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Three-dimensional max-type cyclic system of difference equations

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We show that all solutions of the max-type cyclic system of difference equations $x_{n+1}=\max\{\frac{\alpha}{x_n}, y_n\}$, $y_{n+1}=\max\{\frac{\alpha}{y_n}, z_n\}$, $z_{n+1}=\max\{\frac{\alpha}{z_n}, x_n\}$ where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, α is real positive number, and initial values x_0, y_0 and $z_0\in(0,\infty)$, are periodic with period three. Finally we generalized the results for more general system which is finite dimensional system.

Key words: Max-type system, difference equations, eventually periodic solutions.

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INTRODUCTION

Since the discrete structure underlying quantum gravity leads to 'difference equations' as the basic dynamical laws, new problems to determine properties of their solutions arise. The type of these equations is different from standard ones, and questions asked sometimes differ from what traditional investigations were interested in, which requires new tools to be developed. One issue is how to derive semi-classical properties of solutions in regimes where difference equations are close to continuum behavior. More generally, an important question is how such semi classical behavior can emerge dynamically in a quantum system.

Recently, there has been a great interest in studying nonlinear difference equations since many models describing real life situations in population biology, economics, probability theory, genetics, psychology, sociology etc. are represented by these equations. As we know the investigation of max-type difference equations attracted some attention recently. Some of the max-type difference equations arises naturally in certain models in automatic control theory (Stevic, 2010), and other fields. There has been some recent interest in studying max-type difference equations, see, for instance, (Berenhaut et al., 2006; Cinar et al., 2005; Grove and Ladas, 2005; Iricanin and Stevic, 2009; Mishev et al., 2003) and references cited therein. These papers study mostly some particular cases of the following max-type difference equations

$$x_n=\max\left\{B_n^{(0)}, B_n^{(1)}\frac{x_{n-p_1}^{r_1}}{x_{n-q_1}^{s_1}}, B_n^{(2)}\frac{x_{n-p_2}^{r_2}}{x_{n-q_2}^{s_2}}, \ldots, B_n^{(k)}\frac{x_{n-p_k}^{r_k}}{x_{n-q_k}^{s_k}}\right\}, \quad n\in\mathbb{N}_0, \tag{1}$$

Where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $k\in\mathbb{N}$, p_i, q_i, $i=1,\ldots,k$ are

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natural numbers such that $p_1 < p_2 < ..... < p_k$, $q_1 < q_2 < ..... < q_k$, $r_i, s_i \in (0,\infty), i = 1,2,..k$ and $B_n^{(j)}, j = 0,1,2,...., k$ are real sequences .

It is said that solution $(x_n)_{n=-s_k}^{\infty}$ of Equation (1), where $s_k = \max\{p_k, q_k\}$ is eventually periodic with period $p \in \mathbb{N}$ if there is an $n_1 \geq -s_k$ such that $x_{n+p} = x_n$, for $n \geq n_1$. If $n_1 = -s_k$ then we say that the sequence $(x_n)_{n=-s_k}^{\infty}$ is periodic with period π. Period π is called a minimal one if there is no $p_1 < p$ that is a period for the sequence $(x_n)_{n=-s_k}^{\infty}$ (Berg and Stevic, 2006; Grove and Ladas, 2005). If π= 1 then such solutions are called eventually constant, or trivial ones (Iricanin and Stevic, 2009). The next particular case of Equation (1):

$$x_n = \max\left\{\frac{A_n}{x_{n-k}}, x_{n-s}\right\}; \quad n \in \mathbb{N}_0 \tag{2}$$

where $k, s \in \mathbb{N}$, and $(A_n)_{n\in\mathbb{N}_0}$ is a sequence of real numbers, attracted some attention recently (for example, (Elsayed and Iricanin, 2009; Elsayed et al., 2010; Elsayed and Stevic, 2009; Iricanin and Stevic, 2009). Positive solutions of Equation (2) are usually related with periodicity. If solutions are not of constant sign then it is known that Equation (2) can have non-periodic solutions which could be even unbounded (Elsayed et al., 2010). Simsek et al. (2006) studied the solutions of the difference equation

$$x_{n+1} = \max\{\frac{1}{x_{n-1}}, x_{n-1}\}$$

Elsayed and Stevic (2009) obtained the behavior of the max-type equation

$$x_{n+1} = \max\{\frac{A}{x_n}, x_{n-2}\}$$

Xiao and Shi (2013) satisfied the periodic solutions of the max-type equation

$$x_{n+1} = \max\{\frac{\beta}{x_n}, x_{n-1}\}$$

Iricanin and Touafek (2012) gave the complete solution of the max-type system:

$$x_{n+1} = \max\{\frac{A_n}{y_n}, x_{n-1}\}, \quad y_{n+1} = \max\{\frac{B_n}{x_n}, y_{n-1}\}, n \in \mathbb{N}_0$$

Motivated by above mentioned papers, we investigated the global periodicity of the three-dimensional max-type cyclic difference system of the form

$$x_{n+1} = \max\{\frac{\alpha}{x_n}, y_n\}, \quad y_{n+1} = \max\{\frac{\alpha}{y_n}, z_n\}, \quad z_{n+1} = \max\{\frac{\alpha}{z_n}, x_n\} \tag{3}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, α is real positive number, and initial values x_0, y_0 and $z_0 \in (0,\infty)$.

THE CLOSED FORM SOLUTION

Now we study the behavior of the solutions of system (3). For the sake of easier presentation, we formulate and prove eight theorems.

Case 1: $x_0 y_0 \geq \alpha, y_0 z_0 \geq \alpha, x_0 z_0 \geq \alpha$

Theorem (1)

Suppose that (x_n, y_n, z_n) is a solution of system (3) such that $x_0 y_0 \geq \alpha, y_0 z_0 \geq \alpha, and\ x_0 z_0 \geq \alpha$. Then the solution is given by:

$$x_{3n} = y_{3n-1} = z_{3n-2} = x_0 \quad , n \geq 1 \tag{4}$$

$$y_{3n} = z_{3n-1} = x_{3n-2} = y_0 \quad , n \geq 1 \tag{5}$$

$$z_{3n} = x_{3n-1} = y_{3n-2} = z_0 \quad , n \geq 1 \tag{6}$$

Proof

First, note that the conditions $x_0 y_0 \geq \alpha, y_0 z_0 \geq \alpha, and\ x_0 z_0 \geq \alpha$ imply $x_1 = \max\{\frac{\alpha}{x_0}, y_0\} = y_0$, $y_1 = \max\{\frac{\alpha}{y_0}, z_0\} = z_0$, $z_1 = \max\{\frac{\alpha}{z_0}, x_0\} = x_0$

By the same way we can see that

$$x_2 = \max\{\frac{\alpha}{y_0}, z_0\} = z_0, \quad y_2 = \max\{\frac{\alpha}{z_0}, x_0\} = x_0, \quad z_2 = \max\{\frac{\alpha}{x_0}, y_0\} = y_0$$

$$x_3 = \max\{\frac{\alpha}{z_0}, x_0\} = x_0, \quad y_3 = \max\{\frac{\alpha}{x_0}, y_0\} = y_0, \quad z_3 = \max\{\frac{\alpha}{y_0}, z_0\} = z_0$$

$x_4 = y_0$, $y_4 = z_0$, $z_4 = x_0$

$x_5 = z_0$, $y_5 = x_0$, $z_5 = y_0$

By continuing we will get Equations (4), (5) and (6).

Case 2: $x_0 y_0 \geq \alpha \ , y_0 z_0 \geq \alpha , x_0 z_0 \leq \alpha$

Theorem (2)

Suppose that (x_n, y_n, z_n) is a solution of system (3) such that $x_0 y_0 \geq \alpha \ , y_0 z_0 \geq \alpha , and \ x_0 z_0 \leq \alpha$. Then the solution is given by:

$$x_{3n} = y_{3n-1} = z_{3n-2} = \frac{\alpha}{z_0} \quad , n \geq 1 \tag{7}$$

$$y_{3n} = z_{3n-1} = x_{3n-2} = y_0 \quad , n \geq 1 \tag{8}$$

$$x_1 = \max\{\frac{\alpha}{x_0}, y_0\} = y_0$$

$$z_{3n} = x_{3n-1} = y_{3n-2} = z_0 \quad , n \geq 1 \tag{9}$$

Proof

From conditions $x_0 y_0 \geq \alpha \ , y_0 z_0 \geq \alpha , and \ x_0 z_0 \leq \alpha$ we have that, $y_1 = \max\{\frac{\alpha}{y_0}, z_0\} = z_0$, $z_1 = \max\{\frac{\alpha}{z_0}, x_0\} = \frac{\alpha}{z_0}$

Also from the given conditions we can see that $y_0 \geq z_0$, then we have

$$x_2 = z_0 \ , y_2 = \frac{\alpha}{z_0} \ , z_2 = y_0$$

In this case we find

$$x_3 = \frac{\alpha}{z_0}, \ y_3 = y_0 \ , z_3 = z_0$$

$$x_4 = y_0, \ y_4 = z_0 \ , z_4 = \frac{\alpha}{z_0}$$

$$x_5 = z_0 \ , y_5 = \frac{\alpha}{z_0} \ , z_5 = y_0$$

By continuing we will get Equations (7), (8) and (9).

Case 3: $x_0 y_0 \leq \alpha \ , y_0 z_0 \leq \alpha , x_0 z_0 \geq \alpha$

Theorem (3)

Suppose that (x_n, y_n, z_n) is a solution of system (3) such that $x_0 y_0 \leq \alpha \ , y_0 z_0 \leq \alpha , and \ x_0 z_0 \geq \alpha$. Then the solution is given by:

$$x_{3n} = y_{3n-1} = z_{3n-2} = x_0 \quad , n \geq 1 \tag{10}$$

$$y_{3n} = z_{3n-1} = x_{3n-2} = \frac{\alpha}{x_0} \quad , n \geq 1 \tag{11}$$

$$z_{3n} = x_{3n-1} = y_{3n-2} = \frac{\alpha}{y_0} \quad , n \geq 1 \tag{12}$$

Proof

By using conditions $x_0 y_0 \leq \alpha \ , y_0 z_0 \leq \alpha , and \ x_0 z_0 \geq \alpha$ we have that

$$x_1 = \frac{\alpha}{x_0}, \ y_1 = \frac{\alpha}{y_0} \ , z_1 = x_0$$

Also we have $x_0 \geq y_0$, then

$$x_2 = \frac{\alpha}{y_0}, \ y_2 = x_0 \ , z_2 = \frac{\alpha}{x_0}$$

In this case we can see that

$$x_3 = x_0, \ y_3 = \frac{\alpha}{x_0} \ , z_3 = \frac{\alpha}{y_0}$$

$$x_4 = \frac{\alpha}{x_0}, \ y_4 = \frac{\alpha}{y_0} \ , z_4 = x_0$$

$$z_5 = \frac{\alpha}{x_0}, \ x_5 = \frac{\alpha}{y_0} \ , y_5 = x_0$$

$$y_6 = \frac{\alpha}{x_0}, \ z_6 = \frac{\alpha}{y_0} \ , x_6 = x_0$$

By continuing we will get Equations (10), (11) and (12).

Case 4: $x_0 y_0 \le \alpha \ , y_0 z_0 \le \alpha , x_0 z_0 \le \alpha$

Theorem (4)

Suppose that (x_n , y_n , z_n) is a solution of system (3) such that $x_0 y_0 \le \alpha \ , y_0 z_0 \le \alpha , and \ x_0 z_0 \le \alpha$. Then the solution is given by:

$$x_{3n} = y_{3n-1} = z_{3n-2} = \frac{\alpha}{z_0} \quad , n \ge 1$$

$$y_{3n} = z_{3n-1} = x_{3n-2} = \frac{\alpha}{x_0} \quad , n \ge 1$$

$$z_{3n} = x_{3n-1} = y_{3n-2} = \frac{\alpha}{y_0} \quad , n \ge 1$$

Proof

The proof is given as in theorem (1)

Case 5: $x_0 y_0 \le \alpha \ , y_0 z_0 \ge \alpha , x_0 z_0 \le \alpha$

Theorem (5)

Suppose that (x_n , y_n , z_n) is a solution of system (3) such that $x_0 y_0 \le \alpha \ , y_0 z_0 \ge \alpha , and \ x_0 z_0 \le \alpha$. Then the solution is given by:

$$x_{3n} = y_{3n-1} = z_{3n-2} = \frac{\alpha}{z_0} \quad , n \ge 1$$

$$y_{3n} = z_{3n-1} = x_{3n-2} = \frac{\alpha}{x_0} \quad , n \ge 1$$

$$z_{3n} = x_{3n-1} = y_{3n-2} = z_0 \quad , n \ge 1$$

Proof

The proof is given as in theorem (2).

Case 6: $x_0 y_0 \le \alpha \ , y_0 z_0 \ge \alpha , x_0 z_0 \ge \alpha$

Theorem (6)

Suppose that (x_n , y_n , z_n) is a solution of system (3) such that $x_0 y_0 \le \alpha \ , y_0 z_0 \ge \alpha , and \ x_0 z_0 \ge \alpha$. Then the solution is given by:

$$x_{3n} = y_{3n-1} = z_{3n-2} = x_0 \quad , n \ge 1$$

$$y_{3n} = z_{3n-1} = x_{3n-2} = \frac{\alpha}{x_0} \quad , n \ge 1$$

$$z_{3n} = x_{3n-1} = y_{3n-2} = z_0 \quad , n \ge 1$$

Proof

The proof is given as in theorem (2).

Case 7: $x_0 y_0 \ge \alpha \ , y_0 z_0 \le \alpha , x_0 z_0 \ge \alpha$

Theorem (7)

Suppose that (x_n , y_n , z_n) is a solution of system (3) such that $x_0 y_0 \ge \alpha \ , y_0 z_0 \le \alpha , and \ x_0 z_0 \ge \alpha$. Then the solution is given by:

$$x_{3n} = y_{3n-1} = z_{3n-2} = x_0 \quad , n \ge 1$$

$$y_{3n} = z_{3n-1} = x_{3n-2} = y_0 \quad , n \ge 1$$

$$z_{3n} = x_{3n-1} = y_{3n-2} = \frac{\alpha}{y_0} \quad , n \ge 1$$

Proof

The proof is given as in theorem (2).

Case 8: $x_0 y_0 \ge \alpha \ , y_0 z_0 \le \alpha , x_0 z_0 \le \alpha$

Theorem (8)

Suppose that (x_n , y_n , z_n) is a solution of system (3) such that $x_0 y_0 \ge \alpha \ , y_0 z_0 \le \alpha , and \ x_0 z_0 \le \alpha$. Then the solution is given by:

$$x_{3n} = y_{3n-1} = z_{3n-2} = \frac{\alpha}{z_0} \quad , n \ge 1$$

$$y_{3n} = z_{3n-1} = x_{3n-2} = y_0 \quad , n \ge 1$$

$$z_{3n} = x_{3n-1} = y_{3n-2} = \frac{\alpha}{y_0} \quad , n \ge 1$$

Proof

The proof is given as in theorem (2).

GENERAL M-DIMENSIONAL SYSTEM

The previous results can be generalized to the following more general system which is m-dimensional system :

$$x_{n+1}^{(1)} = \max\{\frac{\alpha}{x_n^{(1)}}, x_n^{(2)}\}$$
$$x_{n+1}^{(2)} = \max\{\frac{\alpha}{x_n^{(2)}}, x_n^{(3)}\}$$
$$x_{n+1}^{(3)} = \max\{\frac{\alpha}{x_n^{(3)}}, x_n^{(4)}\} \qquad (13)$$
$$x_{n+1}^{(m)} = \max\{\frac{\alpha}{x_n^{(m)}}, x_n^{(1)}\}$$

Theorem 9

Suppose that $\{x_n^{(i)}\}_{i=1}^{m}$ is a solution of system (13) such that $x_0^{(j)}x_0^{(k)} \geq \alpha, x_0^{(1)}x_0^{(m)} \geq \alpha$ where j = 2; 3:::; m, and κ=1;2;:::; m–1. Then the solution is given by:

$$x_{mn}^{(1)} = x_{mn-1}^{(2)} = x_{mn-2}^{(3)} = ... = x_{mn-m+1}^{(m)} = x_0^{(1)}$$
$$x_{mn}^{(2)} = x_{mn-1}^{(3)} = x_{mn-2}^{(4)} = ... = x_{mn-m+2}^{(m)} = x_{mn-m+1}^{(1)} = x_0^{(2)}$$
$$x_{mn}^{(m)} = x_{mn-1}^{(1)} = x_{mn-2}^{(2)} = ... = x_{mn-m+1}^{(m-1)} = x_0^{(m)}$$

Theorem 10

Suppose that $\{x_n^{(i)}\}_{i=1}^{m}$ is a solution of system (13) such that $x_0^{(j)}x_0^{(k)} \leq \alpha, x_0^{(1)}x_0^{(m)} \leq \alpha$ where *j* = 2; 3:::;m, and κ=1;2;:::; m-1. Then the solution is given by:

$$x_{mn}^{(1)} = x_{mn-1}^{(2)} = x_{mn-2}^{(3)} = ... = x_{mn-m+1}^{(m)} = \frac{\alpha}{x_0^{(m)}}$$
$$x_{mn}^{(2)} = x_{mn-1}^{(3)} = x_{mn-2}^{(4)} = ... = x_{mn-m+2}^{(m)} = x_{mn-m+1}^{(1)} = \frac{\alpha}{x_0^{(1)}}$$
$$x_{mn}^{(m)} = x_{mn-1}^{(1)} = x_{mn-2}^{(2)} = ... = x_{mn-m+1}^{(m-1)} = \frac{\alpha}{x_0^{(m-1)}}$$

Theorem 11

Suppose that $\{x_n^{(i)}\}_{i=1}^{m}$ is a solution of system (13) such that $x_0^{(j)}x_0^{(k)} \geq \alpha, x_0^{(1)}x_0^{(m)} \leq \alpha$ where j = 2; 3:::;m, and κ=1;2;:::; m-1.

Then the solution is given by:

$$x_{mn}^{(2)} = x_{mn-1}^{(3)} = x_{mn-2}^{(4)} = ... = x_{mn-m+2}^{(m)} = x_{mn-m+1}^{(1)} = x_0^{(2)}$$
$$x_{mn}^{(m)} = x_{mn-1}^{(1)} = x_{mn-2}^{(2)} = ... = x_{mn-m+1}^{(m-1)} = x_0^{(m)}$$

Theorem 12

Suppose that $\{x_n^{(i)}\}_{i=1}^{m}$ is a solution of system (13) such that $x_0^{(j)}x_0^{(k)} \leq \alpha, x_0^{(1)}x_0^{(m)} \geq \alpha$ where j = 2; 3:::;m, and κ=1;2;:::; m-1. Then the solution is given by:

$$x_{mn}^{(1)} = x_{mn-1}^{(2)} = x_{mn-2}^{(3)} = ... = x_{mn-m+1}^{(m)} = x_0^{(1)}$$
$$x_{mn}^{(2)} = x_{mn-1}^{(3)} = x_{mn-2}^{(4)} = ... = x_{mn-m+2}^{(m)} = x_{mn-m+1}^{(1)} = \frac{\alpha}{x_0^{(1)}}$$
$$x_{mn}^{(m)} = x_{mn-1}^{(1)} = x_{mn-2}^{(2)} = ... = x_{mn-m+1}^{(m-1)} = \frac{\alpha}{x_0^{(m-1)}}$$

Theorem 13

Suppose that $\{x_n^{(i)}\}_{i=1}^{m}$ is a solution of system (13) such that $x_0^{(s)}x_0^{(s-1)} \geq \alpha, x_0^{(t)}x_0^{(t-1)} \leq \alpha, x_0^{(1)}x_0^{(m)} \leq \alpha$ where $s,t \in \{2,3,....,m\}$. Then the solution is given by:

$$x_{mn}^{(1)} = x_{mn-1}^{(2)} = x_{mn-2}^{(3)} = ... = x_{mn-m+1}^{(m)} = \frac{\alpha}{x_0^{(m)}}$$
$$x_{mn}^{(s)} = x_{mn-1}^{(s+1)} = x_{mn-2}^{(s+2)} = ... = x_{mn-m+2}^{(m)} = x_{mn-m+1}^{(1)} = x_0^{(s)}$$
$$x_{mn}^{(t)} = x_{mn-1}^{(t+1)} = x_{mn-2}^{(t+2)} = ... = x_{mn-m+2}^{(m)} = x_{mn-m+1}^{(1)} = \frac{\alpha}{x_0^{(t-1)}}$$
$$x_{mn}^{(m)} = x_{mn-1}^{(1)} = x_{mn-2}^{(2)} = ... = x_{mn-m+1}^{(m-1)} = \frac{\alpha}{x_0^{(1)}}$$

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