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Full Length Research Paper

Analytical solution for strongly nonlinear oscillation systems using energy balance method

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This paper applied He's energy balance method (EBM) to solve the non-natural vibrations and oscillations. We find that this method (EBM) works very well for the whole range of initial amplitudes and does not demand small perturbation and also sufficiently accurate to both linear and nonlinear physics and engineering problems. We consider a high nonlinear single degree of freedom to illustrate the effectiveness and convenience of the method. He's energy balance method as approximate method and Runge-Kutta's (RK) algorithm was also implemented to solve the governing equation through a numerical method. Finally, the accuracy of the solution obtained by the approximate method (EBM) has been shown graphically and compared with that of the numerical solution.

Key words: Energy balance method, nonlinear oscillators, mathematical pendulum, Runge-Kutta's algorithm.

INTRODUCTION

Most of engineering problems, especially some oscillation equations are nonlinear, and in most cases, it is difficult to solve such equations, especially analytically. One of the well-known methods to solve nonlinear problems is perturbation method. The traditional perturbation method contains many shortcomings. Recently, considerable attention has been paid towards approximate solutions for analytically solving nonlinear differential equation. Many nonlinear problems do not contain such perturbation quantity, so to overcome the shortcomings, many new techniques have appeared in open literature such as, Homotopy perturbation (Bayat et al., 2010; Bayat et al., 2011a), parameter–expansion (Kimiaeifar et al., 2010), parameterized perturbation (He, 1999), energy balance (Bayat et al., 2011b, c), variational approach (Bayat et al., 2011d; Pakar et al., 2011; He, 2007) and the other analytical and numerical methods (Bayat et al., 2011e, f, g, h; Shahidi et al., 2011; Soleimani et al., 2011; Ghasemi et al., 2011; Ehidiamhen, 2009; Yang, 2010).

In the present paper, Energy balance method (EBM) was used for nonlinear oscillators, which was proposed by He (2002). This method can be seen as a Ritz-like method and leads to a very rapid convergence of the solution, and can be easily extended to other nonlinear oscillations. In short, this method yields extended scope of applicability, simplicity, flexibility in application, and avoidance of complicated numerical and analytical integration as compared to others among the previous approaches, such as, the perturbation methods, and so could widely applicable in engineering and science.

To show the efficiency and accuracy of the methods some comparisons have done with the results obtained by the EBM and Runge-kutta. The EBM has an excellent agreement with the Runge-kutta and they are valid for whole domain.

Mathematical formulation

A conservative nonlinear single degree of freedom is shown in Figure 1. The governing equation of the oscillation is as follow (Nayfeh, 1993);

$$\left(m_1+\frac{m_2 u^2}{l^2-u^2}\right)\ddot{u}+\frac{m_2 l^2 u \dot{u}^2}{\left(l^2-u^2\right)^2}+K u+m_2 g\frac{u}{\sqrt{l^2-u^2}}=0 \tag{1}$$

With the boundary condition of:

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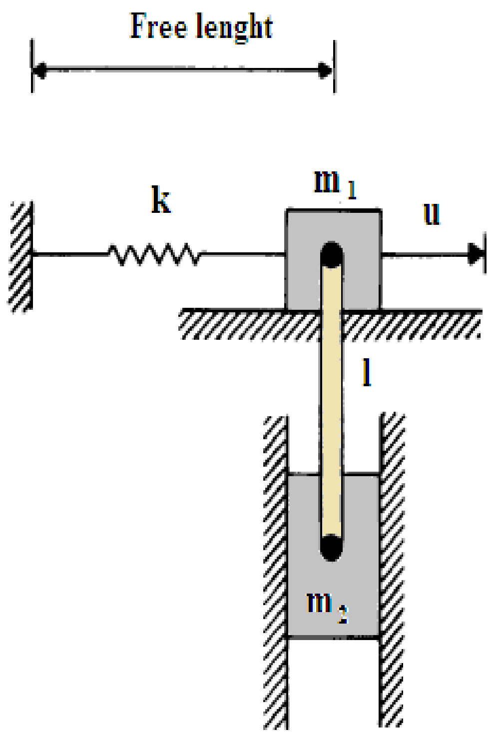

Figure 1. A conservative nonlinear single degree of freedom.

$$u(0)=A \qquad \dot{u}(0)=0 \tag{2}$$

Basic idea of energy balance method (EBM)

In the present paper, we consider a general nonlinear oscillator in the form (He, 2002);

$$u''+f\,(u\;(t))=0, \tag{3}$$

where, u and t are generalized dimensionless displacement and time variables, respectively, and $f\;=f\;(\,u\;,\;u',\;t\,)$.
Its variational principle can be easily obtained:

$$J(u)=\int_0^t(-\frac{1}{2}\,u'^2+F(u))\;dt \tag{4}$$

where $T=\dfrac{2\pi}{\omega}$ is period of the nonlinear oscillator, $F(u)=\int f\;(u)\;du$.
Its Hamiltonian, therefore, can be written in the form:

$$\Delta H\;=\frac{1}{2}u'^2+F(u)=F(A) \tag{5}$$

or

$$R(t)=\frac{1}{2}u'^2+F(u)-F(A)=0 \tag{6}$$

Oscillatory systems contain two important physical parameters, that is, the frequency ω and the amplitude of oscillation, A. So let us consider such initial conditions:

$$u(0)=A \quad , \quad u'(0)=0, \tag{7}$$

We use the following trial function to determine the angular frequency, ω:

$$u=A\;\;\cos\;(\omega\,t), \tag{8}$$

Substituting Equation 18 into Equation 16, we obtain the following residual equation:

$$R(t)=\frac{1}{2}A^2\omega^2\sin^2(\omega\,t)+F(A\cos\;(\omega\,t))-F(A)=0, \tag{9}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω. Since Equation 9, is only an approximation to the exact solution, R, cannot be made zero everywhere. Collocation at $\omega\,t=\pi/4$ gives:

$$\omega=\sqrt{\frac{2F(A)-F(A\cos\;(\omega\,t))}{A^2\sin^2(\omega\,t)}}\;,\qquad \text{rad/sec}, \tag{10}$$

Its period can be written in the form:

$$T\;=\frac{2\pi}{\sqrt{\dfrac{2F(A)-F(A\cos\;(\omega\,t))}{A^2\sin^2(\omega\,t)}}}, \tag{11}$$

Application of energy balance method

Its variational and Hamiltonian formulations of the Equation 1 can be readily obtained as:

$$J(u)=\int_0^t\left(-\frac{1}{2}\left(m_1\dot{u}^2-\frac{m_2l^2\dot{u}^2u^4}{\left(-l^2+u^2\right)^2}+\frac{m_2\dot{u}^2u^2}{\left(-l^2+u^2\right)^2}\right)+\frac{1}{2}Ku^2-m_2g\sqrt{l^2-u^2}\right)dt \tag{12}$$

$$\begin{aligned}H&=\frac{1}{2}\left(m_1\dot{u}^2-\frac{m_2l^2\dot{u}^2u^4}{\left(-l^2+u^2\right)^2}+\frac{m_2\dot{u}^2u^2}{\left(-l^2+u^2\right)^2}\right)+\frac{1}{2}Ku^2-m_2g\sqrt{l^2-u^2}\\&=\frac{1}{2}KA^2-m_2g\sqrt{l^2-A^2}\end{aligned} \tag{13}$$

Table 1. Comparison of frequency corresponding to various parameters of system.

A	m_1	m_2	l	g	K	ω_{EBM}	ω_{NM}	Error (%)
0.1	3	1	2	10	5	1.82579	1.79142	1.918
0.5	3	1	2	10	5	1.82702	1.79215	1.946
1	2	4	4	10	5	2.67577	2.59911	2.949
2	2	4	4	10	5	2.50239	2.43833	2.628
3	2	2	5	10	5	1.99328	1.95544	1.935
4	2	2	5	10	5	1.89981	1.86917	1.639
5	1	4	5	10	5	2.35063	2.29569	2.393
10	1	4	10	10	5	1.80631	1.77553	1.733
15	4	1	20	10	5	1.13556	1.12403	1.026
20	4	1	20	10	5	1.13263	1.12078	1.057

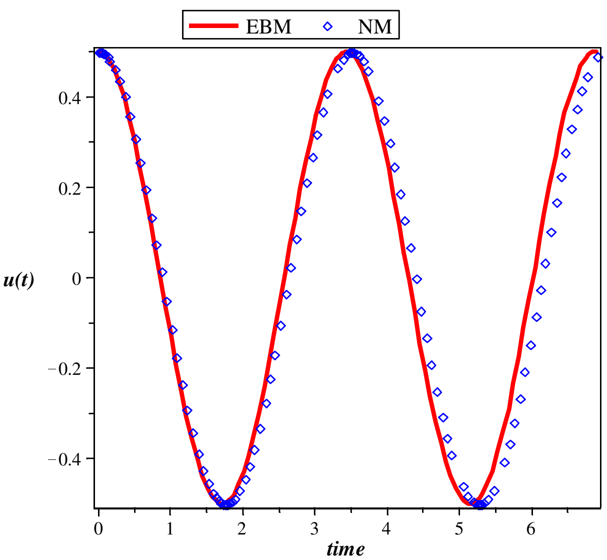

Figure 2. Comparison of analytical solution of $u(t)$ based on time with the numerical solution for $m_1 = 3$, $m_2 = 1$, $l = 2$, $g = 10$, $k = 5$, $A = 0.5$.

or

$$R(t)=\frac{1}{2}\left(m_1\dot{u}^2-\frac{m_2 l^2\dot{u}^2u^4}{\left(-l^2+u^2\right)^2}+\frac{m_2\dot{u}^2u^2}{\left(-l^2+u^2\right)^2}\right)+\frac{1}{2}Ku^2-m_2 g\sqrt{l^2-u^2}-\frac{1}{2}KA^2+m_2 g\sqrt{l^2-A^2}=0 \tag{14}$$

Choosing the trial function $u(t)=A\cos(\omega t)$, we obtain the following residual equation: which trigger the following results:

$$\frac{1}{2}\left(m_1A^2\omega^2\sin^2(\omega t)+\frac{m_2A^4\omega^2\sin^2(\omega t)\cos^2(\omega t)}{\left(l^2-A^2\cos^2(\omega t)\right)^2}-\frac{m_2A^6\omega^2\sin^2(\omega t)\cos^4(\omega t)}{\left(l^2-A^2\cos^2(\omega t)\right)^2}\right)+\frac{1}{2}KA^2\cos^2(\omega t)-m_2 g\sqrt{l^2-A^2\cos^2(\omega t)}-\frac{1}{2}KA^2+m_2 g\sqrt{l^2-A^2}=0$$

If we collocate at $\omega=\frac{\pi}{4}$, we obtain the following result:

$$\frac{1}{2}\left(m_1A^2\omega^2\sin^2(\omega t)+\frac{m_2A^4\omega^2\sin^2(\omega t)\cos^2(\omega t)}{\left(l^2-A^2\cos^2(\omega t)\right)^2}-\frac{m_2A^6\omega^2\sin^2(\omega t)\cos^4(\omega t)}{\left(l^2-A^2\cos^2(\omega t)\right)^2}\right)+\frac{1}{2}KA^2\cos^2(\omega t)-m_2 g\sqrt{l^2-A^2\cos^2(\omega t)}-\frac{1}{2}KA^2+m_2 g\sqrt{l^2-A^2}=0 \tag{15}$$

$$\omega=\frac{\sqrt{\left(A^2-2l^2\right)\left((m_1-m_2)A^2-2m_1l^2\right)\left(KA^2+2\sqrt{4l^2-2A^2}\,g\,m_2-4\sqrt{l^2-A^2}\right)g\,m_2}}{\left(2m_1l^2-(m_1-m_2)A^2\right)A} \tag{16}$$

Its period can be written in the form:

$$T=\frac{2A\pi\left(2m_1l^2-(m_1-m_2)A^2\right)}{\sqrt{\left(A^2-2l^2\right)\left((m_1-m_2)A^2-2m_1l^2\right)\left(KA^2+2\sqrt{4l^2-2A^2}\,g\,m_2-4\sqrt{l^2-A^2}\right)g\,m_2}} \tag{17}$$

We can obtain the following approximate solution:

$$u(t)=A\cos\left(\frac{\sqrt{\left(A^2-2l^2\right)\left((m_1-m_2)A^2-2m_1l^2\right)\left(KA^2+2\sqrt{4l^2-2A^2}\,g\,m_2-4\sqrt{l^2-A^2}\right)g\,m_2}}{\left(2m_1l^2-(m_1-m_2)A^2\right)A}t\right) \tag{18}$$

RESULTS AND DISCUSSION

To illustrate the accuracy of the accuracy of the EBM, the procedures explained in previous sections are applied to obtain natural frequency and corresponding displacement of a high nonlinear single degree of freedom. Comparisons of angular frequencies for different parameters via numerical is presented in Table 1. The maximum relative error between the EBM results and numerical results is 2.949%.To further illustrate and verify the accuracy of the present analytical approach, comparison of EBM and numerical solution are presented

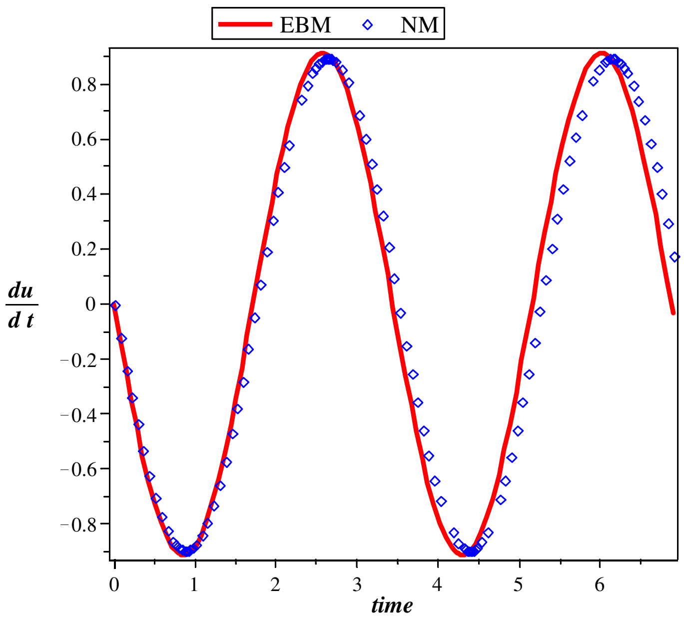

Figure 3. Comparison of analytical solution of du/dt based on time with the numerical solution for $m_1 = 3$, $m_2 = 1$, $l = 2$, $g = 10$, $k = 5$, and $A = 0.5$.

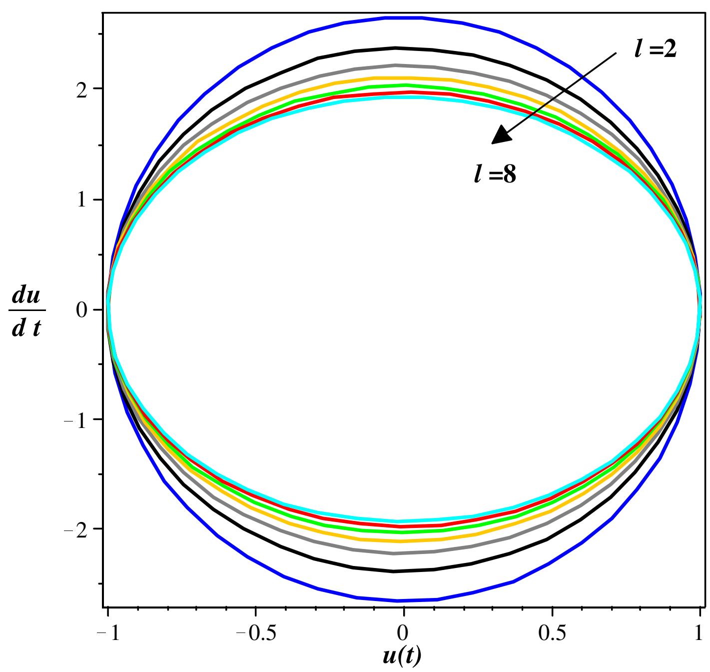

Figure 4. The phase plane, $m_1 = 2$, $m_2 = 2$, A=1, $g = 10$, $k = 5$.

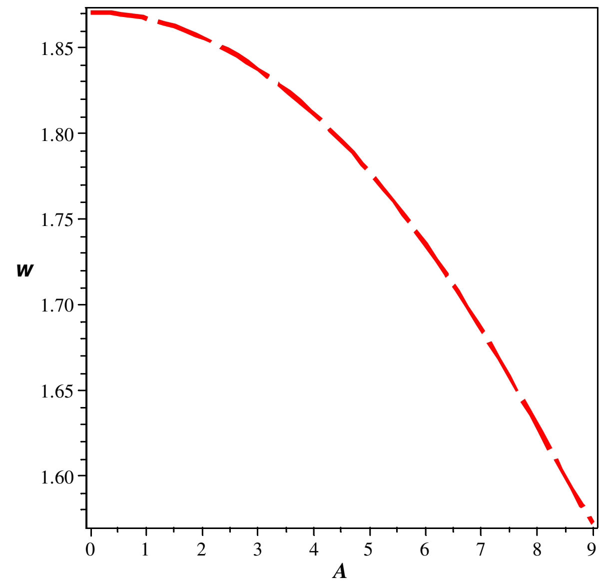

Figure 5. Variation of frequency with respect to various parameters of amplitude (A) for $m_1 = 2$, $m_2 = 2$, $l = 10$, and $g = 10$.

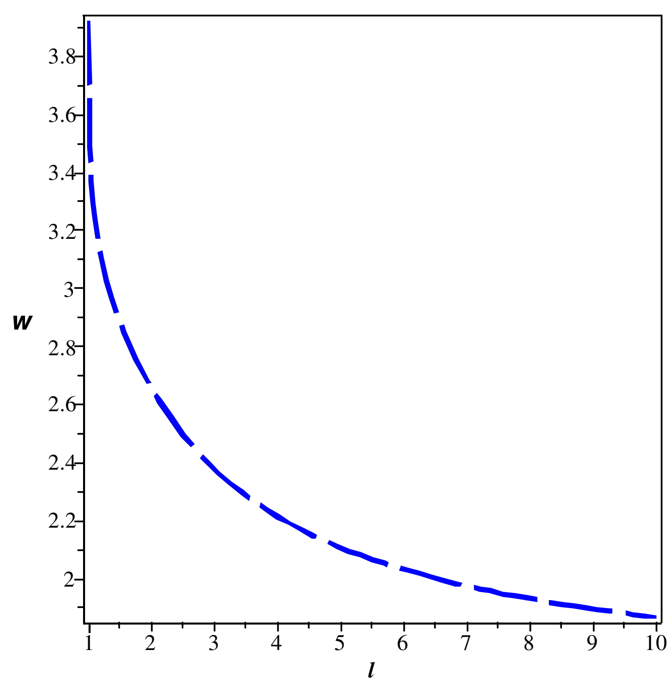

Figure 6. Variation of frequency respect to various parameters of (l) for $m_1 = 2$, $m_2 = 2$, $l = 10$.

in Figures 2 and 3, for $u(t)$ and $\dot{u}(t)$ at different parameters of the systems. Figure 4 represents the phase plane of the problem for $m_1 = 2$, $m_2 = 2$, $A = 1$, g =10.

Comparison of frequency corresponding to various parameters of amplitude (A) and length (L) is shown in the Figures 5 to 7.

As shown in Figures 2 to 7 and Table 1, it is apparent that the EBM has an excellent agreement with the

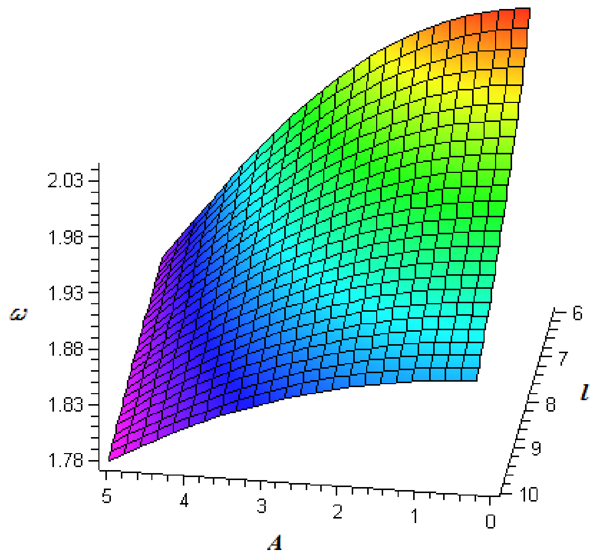

Figure 7. Sensitivity analysis of frequency for m_1 = 2, m_2 = 2, g = 10, k = 5.

numerical solution using Rung-Kutta and these expressions are valid for a wide range.

Conclusion

In this study, the He's Energy Balance method was successfully applied to solve the governing equation of a system of nonlinear conservative oscillation. The accuracy of the method was investigated by a comparison which was made between Runge-Kutta algorithms. The excellent agreement of the EBM solutions and the Runge-Kutta solutions shows the reliability and the efficiency of the method. This new method accelerated the convergence to the solutions. The EBM provides efficient alternative tools in solving nonlinear equations. The method is useful to obtain analytical solution for all oscillators and vibration problems. The EBM is a well-established method for analyzing nonlinear systems which can be easily extended to any nonlinear equation.

Nomenclature: u, Displacement; l, Distance between two mass; m_1, first mass of system; g, gravity; m_2, Second mass of system; A, amplitude; K, spring Stiffness; ω, system frequency.

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