Three-dimensional max-type cyclic system of difference equations

Tarek F. Ibrahim

Department of Mathematics, Faculty of Sciences and Arts (S. A.), King Khalid University, Abha, Saudi Arabia.
Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.

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We show that all solutions of the max-type cyclic system of difference equations

\[ x_{n+1} = \max\{x_n, y_n, z_n\}, \]
\[ y_{n+1} = \max\{\alpha x_n, y_n\}, \]
\[ z_{n+1} = \max\{\alpha x_n, z_n\} \]

where \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), \( \alpha \) is real positive number, and initial values \( x_0, y_0 \) and \( z_0 \in (0, \infty) \), are periodic with period three. Finally we generalized the results for more general system which is finite dimensional system.

Key words: Max-type system, difference equations, eventually periodic solutions.

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INTRODUCTION

Since the discrete structure underlying quantum gravity leads to ‘difference equations’ as the basic dynamical laws, new problems to determine properties of their solutions arise. The type of these equations is different from standard ones, and questions asked sometimes differ from what traditional investigations were interested in, which requires new tools to be developed. One issue is how to derive semi-classical properties of solutions in regimes where difference equations are close to continuum behavior. More generally, an important question is how such semi classical behavior can emerge dynamically in a quantum system.

Recently, there has been a great interest in studying nonlinear difference equations since many models describing real life situations in population biology, economics, probability theory, genetics, psychology, sociology etc. are represented by these equations. As we know the investigation of max-type difference equations attracted some attention recently. Some of the max-type difference equations arises naturally in certain models in automatic control theory (Stevic, 2010), and other fields. There has been some recent interest in studying max-type difference equations, see, for instance, (Berenhaut et al., 2006; Cinar et al., 2005; Grove and Ladas, 2005; Iricanin and Stevic, 2009; Mishev et al., 2003) and references cited therein. These papers study mostly some particular cases of the following max-type difference equations

\[ x_n = \max\{B_n(0), B_n(1), \ldots, B_n(p_n), \ldots, B_n(k), \ldots\}, \quad n \in \mathbb{N}_0, \]

(1)

Where \( \mathbb{N}_0 = \mathbb{N} \cup \{0\}\), \( k \in \mathbb{N} \), \( p_i, q_i \), \( i = 1, \ldots, k \) are
natural numbers such that $p_1 < p_2 < \ldots < p_k$, $q_1 < q_2 < \ldots < q_k$, $r_i, s_i \in (0, \infty), i = 1, 2, \ldots, k$ and $B^{(j)}_n, j = 0, 1, 2, \ldots, k$ are real sequences.

It is said that solution $(x_n)_{n=1}^\infty$ of Equation (1), where $s_k = \max \{ p_k, q_k \}$ is eventually periodic with period $p \in \mathbb{N}$ if there is an $n_1 \geq -s_k$ such that $x_{n+p} = x_n$, for $n \geq n_1$. If $n_1 = -s_k$ then we say that the sequence $(x_n)_{n=-s_k}^\infty$ is periodic with period $\pi$. Period $\pi$ is called a minimal one if there is no $p_1 < p$ that is a period for the sequence $(x_n)_{n=-s_k}^\infty$, (Berg and Stevic, 2006; Grove and Ladas, 2005). If $\pi = 1$ then such solutions are called eventually constant, or trivial ones (Iricanin and Stevic, 2009). The next particular case of Equation (1):

$$x_n = \max \left\{ \frac{A_n}{x_{n-k}} , x_{n-\alpha} \right\}, \ n \in \mathbb{N}_0$$

(2)

where $k, s \in \mathbb{N}$, and $(A_n)_{n \in \mathbb{N}_0}$ is a sequence of real numbers, attracted some attention recently (for example, Elsayed and Iricanin, 2009; Elsayed et al., 2010; Elsayed and Stevic, 2009; Iricanin and Stevic, 2009). Positive solutions of Equation (2) are usually related with periodicity. If solutions are not of constant sign then it is known that Equation (2) can have non-periodic solutions which could be even unbounded (Elsayed et al., 2010). Simsek et al. (2006) studied the solutions of the difference equation

$$x_{n+1} = \max \left\{ \frac{A_n}{x_{n-1}} , x_{n-1} \right\}$$

Elsayed and Stevic (2009) obtained the behavior of the max-type equation

$$x_{n+1} = \max \left\{ \frac{A_n}{x_{n-1}} , x_{n-2} \right\}$$

Xiao and Shi (2013) satisfied the periodic solutions of the max-type equation

$$x_{n+1} = \max \left\{ \frac{\beta}{x_{n}} , x_{n-1} \right\}$$

Iricanin and Touafek (2012) gave the complete solution of the max-type system:

$$x_{n+1} = \max \{ A_n/y_n , x_{n-1} \}, \ y_{n+1} = \max \{ B_n/x_n , y_{n-1} \}, \ n \in \mathbb{N}_0$$

Motivated by above mentioned papers, we investigated the global periodicity of the three-dimensional max-type cyclic difference system of the form

$$x_{n+1} = \max \{ A_n/x_n , y_n \}, \ y_{n+1} = \max \{ A_n/y_n , z_n \}, \ z_{n+1} = \max \{ A_n/z_n , x_n \}, \ n \in \mathbb{N}_0$$

(3)

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\alpha$ is real positive number, and initial values $x_0, y_0$ and $z_0 \in (0, \infty)$.

THE CLOSED FORM SOLUTION

Now we study the behavior of the solutions of system (3). For the sake of easier presentation, we formulate and prove eight theorems.

Case 1: $x_0y_0 \geq \alpha$, $y_0z_0 \geq \alpha$, $x_0z_0 \geq \alpha$

Theorem (1)

Suppose that $(x_n, y_n, z_n)$ is a solution of system (3) such that $x_0y_0 \geq \alpha$, $y_0z_0 \geq \alpha$, and $x_0z_0 \geq \alpha$. Then the solution is given by:

$$x_{3n} = y_{3n-1} = z_{3n-2} = x_0, \ n \geq 1$$

(4)

$$y_{3n} = z_{3n-1} = x_{3n-2} = y_0, \ n \geq 1$$

(5)

$$z_{3n} = x_{3n-1} = y_{3n-2} = z_0, \ n \geq 1$$

(6)

Proof

First, note that the conditions $x_0y_0 \geq \alpha$, $y_0z_0 \geq \alpha$, and $x_0z_0 \geq \alpha$ imply

$$x_1 = \max \{ \frac{A}{x_0} , y_0 \} = y_0, \ y_1 = \max \{ \frac{A}{y_0} , z_0 \} = z_0, \ z_1 = \max \{ \frac{A}{z_0} , x_0 \} = x_0$$

by the same way we can see that

$$x_2 = \max \{ \frac{A}{y_0} , z_0 \} = z_0, \ y_2 = \max \{ \frac{A}{z_0} , x_0 \} = x_0, \ z_2 = \max \{ \frac{A}{x_0} , y_0 \} = y_0$$

$$x_3 = \max \{ \frac{A}{z_0} , x_0 \} = x_0, \ y_3 = \max \{ \frac{A}{x_0} , y_0 \} = y_0, \ z_3 = \max \{ \frac{A}{y_0} , z_0 \} = z_0$$
\[ x_4 = y_0 , \quad y_4 = z_0 , \quad z_4 = x_0 \]
\[ x_5 = z_0 , \quad y_5 = x_0 , \quad z_5 = y_0 \]

By continuing we will get Equations (4), (5) and (6).

**Case 2:** \( x_0 y_0 \geq \alpha , y_0 z_0 \geq \alpha , x_0 z_0 \leq \alpha \)

**Theorem (2)**

Suppose that \( (x_n, y_n, z_n) \) is a solution of system (3) such that \( x_0 y_0 \geq \alpha , y_0 z_0 \geq \alpha , and x_0 z_0 \leq \alpha \). Then the solution is given by:

\[ x_{3n} = y_{3n-1} = z_{3n-2} = \frac{\alpha}{z_0} , \quad n \geq 1 \]

(7)

\[ y_{3n} = z_{3n-1} = x_{3n-2} = y_0 , \quad n \geq 1 \]

(8)

\[ x_i = \max \left\{ \frac{\alpha}{x_0}, y_0 \right\} = y_0 \]

\[ z_{3n} = x_{3n-1} = y_{3n-2} = z_0 , \quad n \geq 1 \]

(9)

**Proof**

From conditions \( x_0 y_0 \geq \alpha , y_0 z_0 \geq \alpha , and x_0 z_0 \leq \alpha \) we have that,

\[ y_i = \max \left\{ \frac{\alpha}{y_0}, z_0 \right\} = z_0 , \quad z_i = \max \left\{ \frac{\alpha}{z_0}, x_0 \right\} = \frac{\alpha}{z_0} \]

Also from the given conditions we can see that \( y_0 \geq z_0 \), then we have

\[ x_2 = z_0 , \quad y_2 = \frac{\alpha}{z_0} , \quad z_2 = y_0 \]

In this case we find

\[ x_3 = \frac{\alpha}{z_0} , \quad y_3 = y_0 , \quad z_3 = z_0 \]

\[ x_4 = y_0 , \quad y_4 = z_0 , \quad z_4 = \frac{\alpha}{z_0} \]

\[ x_5 = z_0 , \quad y_5 = \frac{\alpha}{z_0} , \quad z_5 = y_0 \]

By continuing we will get Equations (7), (8) and (9).

**Case 3:** \( x_0 y_0 \leq \alpha , y_0 z_0 \leq \alpha , x_0 z_0 \geq \alpha \)

**Theorem (3)**

Suppose that \( (x_n, y_n, z_n) \) is a solution of system (3) such that \( x_0 y_0 \leq \alpha , y_0 z_0 \leq \alpha , and x_0 z_0 \geq \alpha \). Then the solution is given by:

\[ x_{3n} = y_{3n-1} = z_{3n-2} = x_0 , \quad n \geq 1 \]

(10)

\[ y_{3n} = z_{3n-1} = x_{3n-2} = \frac{\alpha}{x_0} , \quad n \geq 1 \]

(11)

\[ z_{3n} = x_{3n-1} = y_{3n-2} = \frac{\alpha}{y_0} , \quad n \geq 1 \]

(12)

**Proof**

By using conditions \( x_0 y_0 \leq \alpha , y_0 z_0 \leq \alpha , and x_0 z_0 \geq \alpha \) we have that

\[ x_1 = \frac{\alpha}{x_0} , \quad y_1 = \frac{\alpha}{y_0} , \quad z_1 = x_0 \]

Also we have \( x_0 \geq y_0 \), then

\[ x_2 = \frac{\alpha}{y_0} , \quad y_2 = x_0 , \quad z_2 = \frac{\alpha}{x_0} \]

In this case we can see that

\[ x_3 = x_0 , \quad y_3 = \frac{\alpha}{x_0} , \quad z_3 = \frac{\alpha}{y_0} \]

\[ x_4 = \frac{\alpha}{x_0} , \quad y_4 = \frac{\alpha}{y_0} , \quad z_4 = x_0 \]

\[ z_5 = \frac{\alpha}{x_0} , \quad x_5 = \frac{\alpha}{y_0} , \quad y_5 = x_0 \]

\[ y_6 = \frac{\alpha}{x_0} , \quad z_6 = \frac{\alpha}{y_0} , \quad x_6 = x_0 \]
By continuing we will get Equations (10), (11) and (12).

**Case 4:** \( x_0 y_0 \leq \alpha, \ y_0 z_0 \leq \alpha, \ x_0 z_0 \leq \alpha \)

**Theorem (4)**

Suppose that \( (x_n, y_n, z_n) \) is a solution of system (3) such that \( x_0 y_0 \leq \alpha, \ y_0 z_0 \leq \alpha, \ \text{and} \ x_0 z_0 \leq \alpha \). Then the solution is given by:

\[
\begin{align*}
x_{3n} &= y_{3n-1} = z_{3n-2} = \frac{\alpha}{z_0}, & n \geq 1 \\
y_{3n} &= z_{3n-1} = x_{3n-2} = \frac{\alpha}{x_0}, & n \geq 1 \\
z_{3n} &= x_{3n-1} = y_{3n-2} = \frac{\alpha}{y_0}, & n \geq 1
\end{align*}
\]

**Proof**

The proof is given as in theorem (1).

**Case 5:** \( x_0 y_0 \leq \alpha, \ y_0 z_0 \geq \alpha, \ x_0 z_0 \leq \alpha \)

**Theorem (5)**

Suppose that \( (x_n, y_n, z_n) \) is a solution of system (3) such that \( x_0 y_0 \leq \alpha, \ y_0 z_0 \geq \alpha, \ \text{and} \ x_0 z_0 \leq \alpha \). Then the solution is given by:

\[
\begin{align*}
x_{3n} &= y_{3n-1} = z_{3n-2} = \frac{\alpha}{z_0}, & n \geq 1 \\
y_{3n} &= z_{3n-1} = x_{3n-2} = \frac{\alpha}{x_0}, & n \geq 1 \\
z_{3n} &= x_{3n-1} = y_{3n-2} = \frac{\alpha}{y_0}, & n \geq 1
\end{align*}
\]

**Proof**

The proof is given as in theorem (1).

**Case 6:** \( x_0 y_0 \leq \alpha, \ y_0 z_0 \geq \alpha, \ x_0 z_0 \geq \alpha \)

**Theorem (6)**

Suppose that \( (x_n, y_n, z_n) \) is a solution of system (3) such that \( x_0 y_0 \leq \alpha, \ y_0 z_0 \geq \alpha, \ \text{and} \ x_0 z_0 \geq \alpha \). Then the solution is given by:

\[
\begin{align*}
x_{3n} &= y_{3n-1} = z_{3n-2} = \frac{\alpha}{z_0}, & n \geq 1 \\
y_{3n} &= z_{3n-1} = x_{3n-2} = \frac{\alpha}{x_0}, & n \geq 1 \\
z_{3n} &= x_{3n-1} = y_{3n-2} = \frac{\alpha}{y_0}, & n \geq 1
\end{align*}
\]
Proof

The proof is given as in theorem (2).

GENERAL M-DIMENSIONAL SYSTEM

The previous results can be generalized to the following more general system which is m-dimensional system:

\[
\begin{align*}
x^{(i)}_{n+1} &= \max\left\{ \frac{\alpha}{x^{(i)}_n}, x^{(2)}_n \right\} \\
x^{(2)}_{n+1} &= \max\left\{ \frac{\alpha}{x^{(2)}_n}, x^{(3)}_n \right\} \\
x^{(3)}_{n+1} &= \max\left\{ \frac{\alpha}{x^{(3)}_n}, x^{(4)}_n \right\} \\
x^{(m)}_{n+1} &= \max\left\{ \frac{\alpha}{x^{(m)}_n}, x^{(1)}_n \right\} 
\end{align*}
\]

(13)

Theorem 9

Suppose that \( \left\{ x^{(i)}_n \right\}_{i=1}^\infty \) is a solution of system (13) such that

\[
x^{(i)}_0 x^{(i)}_0 \geq \alpha, x^{(i)}_0 x^{(m)}_0 \geq \alpha \quad \text{where} \quad j = 2, 3, \ldots, m \text{ and } \kappa = 1, 2, \ldots, m-1.
\]

Then the solution is given by:

\[
\begin{align*}
x^{(2)}_{mn} &= x^{(3)}_{mn-1} = x^{(4)}_{mn-2} = \ldots = x^{(m)}_{mn-m+1} = x^{(1)}_0 \\
x^{(2)}_{mn} &= x^{(3)}_{mn-1} = x^{(4)}_{mn-2} = \ldots = x^{(m)}_{mn-m+2} = x^{(1)}_{mn-m+1} = x^{(2)}_0 \\
x^{(3)}_{mn} &= x^{(4)}_{mn-1} = x^{(5)}_{mn-2} = \ldots = x^{(m)}_{mn-m+3} = x^{(1)}_{mn-m+2} = x^{(3)}_0 \\
x^{(m)}_{mn} &= x^{(1)}_{mn-1} = x^{(2)}_{mn-2} = \ldots = x^{(m)}_{mn-m+1} = x^{(m)}_0
\end{align*}
\]

Theorem 10

Suppose that \( \left\{ x^{(i)}_n \right\}_{i=1}^\infty \) is a solution of system (13) such that

\[
x^{(i)}_0 x^{(i)}_0 \leq \alpha, x^{(i)}_0 x^{(m)}_0 \leq \alpha \quad \text{where} \quad j = 2, 3, \ldots, m \text{ and } \kappa = 1, 2, \ldots, m-1.
\]

Then the solution is given by:

\[
\begin{align*}
x^{(2)}_{mn} &= x^{(3)}_{mn-1} = x^{(4)}_{mn-2} = \ldots = x^{(m)}_{mn-m+1} = \frac{\alpha}{x^{(m)}_0} \\
x^{(2)}_{mn} &= x^{(3)}_{mn-1} = x^{(4)}_{mn-2} = \ldots = x^{(m)}_{mn-m+2} = \frac{\alpha}{x^{(m)}_{mn-m+1}} = x^{(1)}_0 \\
x^{(3)}_{mn} &= x^{(4)}_{mn-1} = x^{(5)}_{mn-2} = \ldots = x^{(m)}_{mn-m+3} = \frac{\alpha}{x^{(m)}_{mn-m+2}} = x^{(2)}_0 \\
x^{(m)}_{mn} &= x^{(1)}_{mn-1} = x^{(2)}_{mn-2} = \ldots = x^{(m)}_{mn-m+1} = \frac{\alpha}{x^{(m)}_{mn-m+1}}
\end{align*}
\]

Theorem 11

Suppose that \( \left\{ x^{(i)}_n \right\}_{i=1}^\infty \) is a solution of system (13) such that

\[
x^{(j)}_0 x^{(j)}_0 \geq \alpha, x^{(l)}_0 x^{(m)}_0 \leq \alpha \quad \text{where} \quad j = 2, 3, \ldots, m \text{ and } \kappa = 1, 2, \ldots, m-1.
\]

Then the solution is given by:

\[
\begin{align*}
x^{(2)}_{mn} &= x^{(3)}_{mn-1} = x^{(4)}_{mn-2} = \ldots = x^{(m)}_{mn-m+2} = x^{(l)}_{mn-l+1} = x^{(2)}_0 \\
x^{(m)}_{mn} &= x^{(l)}_{mn-1} = x^{(2)}_{mn-2} = \ldots = x^{(m)}_{mn-m+1} = x^{(m)}_0
\end{align*}
\]

Theorem 12

Suppose that \( \left\{ x^{(i)}_n \right\}_{i=1}^\infty \) is a solution of system (13) such that

\[
x^{(j)}_0 x^{(j)}_0 \leq \alpha, x^{(l)}_0 x^{(m)}_0 \leq \alpha \quad \text{where} \quad j = 2, 3, \ldots, m \text{ and } \kappa = 1, 2, \ldots, m-1.
\]

Then the solution is given by:

\[
\begin{align*}
x^{(1)}_{mn} &= x^{(2)}_{mn-1} = x^{(3)}_{mn-2} = \ldots = x^{(m)}_{mn-m+1} = x^{(1)}_0 \\
x^{(2)}_{mn} &= x^{(3)}_{mn-1} = x^{(4)}_{mn-2} = \ldots = x^{(m)}_{mn-m+2} = x^{(2)}_0 \\
x^{(3)}_{mn} &= x^{(4)}_{mn-1} = x^{(5)}_{mn-2} = \ldots = x^{(m)}_{mn-m+3} = x^{(3)}_0 \\
x^{(m)}_{mn} &= x^{(1)}_{mn-1} = x^{(2)}_{mn-2} = \ldots = x^{(m)}_{mn-m+1} = \frac{\alpha}{x^{(m)}_0}
\end{align*}
\]

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