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Full Length Research Paper

Optimal control of inventory-production systems with gumbel distributed deterioration

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This paper is concerned with the development of an inventory-production model for a special class of generalized extreme value, that is, gumbel distributed deterioration items. The production system with inventory-level-dependent demand is considered and Pontryagin maximum principle is used to determine the optimal control, which is the production rate that minimizes the optimal control model, while satisfying the system dynamics. The necessary optimality conditions are also derived in this case. It is then illustrated with the help of examples.

Key words: Inventory-production systems, gumbel distributed deterioration, optimal control, pontryagin maximum principle.

INTRODUCTION

Inventory problems for deteriorating items and the variations in the demand rate with time have been studied extensively by many researchers from time to time. This type of research started with the work of Whitin (1957) who considered the deterioration of the fashion goods at the end of a prescribed shortage period. Importance of items deteriorating in inventory modeling is now widely acknowledged and has received a lot of attention (Raafat, 1991; Shah and Shah, 2000; Goyal and Giri, 2001). A reasonable model of an inventory system was developed by Ghare and Schrader (1963), considering the inventory depletion not only by demand but also by item's deterioration. Their observation led to the modeling of the inventory items with decaying processes by the differential equation:

$$\frac{dx(t)}{dt} + \theta x(t) = -y(t),$$

where θ is the constant decay rate, $x(t)$ is the inventory level at time t, and $y(t)$ is the demand rate at time t.

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This paper develops an optimal control model and utilizes Pontryagin maximum principle by Pontryagin et al. (1962) to derive the necessary optimality conditions for inventory-production systems, which, to the best of our knowledge, is an optimal control theory that has never been applied in conjunction with a special class of generalized extreme value, which is, gumbel distributed deterioration items. During the last two decades, various researches attacked on inventory-production problem with the application of optimal control theory. It has been successfully applied in production planning when only deterioration items were involved (Bounkhel and Tadj, 2005; Bounkhel et al., 2005; Tadj et al., 2006; Benhadid et al., 2008; Awad et al., 2009). In this context, a few researches are found for Weibull distributed deterioration items (Ghosh and Chaudhuri, 2004; Al-khedhairi and Tadj, 2007; Baten and Kamil, 2009) for Pareto distribution deterioration rate (Srinivasa et al., 2005, 2007; Baten and Kamil, 2010). But no attempt has been made to develop the inventory model as an optimal control model and to derive an explicit solution of an inventory model with gumbel distribution deterioration using Pontryagin maximum principle. The continuous review policy of optimal control approach is to be novel in this framework. There seems to be no literature on the optimal control of continuous review manufacturing systems with this gumbel distribution deterioration items rate.

The novelty we take into consideration in this study is that, the time of deterioration is a random variable followed by three-parameter generalized extreme value distribution. The probability density function of a generalized extreme value distribution having probability distribution of the form:

$$f(t)=\frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}^{\xi+1}\exp\left\{-\exp\{-(t-\mu)/\sigma\}^{\xi+1}\right\}$$

where $\mu\in R$ is the location parameter and $\sigma\in(0,\infty)$ is the scale parameter and $\xi\in(-\infty,\infty)$ is the shape parameter.

The shape parameter ξ governs the tail behavior of the distribution. The family defined by $\xi\to 0$ corresponds to a special case of generalized extreme value distribution, that is, Gumbel distribution. This distribution can be used to model either maximum or minimum rate of deterioration. The probability density function and probability distribution function of Gumbel corresponds to a special class of extreme (here maximum) value distribution

$$f_{\max}(t)=\frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}\exp\{-\exp\{-(t-\mu)/\sigma\}\},\quad t>0,$$

and

$$F_{\max}(t)=1-\exp\{-\exp\{-(t-\mu)/\sigma\}\},\quad t>0$$

respectively.

The instantaneous rate of deterioration of Gumbel distribution corresponds to a maximum value of the on-hand inventory and is given by:

$$\theta(t)=\frac{f_{\max}(t)}{1-F_{\max}(t)}=\frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\},\quad t>0.$$

In developing inventory models, continuous-time models of time-varying demands have been considered in this study. The time of deterioration rate is assumed to follow Gumbel distribution as well as a non-negative discount rate is considered for the inventory-production systems. We assume that all the functional forms are non-negative continuous and differentiable on $[0,\infty]$.

This paper develops a first model in which the dynamic demand is a function of time. We then extend this first model to an even more general model in which items deterioration are taken into account which refer to a special class of generalized extreme value, which is, Gumbel distribution. The paper also derives explicit optimal policies for the inventory models where items are deteriorating with this type of Gumbel distribution that can be used in the decision making process.

MATERIALS AND METHODS

Model without item deterioration and notations

We assume that an inventory goal level and a production goal rate are set, and penalties are incurred when the inventory level and the production rate deviate from these goals. We introduce the following notations to write the optimal control model:

q: Inventory holding cost incurred for the inventory level to deviate from its goal.
r: Production unit cost incurred for the production rate to deviate from its goal.
$\hat{x}(t)$: Inventory goal level.
$\hat{u}(t)$: Production goal rate.
$\rho\geq 0$: Constant non-negative discount rate.

We want to keep the inventory $x(t)$ as close as possible to its goal $\hat{x}(t)$, and also keep the production rate $u(t)$ as close to its goal level $\hat{u}(t)$. The quadratic terms $q[x(t)-\hat{x}(t)]^2$ and $r[u(t)-\hat{u}(t)]^2$ impose 'penalties' for having either x or u not being close to its corresponding goal level.

The optimal control model can be expressed as the quadratic form that we need to minimize

$$\text{minimize}\quad J(u,x,\hat{u})=\frac{1}{2}\int_0^T e^{-\rho t}\left\{q[x(t)-\hat{x}(t)]^2+r[u(t)-\hat{u}(t)]^2\right\}dt \qquad (1)$$

subject to the dynamics of the inventory level of the state equation which says that the inventory at time t is increased by the production rate $u(t)$ and decreased by the demand rate $y(t)$ can be written as:

$$dx(t)=[u(t)-y(t)]dt \qquad (2)$$

with initial condition $x(T)=0$ and the non-negativity constraint

$$u(t)\geq 0,\quad \text{for all } t\in[0,T] \qquad (3)$$

where the fixed length of the planning horizon is T, $x(t)$: inventory level function at any instant of time $t\in[0,T]$, $u(t)$: production rate at any instant of time $t\in[0,T]$ and $y(t)$: demand rate at any instant of time $t\in[0,T]$.

The current-value Hamiltonian of the model is defined as

$$H\left(t,x(t),u(t),\hat{u}(t),\gamma(t)\right)$$
$$=-\frac{1}{2}\int_0^T e^{-\rho t}\left\{q\left[x(t)-\hat{x}(t)\right]^2+r\left[u(t)-\hat{u}(t)\right]^2\right\}+\gamma(t)\left[u(t)-y(t)\right]. \quad (4)$$

Model with item deterioration

Consider a system where items, subject to Gumbel distributed deterioration, corresponds to a special class of extreme value distribution. For $t \geq 0,$ let $\theta(t)=\frac{1}{\sigma}\exp\left\{-\left(t-\mu\right)/\sigma\right\}$ be the deterioration rate at the inventory level $x(t)$ at time t. Keeping same notation and the same optimal control model as in the previous section, the dynamics of the inventory level of the state equation which says that the inventory at time t is increased by the production rate $u(t)$ and decreased by the demand rate $y(t)$ and the rate of deterioration $\frac{1}{\sigma}\exp\left\{-\left(t-\mu\right)/\sigma\right\}$ of Gumbel distribution corresponds to a special class of extreme value distribution can be written as according to

$$dx(t)=[u(t)-y(t)-\frac{1}{\sigma}\exp\left\{-\left(t-\mu\right)/\sigma\right\}x(t)]dt \quad (5)$$

with initial condition $x(T)=0$ and the non-negativity constraint $u(t)\geq 0, \quad \text{for all } t\in\left[0,T\right].$

The current-value Hamiltonian of the model is defined as

$$H\left(t,x(t),u(t),\hat{u}(t),\gamma(t)\right)$$
$$=-\frac{1}{2}\int_0^T e^{-\rho t}\left\{q\left[x(t)-\hat{x}(t)\right]^2+r\left[u(t)-\hat{u}(t)\right]^2\right\}+\gamma(t)\left[u(t)-y(t)-\frac{1}{\sigma}\exp\left\{-\left(t-\mu\right)/\sigma\right\}x(t)\right] \quad (6)$$

RESULTS

Development of the optimal control models

Let us consider a manufacturing firm, producing a single product, selling some and stocking the rest in a warehouse. We assume that the production deteriorates while in stock and the demand rate varies with time. The firm has set an inventory goal level and production goal rate. Since the constraint $u(t)-y\left(t\right)\geq 0, \quad \text{for all } t\in\left[0,T\right]$ with the state equation x is nondecreasing. Therefore, shortages are not allowed in this study.

Define the variables $z(t), \tilde{z}(t) \text{ and } \eta(t)$ such that:

$$z(t)=x(t)-\hat{x}(t), \quad (7) \qquad \tilde{z}(t)=u(t)-\hat{u}(t), \quad (8)$$

and $$\eta(t)=\hat{u}(t)-y(t)-\frac{1}{\sigma}\exp\left\{-\left(t-\mu\right)/\sigma\right\}\hat{x}(t). \quad (9)$$

Adding and subtracting the last term $\frac{1}{\sigma}\exp\left\{-\left(t-\mu\right)/\sigma\right\}\hat{x}(t)$ from the right hand side of Equation (9) to Equation (5) and rearranging the terms, we have:

$$d\left(x(t)-\hat{x}(t)\right)=[-\frac{1}{\sigma}\exp\left\{-\left(t-\mu\right)/\sigma\right\}\left(x(t)-\hat{x}(t)\right)+u(t)-y(t)-\frac{1}{\sigma}\exp\left\{-\left(t-\mu\right)/\sigma\right\}\hat{x}(t)]dt.$$

Hence by Equation (7)

$$dz(t)=[-\frac{1}{\sigma}\exp\left\{-\left(t-\mu\right)/\sigma\right\}z(t)+u(t)-y(t)-\frac{1}{\sigma}\exp\left\{-\left(t-\mu\right)/\sigma\right\}\hat{x}(t)]dt. \quad (10)$$

Now, substituting Equations (8) and (9) in (10) yields:

$$dz(t)=[-\frac{1}{\sigma}\exp\left\{-\left(t-\mu\right)/\sigma\right\}z(t)+\tilde{z}(t)+\eta(t)]dt. \quad (11)$$

The optimal control model (1) becomes:

$$\text{minimize} \quad J\left(z,\tilde{z}\right)=\frac{1}{2}\int_0^T e^{-\rho t}\{q[z(t)^2]+r[\tilde{z}(t)^2]\}dt \quad (12)$$

subject to an ordinary differential Equation (11) and the non-negativity constraint $\tilde{z}(t)\geq 0, \quad \text{for all } t\in\left[0,T\right].$

By the virtue of (2), the instantaneous state of the inventory level $x(t)$ at any time t is governed by the differential equation:

$$\frac{dx(t)}{dt}=u(t)-y(t), \quad 0\leq t\leq T, \quad x(T)=0 \quad (13)$$

The boundary conditions with Equation (13) are: at $x(0)=0,\ x\left(T\right)=0$

$$x(t)=\left[u(t)-y\left(t\right)\right]t, \quad \text{for } 0\leq t\leq T. \quad (14)$$

Assuming that $x(0)=x$ is known and note that the production goal rate $\hat{u}(t)$ can be computed using the state Equation (13) as:

$$\hat{u}(t)=y(t) \quad (15)$$

By the virtue of Equation (5), the instantaneous state of the inventory level $x(t)$ at any time t is governed by the differential equation:

$$\frac{dx(t)}{dt}+\frac{1}{\sigma}\exp\left\{-\left(t-\mu\right)/\sigma\right\}x(t)=u(t)-y(t), \quad 0\leq t\leq T, \quad x(T)=0 \quad (16)$$

This is a linear ordinary differential equation of first order and its integrating factor is:

$$=\exp\{\int \frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}dt\}=\exp\left[\exp\{-(t-\mu)/\sigma\}\right].$$

Multiplying both sides of Equation (16) by $\exp\left[\exp\{-(t-\mu)/\sigma\}\right]$ and then integrating over $[0,T]$, we have:

$$x(t)\exp\left[\exp\{-(t-\mu)/\sigma\}\right]-x(0)=-\int_0^T[y(t)-u(t)]\exp\left[\exp\{-(t-\mu)/\sigma\}\right]dt. \quad (17)$$

Substituting this value of $x(0)$ in Equation (16), we obtain the instantaneous level of inventory at any time $t\in[0,T]$ is given by

$$x(t)=\frac{\int_0^{t_1}[y(t)-u(t)]\exp\left[\exp\{-(t-\mu)/\sigma\}\right]dt-\int_0^T[y(t)-u(t)]\exp\left[\exp\{-(t-\mu)/\sigma\}\right]dt}{\exp\left[\exp\{-(t-\mu)/\sigma\}\right]}.$$

Solving the differential equation, the on-hand inventory at time t is obtained as

$$x(t)=x(0)\exp\left[\exp\{-(t-\mu)/\sigma\}\right]-\int_0^T[y(t)-u(t)]dt \quad 0\le t\le T. \quad (18)$$

Assuming that $x(0)=x$ is known and note that the production goal rate $\hat{u}(t)$ can be computed using the state Equation (16) as:

$$\hat{u}(t)=y(t)+\frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}\hat{x}(t) \quad (19)$$

Solution to the optimal control models

In order to solve the optimal control model (1) subject to state Equations (2) and (5), we derive the necessary optimality conditions using Pontryagin maximum principle developed by Pontryagin (1962), also, Sethi and Thompson (2000).

Solution of the optimal control model without item deterioration

The optimal control approach consists in determining the optimal control $\hat{u}(t)$ that minimizes the optimal control model (1) subject to the state Equation (2). By the maximum principle of Pontryagin (1962), there exists adjoint function $\gamma(t)$ such that the Hamiltonian functional form (4) satisfies the control equation:

$$\frac{\partial}{\partial u(t)}H\left(t,x(t),u(t),\hat{u}(t),\gamma(t)\right)=0, \quad (20)$$

the adjoint equation

$$\frac{\partial}{\partial x(t)}H\left(t,x(t),u(t),\hat{u}(t),\gamma(t)\right)=-\frac{d}{dt}\gamma(t), \quad \gamma(T)=0 \quad (21)$$

and the state equation

$$\frac{\partial}{\partial \gamma(t)}H\left(t,x(t),u(t),\hat{u}(t),\gamma(t)\right)=\frac{d}{dt}x(t), \quad x(0)=0. \quad (22)$$

Then the control Equation (20) is equivalent to:

$$u(t)=\hat{u}(t)+\frac{e^{\rho t}}{r}\gamma(t). \quad (23)$$

The adjoint Equation (21) is equivalent to:

$$\frac{d}{dt}\gamma(t)=qe^{-\rho t}\left[x(t)-\hat{x}(t)\right], \quad (24)$$

and the state Equation (22) is similar to (2).

Substitution expression (23) into the state Equation (2) yields

$$\frac{d}{dt}x(t)=\hat{u}(t)+\frac{\gamma(t)e^{\rho t}}{r}-y(t). \quad (25)$$

From Equation (25) we have

$$\frac{\gamma(t)e^{\rho t}}{r}=\frac{d}{dt}x(t)-\hat{u}(t)+y(t). \quad (26)$$

By differentiating Equation (25), we obtain:

$$\frac{d^2}{dt^2}x(t)=\frac{d}{dt}\hat{u}(t)-\frac{d}{dt}y(t)+\frac{1}{r}\left[e^{\rho t}\frac{d}{dt}\gamma(t)+\rho\gamma(t)e^{\rho t}\right]. \quad (27)$$

Substitution expression (24) into Equation (27) yields

$$\frac{d^2}{dt^2}x(t)=\frac{d}{dt}\hat{u}(t)-\frac{d}{dt}y(t)+\frac{q}{r}\left[x(t)-\hat{x}(t)\right]+\frac{\rho}{r}e^{\rho t}\gamma(t). \quad (28)$$

Finally, substituting expression (26) into (28) to Obtain

$$\frac{d^2}{dt^2}x(t)-\frac{q}{r}x(t)=\frac{d}{dt}\hat{u}(t)-\frac{d}{dt}y(t)-\frac{q}{r}\hat{x}(t)+\rho\left[y(t)-\hat{u}(t)\right]. \qquad (29)$$

Since a closed form solution is not possible, so this boundary value problem can be solvednumerically together with initial condition $x(0)=0$ and the terminal condition $\gamma(T)=0.$

Solution of the optimal control model with item deterioration

The optimal control approach consists in determining the optimal control $\hat{u}(t)$ that minimizes the optimal control model (1) subject to the state Equation (5). By the maximum principle of Pontryagin (1962), there exists adjoint function $\gamma(t)$ such that the Hamiltonian functional form (6) satisfies the necessary conditions (20), (21) and (22). Then, here, the control Equation (20) is equivalent to (23) also.

The adjoint Equation (21) is equivalent to:

$$\frac{d}{dt}\gamma(t)=\left[qe^{-\rho t}+\gamma(t)\exp\{-(t-\mu)/\sigma\}\right]-qe^{-\rho t}\hat{x}(t), \qquad (30)$$

And the state Equation (22) is similar to (5).

Substituting expression (23) into the state Equation (5) yields

$$\frac{d}{dt}x(t)=\hat{u}(t)+\frac{\gamma(t)e^{\rho t}}{r}-y(t)-\frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}x(t), \qquad (31)$$

From Equation (31) we have

$$\frac{\gamma(t)e^{\rho t}}{r}=\frac{d}{dt}x(t)-\hat{u}(t)+y(t)+\frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}x(t). \qquad (32)$$

By differentiating (31), we obtain

$$\frac{d^2}{dt^2}x(t)=\frac{d}{dt}\hat{u}(t)-\frac{d}{dt}y(t)+\frac{1}{r}\left[e^{\rho t}\frac{d}{dt}\gamma(t)+\rho\gamma(t)e^{\rho t}\right]-\exp\{-(t-\mu)/\sigma\}\{-(t-\mu)/\sigma\}. \qquad (33)$$

Substituting expression (30) into the Equation (33) yields

$$\frac{d^2}{dt^2}x(t)=\frac{d}{dt}\hat{u}(t)-\frac{d}{dt}y(t)+\frac{q}{r}[x(t)-\hat{x}(t)]+\frac{1}{r}e^{\rho t}\gamma(t)\left[x(t)\exp\{-(t-\mu)/\sigma\}+\rho\right]$$
$$-\exp\{-(t-\mu)/\sigma\}\{-(t-\mu)/\sigma\}. \qquad (34)$$

Finally, substituting expression (32) into (34) to obtain

$$\frac{d^2}{dt^2}x(t)-\left[\frac{q}{r}+\frac{\gamma(t)}{r}e^{\rho t}\exp\{-(t-\mu)/\sigma\}-\exp\{-(t-\mu)/\sigma\}\right]x(t)$$
$$=\frac{d}{dt}\hat{u}(t)-\frac{d}{dt}y(t)-\frac{q}{r}\hat{x}(t)+\rho\left[y(t)-\hat{u}(t)\right]. \qquad (35)$$

Since a closed form solution is not possible, this boundary value problem can be solved numerically together with initial condition $x(0)=0$ and the terminal condition $\gamma(T)=0.$

DISCUSSION

Illustrative examples

In order to present illustrative examples of the results obtained, we use the following parameters where the planning horizon has length T=12 months, $\rho=0.001,$ the inventory holding cost coefficient $q=5$ the production cost coefficient $r=5.$ The goal inventory level is considered $\hat{x}(t)=1+t+\sin(t),$ and the location and scale parameters of the Gumbel distribution rate are considered as $\mu=1$ and $\sigma=1$ respectively. Then the deterioration rate of Gumbel distribution becomes $\theta(t)=\exp\{-(t-1)\}, \quad t\in[0,T].$

Numerical examples are given for different cases of demand rates.

1. Demand rate is constant: $y(t)=y=20,$
2. Demand rate is linear function of time: $\text{y(t)=y}_1(t)\,t+y_2(t)=t+15,$
3. Demand rate is quadratic function of time: $y(t)=30+0.1t+0.001t^2.$
4. Demand is sinusoidal function of time: $y(t)=1+\sin(t).$
5. Demand is co-sinusoidal function of time: $y(t)=1+\cos(t).$
6. Demand is exponential increasing function of time: $y(t)=\exp(t).$
7. Demand is exponential decreasing function of time: $y(t)=\exp(-t).$

The inventory level $x(t)$ in-terms of the first-order differential equation from (5) and the second-order differential Equation (35) considering the above demand functions are solved numerically using the version 6.5 of the mathematical package MATLAB.

The constant demand rate is assumed to have fixed value 20 units per unit time. Note that here demand and deterioration does not decrease the inventory level displayed in Figure 1. From Figure 2, it is clear that the production rate is following the constant demand rate. In case of linear demand, it is the form $\text{y}_2\text{(t)}=t+15.$ The inventory level x(t) in-terms of the first-order differential equation in terms of linear demand is displayed in Figure 3. The result is shown in Figure 4 and it is found that the

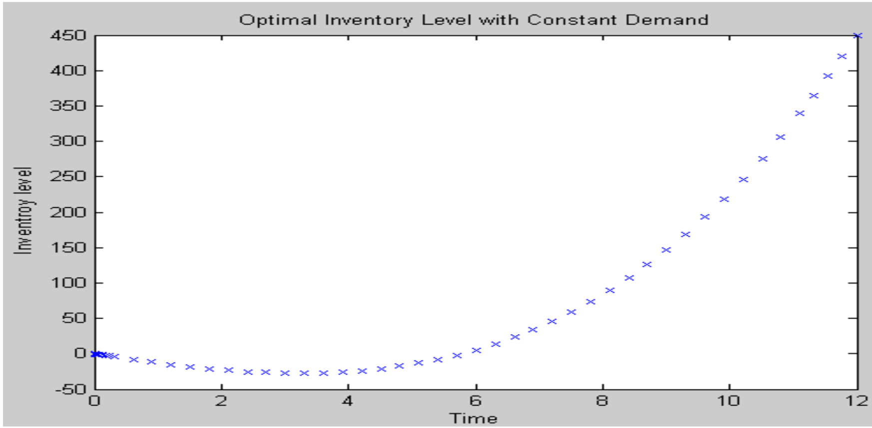

Figure 1. The inventory level $x(t)$ in-terms of the first-order differential equation in terms of constant demand.

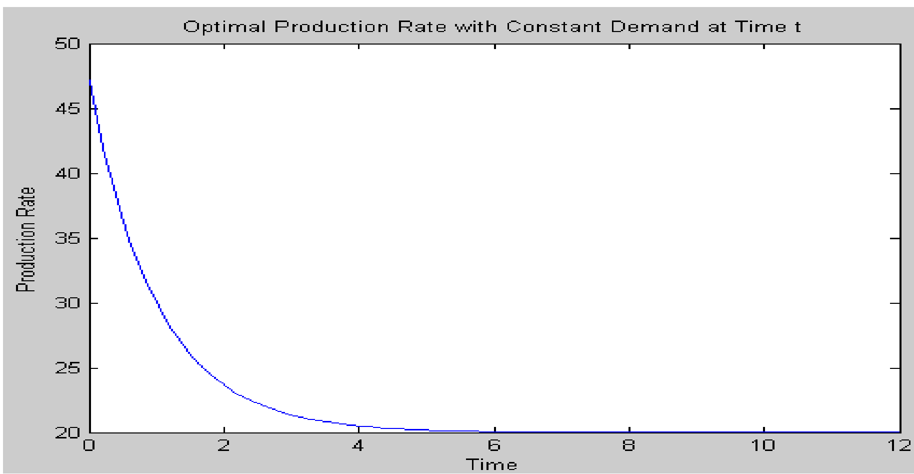

Figure 2. Optimal production policy with constant demand.

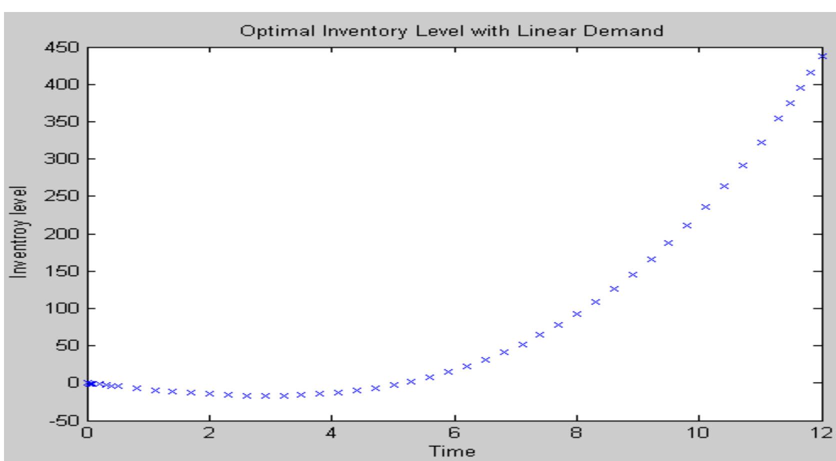

Figure 3. The inventory level $x(t)$ in-terms of the first-order differential equation in terms of linear demand.

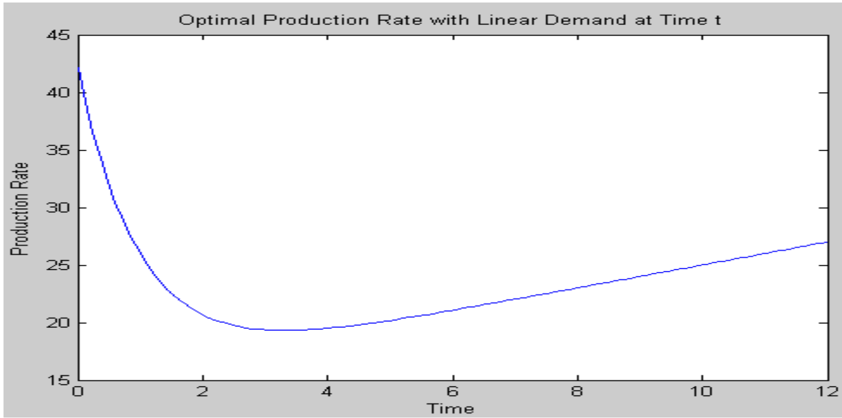

Figure 4. Optimal production policy with linear demand.

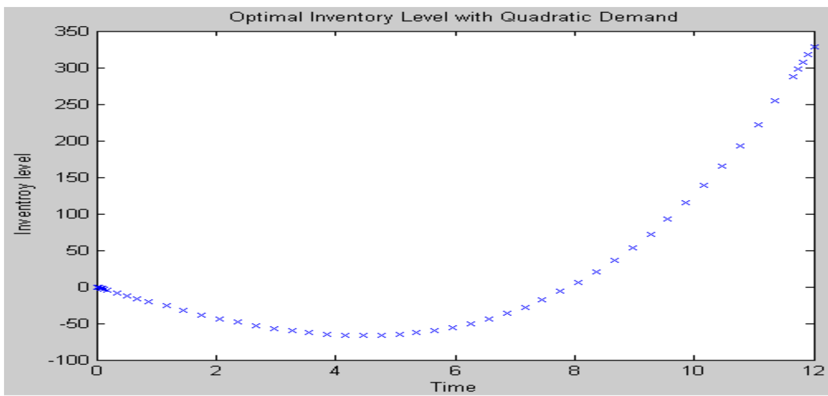

Figure 5. The inventory level $x(t)$ in-terms of the first-order differential equation in terms of quadratic demand.

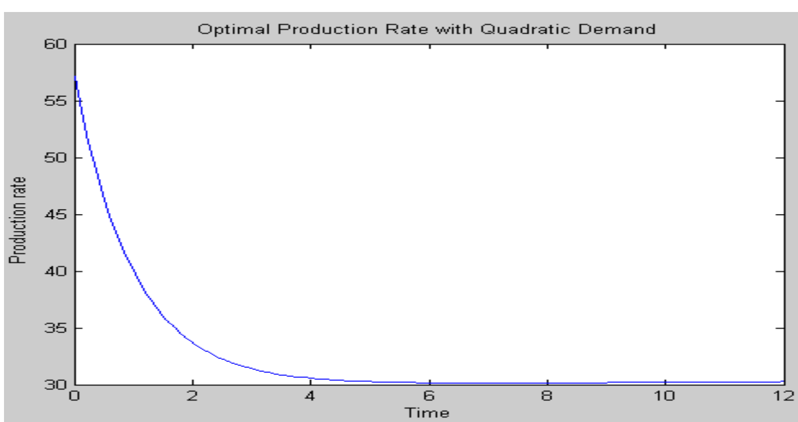

Figure 6. Optimal production policy with quadratic demand.

production rate is following the linear demand rate. The production rate starts with quite large amount due to the large desired production rate. The inventory level x(t) in-terms of the first-order differential equation in terms of quadratic demand is revealed by Figure 5. From Figure 6, it is displayed that the production rate tracks the quadratic demand. The inventory is again increasing because the initial production which is a function of

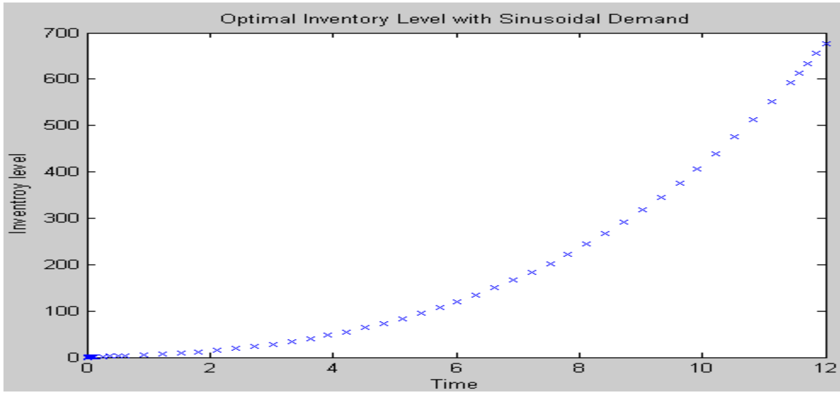

Figure 7. The inventory level $x(t)$ in-terms of the first-order differential equation in terms of sinusoidal demand.

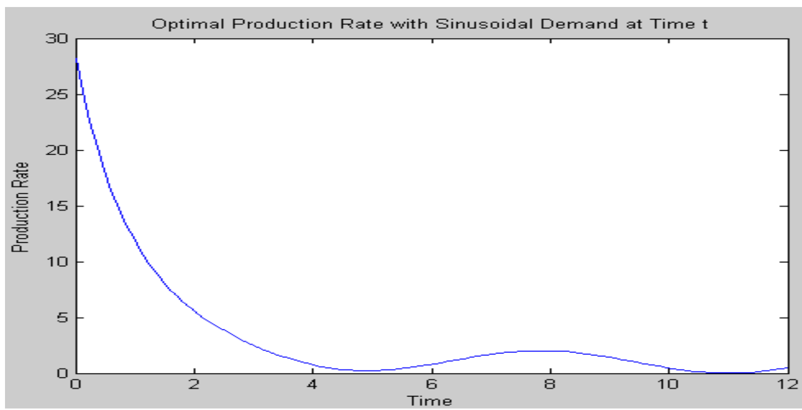

Figure 8. Optimal production policy with sinusoidal demand.

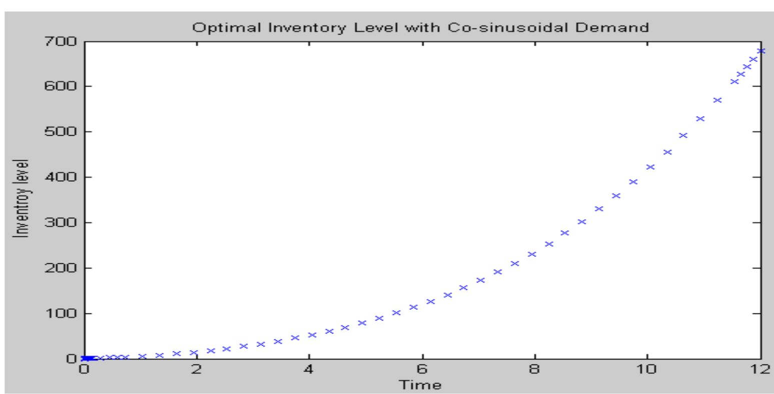

Figure 9. The inventory level $x(t)$ in-terms of the first-order differential equation in terms of co-sinusoidal demand.

desired production rate is high.

Figures 7 to 9 show the slight variations of the inventory and optimal production level with time with changing the shape of the demand functions. In case of sinusoidal, co-sinusoidal and exponential decreasing demand oriented optimal inventory levels over time

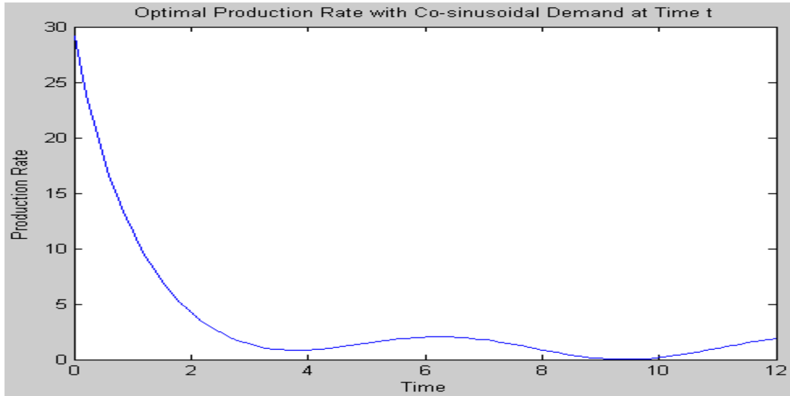

Figure 10. Optimal production policy with co-sinusoidal demand.

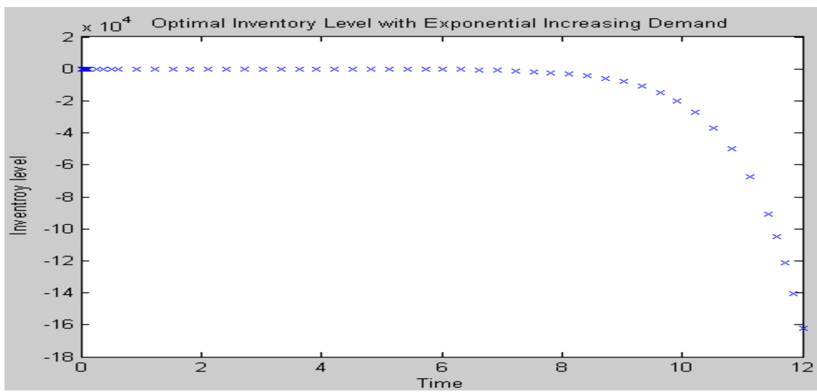

Figure 11. The inventory level $x(t)$ in-terms of the first-order differential equation in terms of exponential increasing demand.

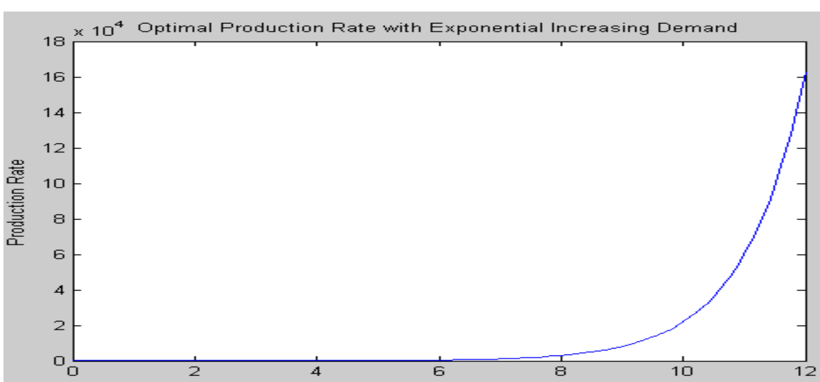

Figure 12. Optimal production policy with exponential increasing demand.

almost have no variations that support the findings of Baten and Kamil (2009). It is observed that the optimal production rates are not very sensitive to changes in the demand functions in case of Gumbel distribution. Optimal production policy with co-sinusoidal demand was shown by Figure 10. On the other hand, Figures 11 and 13 show the large variations of the optimal inventory level with time with changing the shape of the demand functions. From Figures 12 and 14, it is observed that the optimal production rates are very sensitive to changes in the

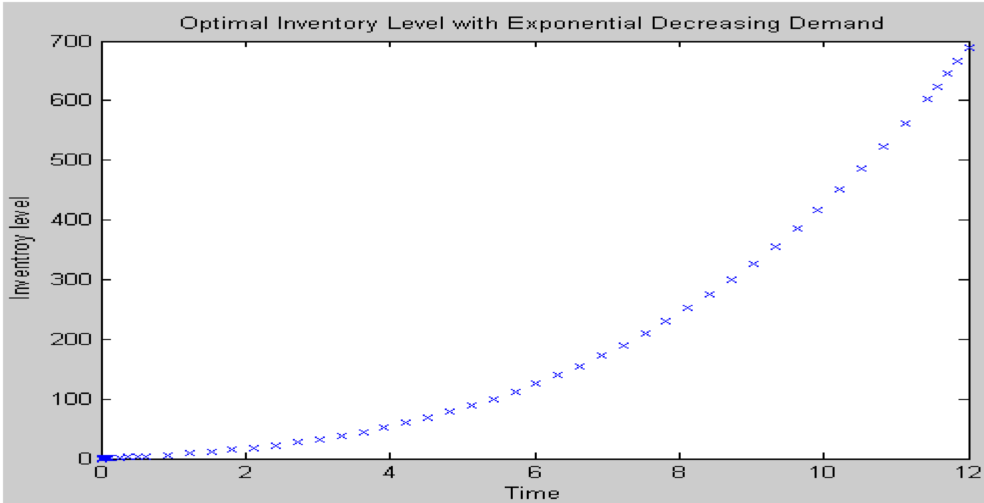

Figure 13. The inventory level $x(t)$ in-terms of the first-order differential equation in terms of exponential decreasing demand.

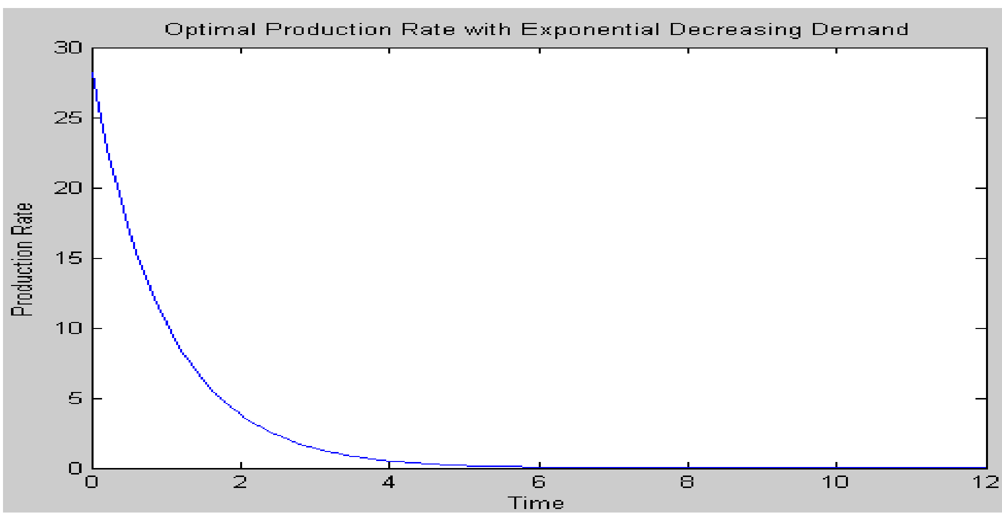

Figure 14. Optimal production policy with exponential decreasing demand.

demand functions. The similar results also available in case of Pareto distributed deterioration study of Baten and Kamil (2010).

The solution of the second-order differential equation is represented by Figure 15 and shows the state of optimal inventory level is increasing.

However, in further discussions, we present the model to measure the performance using different demand patterns. The production level with time t given $\hat{u}(t)$ from the Equation (19) considering the mentioned above different demand rates and we take the inventory goal level is as $\hat{x}(t)=10$ keeping all other parameters unchanged.

Constant demand function

Here, we present the model with constant demand function. Substituting constant $y_1(t)=y_1=20$ instead of $y(t)$ in the controlled system we have:

$$\frac{dx_1(t)}{dt}=u_1(t)-y_1(t)-\frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}x_1(t), \quad 0\le t\le T, \quad x(T)=0$$

from which the production goal rate $\hat{u}(t)$ can be computed (assuming $x(0)=x$) as:

$$\hat{u}_1(t) = y_1(t)+\frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}\hat{x}_1(t).$$

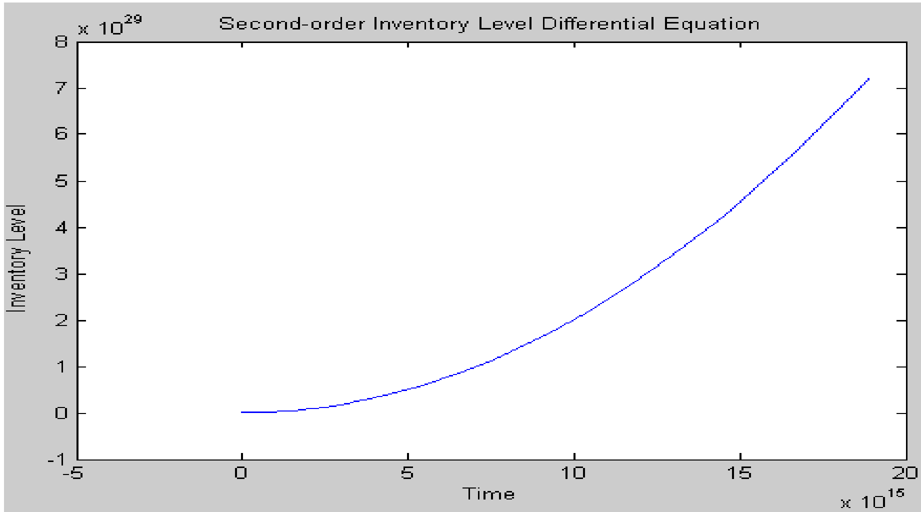

Figure 15. The inventory level $x(t)$ in-terms of the second-order differential equation.

Linear demand function

Here, we present the model with linear demand function. Substituting linear $y_2(t) = t + 15$ instead of $y(t)$ in the controlled system we have:

$$\frac{dx_2(t)}{dt} = u_2(t) - y_2(t) - \frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}x_2(t), \quad 0 \le t \le T, \quad x(T) = 0$$

from which the production goal rate $\hat{u}(t)$ can be computed (assuming $x(0) = x$) as:

$$\hat{u}_2(t) = y_2(t) + \frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}\hat{x}_2(t).$$

Quadratic demand function

Here, we present the model with linear demand function. Substituting Quadratic $y_3(t) = 30 + 0.1t + 0.001t^2$ instead of $y(t)$ in the controlled system we have:

$$\frac{dx_3(t)}{dt} = u_3(t) - y_3(t) - \frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}x_3(t), \quad 0 \le t \le T, \quad x(T) = 0$$

from which the production goal rate $\hat{u}(t)$ can be computed (assuming $x(0) = x$) as:

$$\hat{u}_3(t) = y_3(t) + \frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}\hat{x}_3(t).$$

Sinusoidal demand function

Here, we present the model with sinusoidal demand function. Substituting $y_4(t) = 1 + \sin(t)$ instead of $y(t)$ in the controlled system we have:

$$\frac{dx_4(t)}{dt} = u_4(t) - y_4(t) - \frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}x_4(t), \quad 0 \le t \le T, \quad x(T) = 0$$

from which the production goal rate $\hat{u}(t)$ can be computed (assuming $x(0) = x$) as:

$$\hat{u}_4(t) = y_4(t) + \frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}\hat{x}_4(t).$$

Co-sinusoidal demand function

Here, we present the model with sinusoidal demand function. Substituting $y_5(t) = 1 + \cos(t)$ instead of $y(t)$ in the controlled system we have:

$$\frac{dx_5(t)}{dt} = u_5(t) - y_5(t) - \frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}x_5(t), \quad 0 \le t \le T, \quad x(T) = 0$$

from which the production goal rate $\hat{u}(t)$ can be computed (assuming $x(0) = x$) as:

$$\hat{u}_5(t) = y_5(t) + \frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}\hat{x}_5(t).$$

Exponential increasing demand function

Here, we present the model with sinusoidal demand function. Substituting $y_6(t) = \exp(t)$ instead of $y(t)$ in the controlled system we have:

$$\frac{dx_6(t)}{dt} = u_6(t) - y_6(t) - \frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}x_6(t), \quad 0 \le t \le T, \quad x(T) = 0$$

from which the production goal rate $\hat{u}(t)$ can be computed (assuming $x(0) = x$) as:

$$\hat{u}_6(t) = y_6(t) + \frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}\hat{x}_6(t).$$

Exponential decreasing demand function

Here, we present the model with sinusoidal demand function. Substituting $y_6(t) = \exp(-t)$ instead of $y(t)$ in the controlled system we have:

$$\frac{dx_7(t)}{dt} = u_7(t) - y_7(t) - \frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}x_7(t), \quad 0 \le t \le T, \quad x(T) = 0$$

from which the production goal rate $\hat{u}(t)$ can be computed (assuming $x(0) = x$) as:

$$\hat{u}_7(t) = y_7(t) + \frac{1}{\sigma}\exp\{-(t-\mu)/\sigma\}\hat{x}_7(t).$$

Conclusion

In this paper, we developed an optimal control model in inventory-production system with a special class of generalized extreme value, which is, Gumbel distribution deteriorating items. This paper derived the explicit solution of the optimal control models of an inventory-production system under a continuous review-policy using Pontryagin maximum principle. However, we gave numerical illustrative examples and numerical solution of optimal inventory-production system with Gumbel distribution deteriorating items using different types of demand functions.

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