{i=1}^{k} T_i^{q}\right)^{1/q}\right\}\left(U\Theta V\right)$$
$$= \left\{\left(U^*\left(\sum_{i=1}^{k} D_i^{-p}\right)^{-1/p} U\right)\Theta\left(V^*\left(\sum_{i=1}^{k} T_i^{q}\right)^{1/q} V\right)\right\}$$
$$= \left(\sum_{i=1}^{k} A_i^{-p}\right)^{-1/p} \Theta \left(\sum_{i=1}^{k} B_i^{q}\right)^{1/q} . \ \Upsilon$$

Corollary 5

Let $A_i \in H_n^+$ and $B_i \in H_m^+$ be partitioned matrices ($1 \le i \le k$). If $0 < p < \infty$ and $0 < q < 1$ satisfy $(1/q) - (1/p) = 1$. Then:

$$\sum_{i=1}^{k} A_i * B_i \ge \left(\sum_{i=1}^{k} A_i^{-p}\right)^{-1/p} * \left(\sum_{i=1}^{k} B_i^{q}\right)^{1/q} \quad (16)$$

Proof: Follows immediately by Lemma 1 and Theorem 2. Υ

Remark: The results obtained in this section are quite general. Now, as a special case, consider if the matrices in the main result are non-partitioned, we then have Holder type inequalities involving Kronecker and Hadamard products by replacing Θ by $\otimes$ and $*$ by $\circ$.

Conclusion

The problem may occur that we can't find Holder-type inequalities for usual product of positive (semi) definite matrices, but we established some Holder-type inequalities for the Tracy-Singh, Khatri-Rao, Kronecker and Hadamard products of positive semi-definite matrices which are very important for applications. How to extend the use of Holder-type inequalities in the main result for estimating, for example, ranks, determinants, eigen values, norms of sums of the mentioned four matrix products of finite number of large positive semi-definite matrices, requires further research.

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