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Full Length Research Paper

Complex variable solution of elastic tunneling problems

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The ground loss problem occurs when a cylindrical tunnel is constructed in a soil with the radius of the tunnel being somewhat smaller than the radius of the cavity. The method used in this paper is Muskhelishvilli's complex variable method considering conformal mapping of the elastic region onto a circular ring. The problem of an elastic half plane with a circular cavity was investigated, regarding the case that along the boundary of the cavity, the surface tractions were prescribed. The computer program (ground loss) was used. The program worked interactively, on the basis of values of Poisson's ratio and the ratio of the radius of the cavity to its depth (r/h). It was investigated whether certain problems of stresses and deformations caused by deformation of a tunnel in an elastic half plane could be solved by the complex variable method. For this purpose, two elementary boundary value problems were considered in detail. These include the problem of a half plane with a circular cavity loaded by a uniform radial stress, and the problem in which a uniform radial displacement is imposed on the cavity boundary (this is usually called the ground loss problem). It was concluded that the displacement of the bottom of the tunnel was always smaller than the value, u_o (the displacement of the cavity). For large values of r/h, the displacement may even be negative, that is, downward. The displacement of the bottom was always equal to the average displacement of the tunnel plus a constant value Mo which is the imposed radial displacement.

Key words: Complex variable, tunnel, boundary value problem, elastic.

INTRODUCTION

Certain problems of stresses and deformations caused by deformation of a tunnel in an elastic half plane can be solved by the complex variable method, as described by Muskhelishvili (1953) and developed by Verruijt (2003). The geometry of the problem is that of a half plane with a circular cavity as shown in Figure 1. The boundary conditions are, that the upper boundary of the half plane is free of stress, and that the boundary of the cavity undergoes a certain prescribed displacement, for instance a uniform radial displacement (the ground loss problem) or an ovalisation.

It should be noted that in the classical treatises of Muskhelishvili (1953) and Sokolnikoff (1956) on the complex variable method in elasticity, the problems studied here are briefly mentioned, but it is stated that "difficulties" arise in the solution of these problems, and Verruijt (2003) suggested using another method of solution, such as the method using bipolar coordinates.

In this paper, an elementary problem will be considered in detail. This is the problem of a half plane with a circular cavity loaded by a uniform radial stress, and the problem in which a uniform radial displacement is imposed on the cavity boundary (this is usually called the ground loss problem). It is also planned to consider Mindlin's problem of a circular cavity in an elastic half plane loaded by gravity.

Basic equations

In this section the basic equations of a plane strain elasticity theory were presented, using the complex variable approach (Muskhelishvili, 1953). (Figure 1)

Plane strain elasticity

Consider a homogeneous linear elastic material, deforming under

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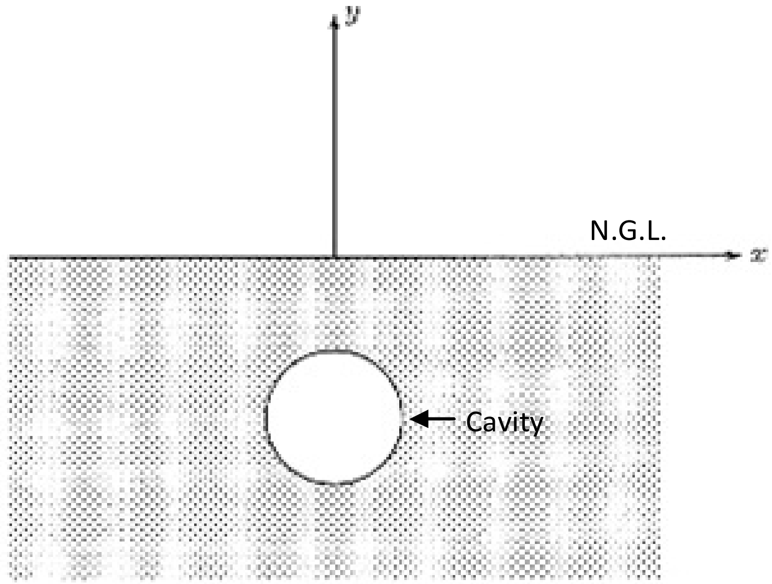

Figure 1. Half plane with circular cavity.

plane strain conditions. In the absence of body forces, the stresses can be expressed in terms of the displacements u_x and u_y by Hook's law

$$\nabla^2\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\right)=0 \tag{1}$$

$$\sigma_{xx}+\sigma_{yy}=2(\lambda+\mu)\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\right) \tag{2}$$

Airy's function

There must exist a single-valued function B(x, y) such that:

$$\sigma_{xx}=\frac{\partial B}{\partial y}, \qquad \sigma_{yx}=-\frac{\partial B}{\partial x}. \tag{3}$$

Similarly, it follows from Equation 2 that there must exist a single-valued function A(x, y) such that:

$$\sigma_{xy}=\frac{\partial A}{\partial y}, \qquad \sigma_{yy}=-\frac{\partial A}{\partial x}. \tag{4}$$

Because $\sigma_{xy} = \sigma_{yx}$, it follows that:

$$\frac{\partial B}{\partial x}=\frac{\partial A}{\partial y} \tag{5}$$

This means that there must exist a single-valued function U such that:

$$A=\frac{\partial U}{\partial x}, \qquad B=\frac{\partial U}{\partial y} \tag{6}$$

The stresses can be expressed in the function U, Airy's stress function, by the relations:

(1)

$$\sigma_{xx}=\frac{\partial^2 U}{\partial y^2}, \quad \sigma_{yx}=\frac{\partial^2 U}{\partial xy}, \quad \sigma_{yy}=\frac{\partial^2 U}{\partial x^2} \tag{7}$$

(2)

In the next section, a general form of the solution will be derived in terms of complex functions.

The Goursat solution

In order to solve Equation 7, we will write:

$\nabla^2 U = P$ (3) (8)

Because U is biharmonic, the function P must be harmonic;

$$\nabla^2 P = 0 \tag{9}$$

The general solution of Equation 9 in terms of an analytic function is:

(4)

$$P = \text{Re}\,\{f(z)\} \tag{10}$$

where f is an analytic function of the complex variable $z = x + iy$. We will write:

$Q = \text{Im}\,\{f(z)\}$, (5) (11)

so that:

$$f(z) = P + iQ \tag{12}$$

because $f(z)$ is analytic, it follows that the functions P *and* Q satisfy the Cauchy-Riemann conditions:

$$\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial y}, \qquad \frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}. \tag{13}$$

A function $\phi(z)$ is introduced as the integral of $f(z)$, apart from a factor 4, so that:

$$\frac{d\phi}{dz}=\frac{1}{4}f(z) \tag{14}$$

The function $\phi(z)$ is also an analytic function of z. If we write:

$$\phi(z) = p + iq \tag{15}$$

it follows that:

$$\frac{d\phi}{dz}=\frac{\partial p}{\partial x}+i\frac{\partial q}{\partial x}=\tfrac{1}{4}f(z)=\tfrac{1}{4}P+\tfrac{1}{4}iQ \tag{16}$$

Thus, using the Cauchy-Riemann conditions for p and q.

$$\frac{\partial p}{\partial x}=\frac{\partial q}{\partial y}=\tfrac{1}{4}P, \qquad \frac{\partial p}{\partial y}=-\frac{\partial q}{\partial y}=-\tfrac{1}{4}Q \tag{17}$$

We now consider the function:

$$F=U-\tfrac{1}{2}(x-iy)(p+iq)-\tfrac{1}{2}(x+iy)(p-iq)=U-xpyq \tag{18}$$

taking the Laplacian of this expression gives:

$$\nabla^2 F=\nabla^2 U-x\nabla^2 p-y\nabla^2 q-2\frac{\partial p}{\partial x}-2\frac{\partial q}{\partial y}. \tag{19}$$

Because p and q are the real and imaginary parts of an analytic function, their Laplacian is zero. Thus, using Equation 17 becomes:

$$\nabla^2 F=\nabla^2 U - P \tag{20}$$

Finally using Equation 8, it follows that the Laplacion F is zero

$$\nabla^2 F = 0 \tag{21}$$

This means that we may write:

$$F=\mathrm{Re}\{x(z)\}=\tfrac{1}{2}\{x(z)+\overline{x(z)}\}, \tag{22}$$

where $x(z)$ is another analytic function of z. The imaginary part of $x(z)$ will be denoted by G, so that:

$$x(z) = F + iG. \tag{23}$$

From Equations 18 and 22, it follows that:

$$2U=\bar{z}\phi(z)+z\overline{\phi(z)}+x(z)+\overline{x(z)}, \tag{24}$$

or

$$U=\mathrm{Re}\{\bar{z}\phi(z)+x(z)\}. \tag{25}$$

This is the general solution of the biharmonic equation, first given by Goursat. In the next section, the stresses and the displacements will be expressed into the two functions $\phi(z)$ and $x(z)$.

Stresses

The stresses are expressed in the second derivatives of Airy's function U. It follows that by using the expressions for the stresses in terms of Airy's function, the following equations can be derived (Verruijt, 2003):

$$\sigma_{xx}+\sigma_{yy}=2\{\phi'(z)+\overline{\phi'(z)}\} \tag{26}$$

$$\sigma_{yy}-\sigma_{xx}+2i\sigma_{xy}=2\{\bar{z}\phi''(z)+\psi'(z)\} \tag{27}$$

These are the equations of Kolosov-Muskhelishvili (Verruijt, 2003).

Displacements

In order to express the displacement components into the complex functions ϕ and ψ, we will start with the basic equations expressing the stresses into the displacements.

Whenever necessary rigid body displacements are added to the displacement field, we may write:

$$2\mu(u_x+iu_y)=k\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)} \tag{28}$$

where

$$k=\frac{\lambda+3\mu}{\lambda+\mu}=3-4v \tag{29}$$

In order to determine the solution of these equations, boundary conditions have to be applied.

Solution of boundary value problems

In this section, the general technique for the solution of boundary value problems for simply connected regions, in particular regions that can be mapped conformally onto a circle (such as a half plane) are discussed. In later section the theory will be applied to multiple connected regions, with circular boundaries (a ring) and to problems for the half plane with a circular hole. Many of the solutions have been presented also by Muskhelishvili (1953) and Sokolnikoff (1956).

Conformal mapping onto the unit circle

Suppose that we wish to solve a problem for an elastic body inside the region R in the complex z-plane. Let there be a conformal transformation of R onto the unit circle γ in the ζ-plane, denoted by:

$$z = \omega(\zeta). \tag{30}$$

we now write:

$$\phi(z) = \phi(\omega(\zeta)) = \phi_*(\zeta) \tag{31}$$

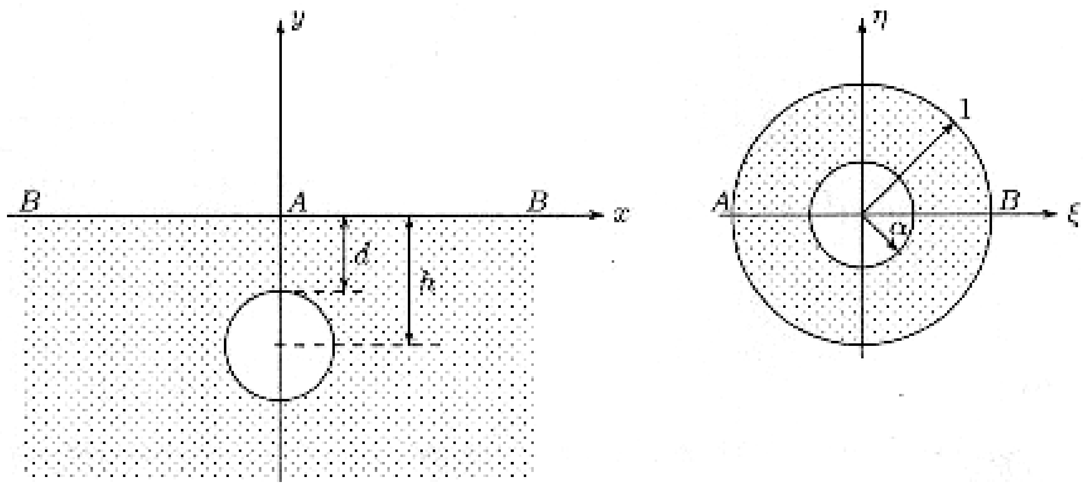

Figure 2. Conformal transformation.

$\psi(z) = \psi(\omega(\zeta)) = \psi_*(\zeta)$ (32)

where the symbol *, indicates that the form of the function ϕ_* is different from that of the function ϕ. The derivative of ϕ is :

$$\phi'(z)=\frac{d\phi}{dz}=\frac{d\phi}{d\zeta}\frac{d\zeta}{dz}=\frac{\phi'_*(\zeta)}{\omega'(\zeta)} \tag{33}$$

The application of surface traction and displacement boundary conditions is found in Verruijt (2003).

Elastic half plane with circular cavity

In this section we will study the problem of an elastic half plane with a circular cavity as shown in Figure 2. The upper boundary of the half plane is assumed to be free of stress, and loading takes place along the boundary of the circular cavity, in the form of a given stress distribution or a given displacement distribution.

It is assumed that the region in the z-plane can be mapped conformally onto a ring in the ζ-plane, bounded by the circles $|\zeta|=1$ and $|\zeta|=\alpha$, where $\alpha<1$.

The Inner Boundary

The conformal transformation is

$$z=\omega(\zeta)=-ia\frac{1+\zeta}{1-\zeta} \tag{34}$$

where a is a certain length. The origin in the z-plane is mapped onto $\zeta=-1$, and the point at infinity in the z-plane is mapped onto $\zeta = 1$, as shown in Figure 2. Differentiation of equation 34 with respect to ζ gives:

$$\omega'(\zeta)=-\frac{2ia}{(1-\zeta)^2} \tag{35}$$

It will be shown that concentric circles in the ζ-plane are mapped on circles in the z-plane, and the relation between the depth of the circle and its radius with the parameter a, which is the radius of the circle in the ζ-plane, will be derived.

For a circle with radius α in the ζ-plane we have:

$$\zeta = \alpha \exp(i\theta) \tag{36}$$

where α is a constant, and θ is a variable. From Equation 34, this will give us:

$$x=\frac{2a\alpha\sin\theta}{1+\alpha^2-2\alpha\cos\theta} \tag{37}$$

$$y=-\frac{a(1+\alpha^2)}{1+\alpha^2-2\alpha\cos\theta}. \tag{38}$$

It is now postulated that these formulas represent a circle, at depth h, having a radius r. This means that it is assumed that there exist constants h and r such that:

$$x^2+(y+h)^2=r^2 \tag{39}$$

In order to prove this we will demonstrate that $\partial r^2/\partial\theta = 0$. This is the case if:

$$\frac{\partial r^2}{\partial\theta}=2x\frac{\partial x}{\partial\theta}+2(y+h)\frac{\partial y}{\partial r}=0. \tag{40}$$

this means that:

$$h=-y-x\frac{\partial x/\partial\theta}{\partial y/\partial\theta} \tag{41}$$

It follows from Equation 37 that:

$$\frac{\partial x}{\partial\theta}=a\frac{(1+\alpha^2)2\alpha\cos\theta-4\alpha^2}{(1+\alpha^2-2\alpha\cos\theta)^2}, \tag{42}$$

and from Equation 38, it follows that:

$$\frac{\partial y}{\partial \theta} = a\frac{(1-\alpha^2)2\alpha\sin\theta}{(1+\alpha^2-2\alpha\cos\theta)^2}. \qquad (43)$$

Substitution of these two, results into Equation 41 which gives some algebraic manipulations:

$$h = a\frac{1+a^2}{1-a^2} \qquad (44)$$

which is indeed a constant, and which also proves that *r is* a constant. With equation 39, the corresponding value of *r* is found to be:

$$r = a\frac{2\alpha}{1-\alpha^2} \qquad (45)$$

If the covering depth of the circular cavity in the z-plane is denoted by *d, (*see Figure 2), it follows that:

$$d = a\frac{1-\alpha}{1+\alpha} \qquad (46)$$

The ratio of depth and cover is:

$$\frac{h}{d} = \frac{1+\alpha^2}{(1-\alpha)^2} \qquad (47)$$

If $\alpha \rightarrow 0$, the radius of the circular cavity will be practically zero, this indicates a very deep tunnel, or a very large covering depth. If $\alpha \rightarrow 1$, the covering depth will be very small. For every value of *h/d* the corresponding value of *a* can be determined from Equation 47.

A displacement boundary condition

A simple boundary condition along the inner boundary in the z-plane is that the normal stress, or the radial displacement, is constant along this boundary. In terms of the displacement this means:

$$u_x = -u_0\frac{(1-\alpha^2)\sin\theta}{1+\alpha^2-2\alpha\cos\theta}, \qquad (48)$$

$$u_y = -u_0\frac{2\alpha-(1+\alpha^2)\cos\theta}{1+\alpha^2-2\alpha\cos\theta}. \qquad (49)$$

It may be noted that for $\alpha \rightarrow 1$, this reduces to $u_x + iu_y = iu_0 \exp(i\theta)$. In the complex variable method, the boundary values have to be expanded into Fourier series. This is given by Verruijt (2003) and Raheem (2006).

A stress boundary condition

A simple boundary condition along the cavity boundary in which the stresses are prescribed, is the case of a uniform radial stress *t*. Then:

$$t_x = t\frac{x}{r} \qquad (50)$$

$$t_y = t\frac{y+h}{r} \qquad (51)$$

Along the boundary of the cavity, we may write $z + ih = r\exp(i\beta)$, where r is the constant radius of the circle and β is a variable angle. Along that path, $ds = rd\beta$, so that:

$$F = it\int_0^{\beta}\exp(i\beta)rd\beta = tr[\exp(i\beta)-1] = t(z+ih-r), \qquad (52)$$

where it has been assumed that the initial point sq corresponds to $\beta = 0$.

Expressed into the value of $\zeta = \alpha\sigma$ along the boundary in the ζ-plane, Equation 52 becomes:

$$F = \frac{2ith\alpha}{(1+\alpha^2)(1-\alpha\sigma)}[\alpha-\sigma+i(1-\alpha\sigma)] \qquad (53)$$

This is the form of the boundary stress function that will be considered later.

A boundary value problem

In this section, the problem of an elastic half plane with a circular cavity was investigated, regarding the case that along the boundary of the cavity, the surface tractions were prescribed.

The complex stress functions $\phi(\zeta)$ and $\psi(\zeta)$ were analyzed throughout the ring-shaped region in the ζ-plane. It was assumed that they were also single-valued, so that logarithmic singularities could be ignored. This means that they can be represented by the Laurent series expansions:

$$\phi(\zeta) = a_0 + \sum_{k=1}^{\infty} a_k\zeta^k + \sum_{k=1}^{\infty} b_k\zeta^{-k} \qquad (54)$$

$$\psi(\zeta) = c_0 + \sum_{k=1}^{\infty} c_k\zeta^k + \sum_{k=1}^{\infty} d_k\zeta^{-k} \qquad (55)$$

These series expansions will converge up to the boundaries $|\zeta| = 1$ and $|\zeta| = \alpha$. The coefficients a_k, b_k, c_k and d_k must be determined from the boundary conditions.

Validation of the solution

In order to validate the solution, it has to be implemented in a computer program (Ground Loss) developed by (Verruijt, 2002).

The program works interactively, on the basis of values of Poisson's ratio *v* and the ratio of the radius of the cavity to its depth (r/h), which must be entered by the user.

The program first calculates the coefficients of the series expansions (taking a maximum of *m* terms), and then calculates stresses and displacements along the boundaries. This enables one to verify whether the boundary conditions are indeed satisfied. In the program, the value of m was taken as 10000. This is usually too large for sufficient convergence.

A special problem is the determination of the constant a_0, which is not explicitly determined by the two boundary conditions. It was found that when an arbitrary value of a_0 was used as a starting value, all the coefficients p_k and q_k became equal (and unequal to zero) for large values of k. This suggestion is to determine the precise value of a_0 such that these coefficients tend towards zero for $k \to\infty$. This appears to work well. The actual procedure used is to first assume $a_0 = 0$, calculate the last coefficient q_{nn}, repeat the calculations with $a_0 = 1$, again calculate the last coefficient q_{nn}, and then determine the value of a_0 by linear interpolation, such that $q_{nn}= 0$. Because of the linearity of the system, it should work well as indeed it appears to do.

NUMERICAL RESULTS

Numerical results of all the displacements and stresses were presented for values of x and y to be entered. The value of y must be negative because the half plane considered is $y < 0$.

Validations

The first validation of the program was the boundary condition at the cavity boundary. The displacements there were calculated, and it was found that the radial displacement was indeed -1, and that the tangential displacement was indeed 0 (both up to six significant numbers). The same was true for the surface tractions along the horizontal upper boundary. It was found that along this boundary $\sigma_{yy} = \sigma_{yx} = 0$, the lateral stress σ_{xx} was not found to be zero, but of course this was not necessary.

By considering points in the complex ζ-plane very close to ζ=1, it is possible to calculate the stresses near infinity. These appear to be zero, as they should be.

In the same way, by taking $\zeta = 1 + \varepsilon$, with $|\varepsilon| \ll 1$, it is possible to calculate the displacements near infinity. It was found that the horizontal displacement was zero, but that the vertical displacement unequal to zero. Although this may be somewhat unexpected, it seems to be very well possible because of the conditions that the displacements at the cavity boundary are rigidly imposed and the stresses at infinity have been assumed to vanish. It has been verified that this displacement at infinity is uniform, by checking the displacements at a great number of points, for various complex values of ε. It appears that a contraction of the cavity (a positive ground loss in tunnel engineering) leads to an upward displacement at infinity. Of course a rigid body displacement of the entire half plane, including the cavity, can take place without inducing any stresses. Thus the displacement at infinity can be made equal to zero by subtracting a constant from all displacements. This means that the cavity itself will also undergo this rigid body displacement. It can be concluded that a contracting cavity will undergo a downward displacement, with respect to the points at infinity.

The program Ground Loss also calculates the stresses along the cavity boundary. It appears that the radial stresses are not uniformly distributed, as they are in an infinite medium, or if $r/h \to 0$, but that the radial stress is larger than average near the bottom, and smaller than average near the top of the tunnel. This does not mean that there is a resulting force, however, this is also determined by the shear stresses. Actually, the validating part of the program Ground Loss also calculates the resulting force of the surface tractions along the cavity boundary, by numerical integration. This resulting force is indeed found to be zero.

An interesting quantity is the total volume of the settlement trough. This can be calculated by integrating the vertical displacements along the surface:

$$\Delta V = -\int_{-\infty}^{+\infty} \upsilon\, dx, \tag{56}$$

where the displacements should be determined along the upper boundary y = 0.

This integral can be transformed into an integral in the ζ-plane, along the unit circle, taking into account the scale factor $|\omega'(\zeta)|$. In this case, this factor appears to be:

$$|\omega'(\zeta)| = \frac{a}{1-\cos\theta} = \frac{1-\alpha^2}{1+\alpha^2}\frac{h}{1-\cos\theta}. \tag{57}$$

thus the integral can be evaluated as:

$$\Delta V = \frac{1-\alpha^2}{1+\alpha^2} h \int_0^{2\pi} \upsilon\, d\theta \tag{58}$$

The result may be compared with the total ground loss at the circumference of the cavity,

$$\Delta V_0 = 2\pi r u_0 = 4\pi h \frac{\alpha}{1+\alpha^2} \tag{59}$$

For smaller values of Poisson's ratio, it appears that the total volume below the settlement trough is larger than the total ground loss. This property is also predicted by the approximate method of Sagaseta (1987), which was generalized by Verruijt and Booker (1996). This approximate method gives:

$$\Delta V = 2(1-v)\, \Delta V_0 \tag{60}$$

The calculations using the program Ground Loss does not confirm this result. Actually the ratio $\Delta V/\Delta V_0$ appears to be smaller than $2(1 - \upsilon)$ in the exact solution. It is only for very small tunnels that the results of the approximate

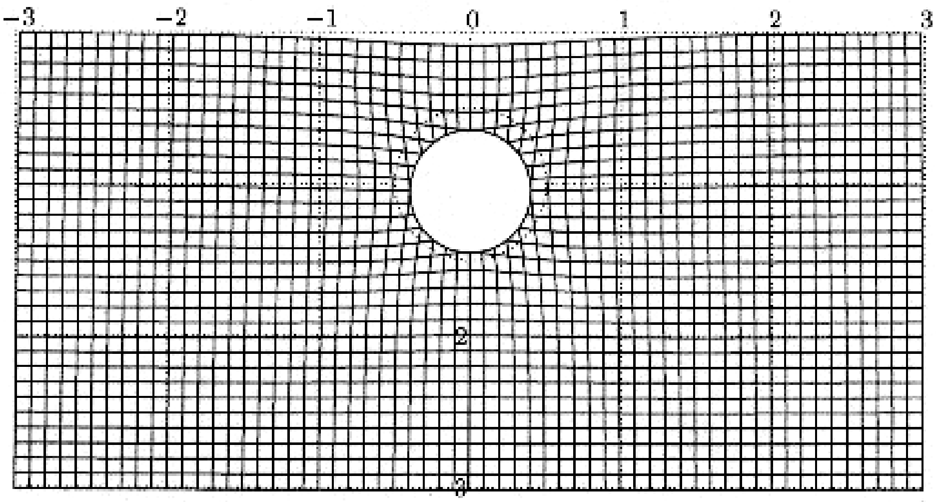

Figure 3. Uniform radial displacement of cavity, in elastic half plane, complex variable solution, multiplication factor = 0.1, $\nu = 0.5$.

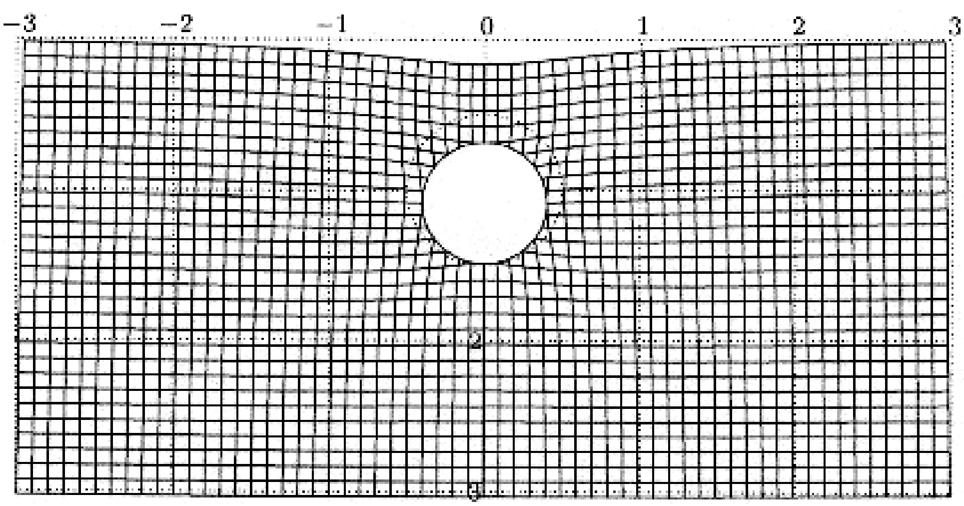

Figure 4. Uniform radial displacement of cavity boundary in elastic half plane, complex variable solution, multiplication factor = 0.1, $\nu = 0.0$.

solution and the exact solution are practically identical, for all values of Poisson's ratio.

Applications

The deformations of the mesh are shown in Figures 3 and 4. These two figures show the vertical displacement of the tunnel as a whole. They also show that the displacement of the surface increases when ν decreases from 0.5 to 0.0.

Figure 5 shows the average vertical displacement of the tunnel as a whole v_c, as a function of v and *r/h* (*r* is the radius of the cavity, *h* is a depth of cavity center line). The displacements are normalized by dividing each displacement by the displacement of cavity, u_o. It appears that for small values of *r/h,* the displacement of the tunnel is practically zero. This case corresponds to the case of a tunnel in an infinite medium, in which there is indeed no average displacement. For larger values of *r/h* (or in other words, tunnels closer to the soil surface), there is a marked vertical displacement of the tunnel. Its value is negative, indicating a downward displacement. For certain combinations of *v* and *r/h,* the displacement may even be larger than twice the imposed radial displacement (Figure 3)

Figure 6 shows the vertical displacement of the bottom of the tunnel v_b. This displacement is usually upward, but

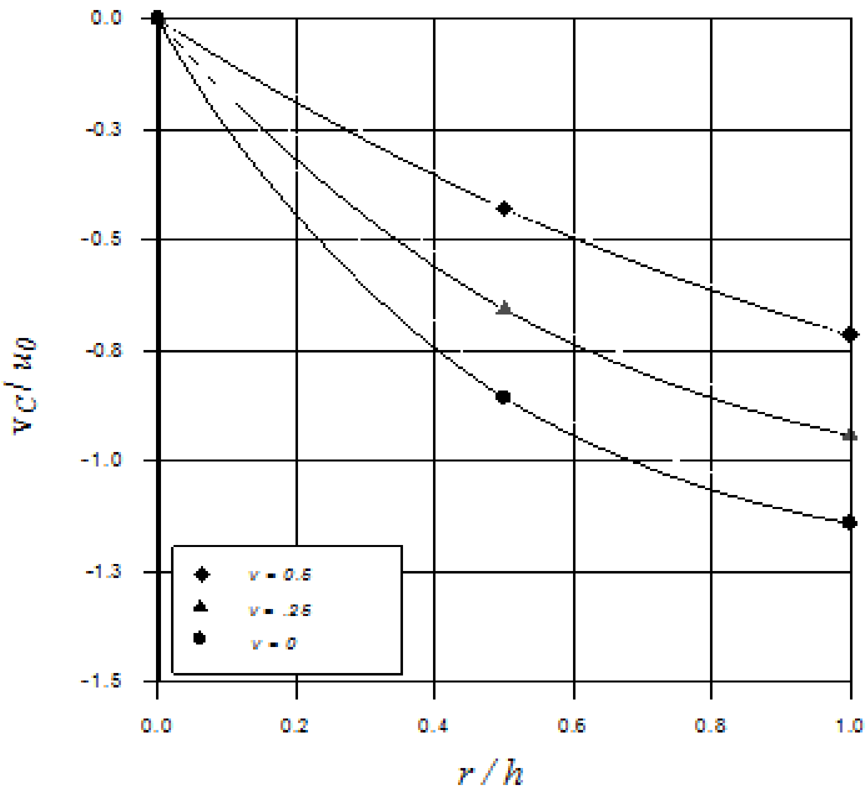

Figure 5. Vertical displacement of tunnel as predicted by the complex variable solution.

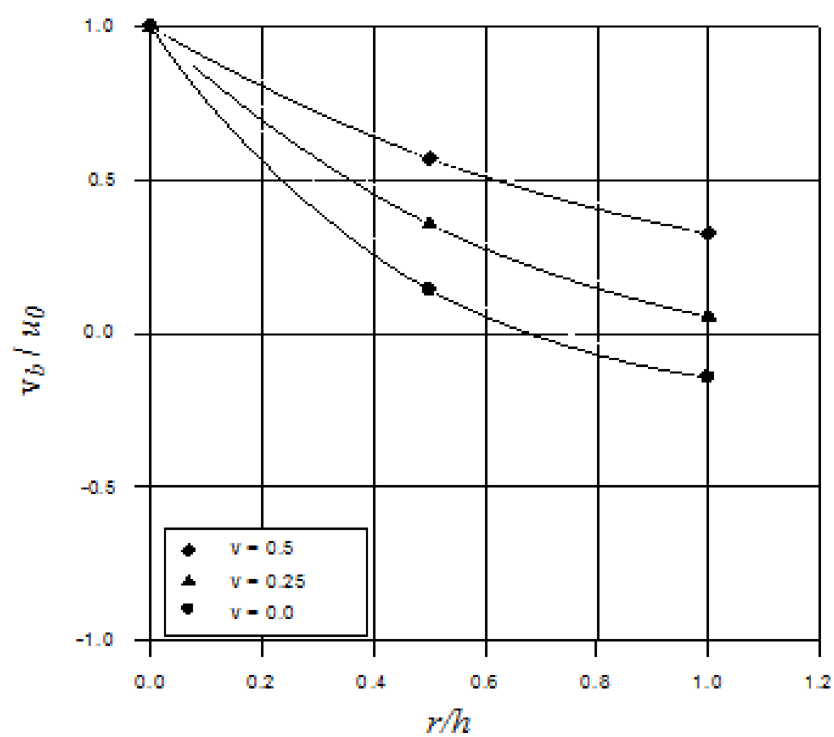

Figure 6. Vertical displacement of bottom of the tunnel as predicted by the complex variable solution.

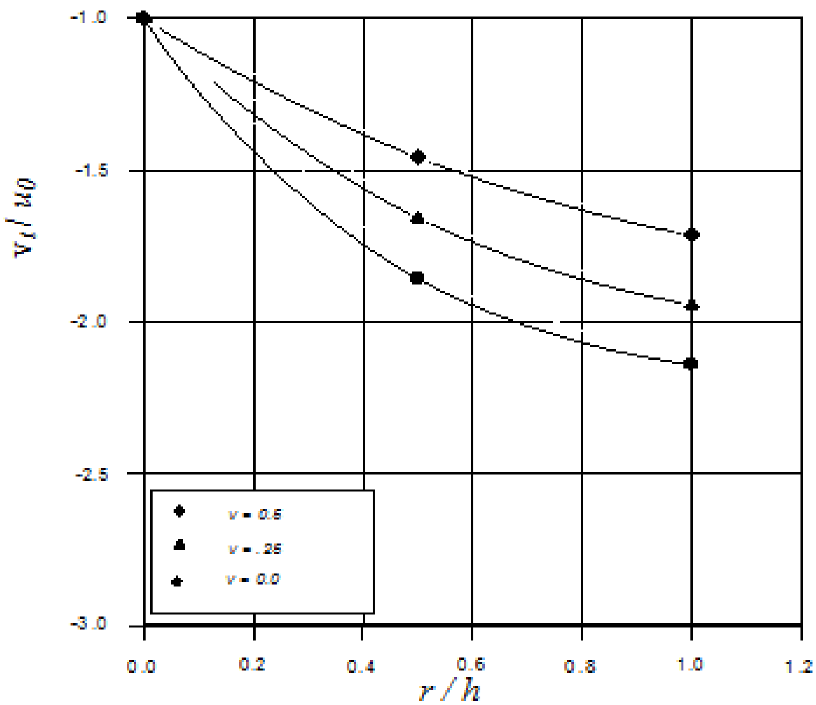

Figure 7. Vertical displacement of top of the tunnel as predicted by the complex variable solution.

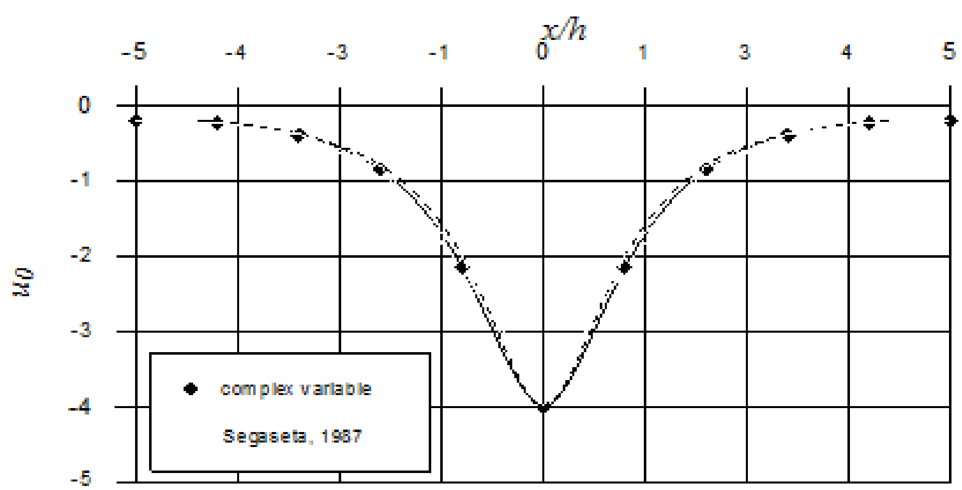

Figure 8. Surface settlement as predicted by the complex variable solution.

because of the average downward displacement of the tunnel, the displacement of the bottom is always smaller than the value u_0 (the displacement of the cavity). For large values of *r/h,* the displacement may even be negative, that is, downward. It may be noted that the displacement of the bottom is equal to the average displacement of the tunnel plus the constant value M_0; the imposed radial displacement.

Figure 7 shows the vertical displacement of the top of the tunnel *vt*. This displacement is equal to the average displacement of the tunnel, shown in Figure 5 minus the constant value u_0. This is indicated by the fact that Figures 5 and 7 differ only in the vertical scale.

Figure 8 shows the actual settlement trough, for r/h = 0.5 and v = 0. The dotted line shows the shape of the settlement trough obtained from Sagaseta's simplified

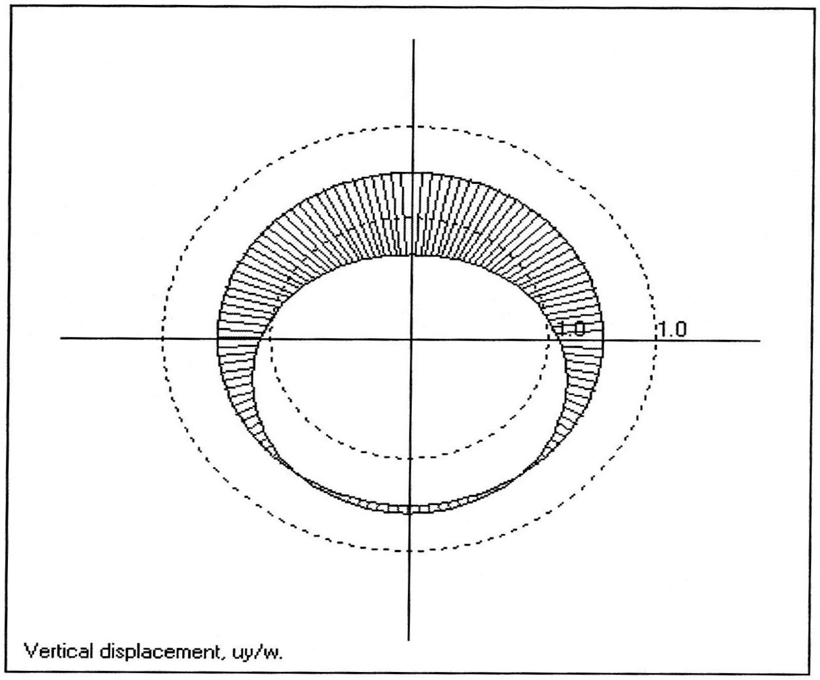

Figure 9. Vertical displacement of cavity boundary u_y/w.

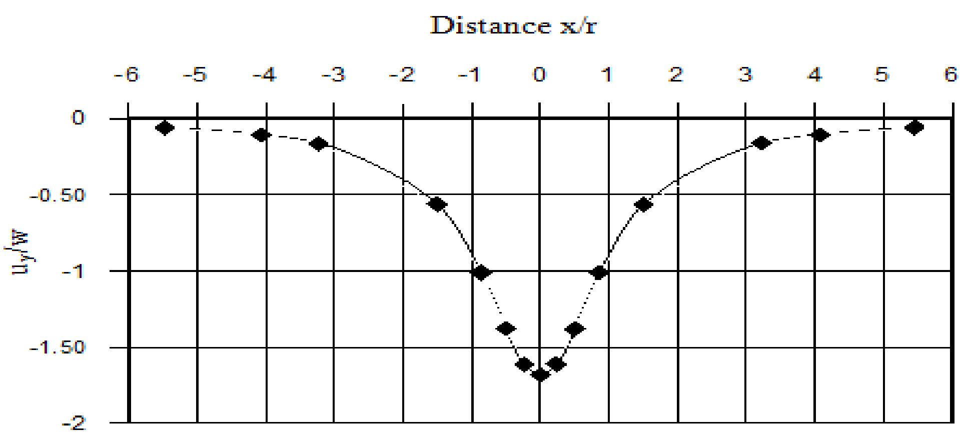

Figure 10. Upper boundary vertical settlement u_y/w.

method (Sagaseta, 1987; Verruijt and Booker, 1996), using a scale factor to let the maximum displacements coincide. (Figures 4 and 5)

Another application is analyzed using the following parameters:

u→0.3
r/h→0.5

where: r/h = radius of circle over depth of center of circle w = radial displacement of cavity, G = shear modulus of material.

The results are shown in Figures 9 to 13.

In Figure 9, the normalized vertical displacement u_y/w was plotted along the cavity boundary. It could be noticed that the vertical displacement vectors of the tunnel's crown was several times larger than those at its bottom. There were points at tunnels bottom at which no displacement took place (Figures 6 and 7).

In Figure 10, the vertical settlement u_y/w along the upper boundary at the problem is drawn. It can be seen that the settlement trough extends to a distance about twice the diameter (4r).

In Figure 11, the horizontal displacement u_x/w along the

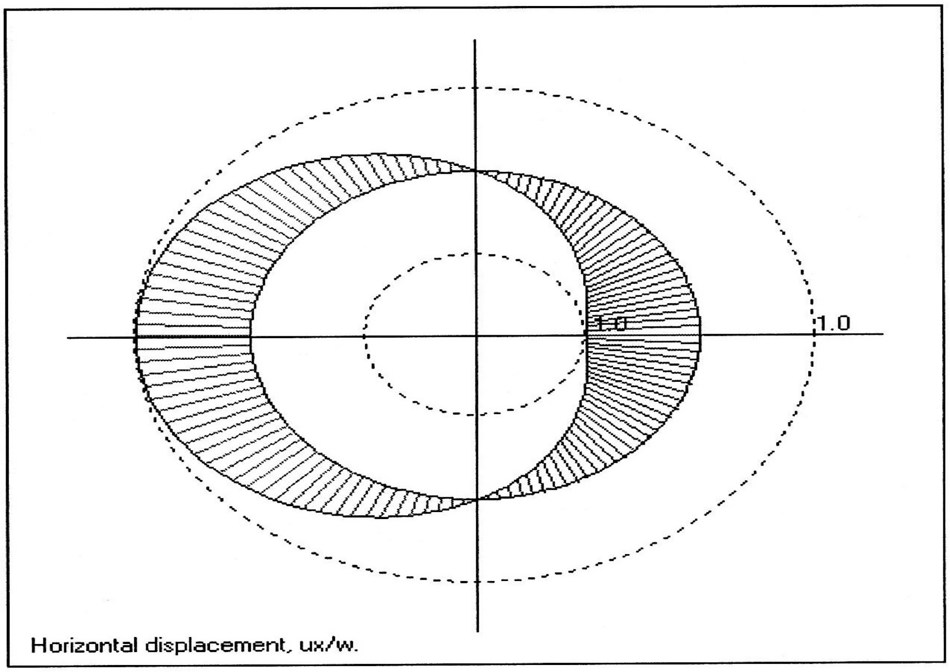

Figure 11. Cavity boundary horizontal displacement u_x/w.

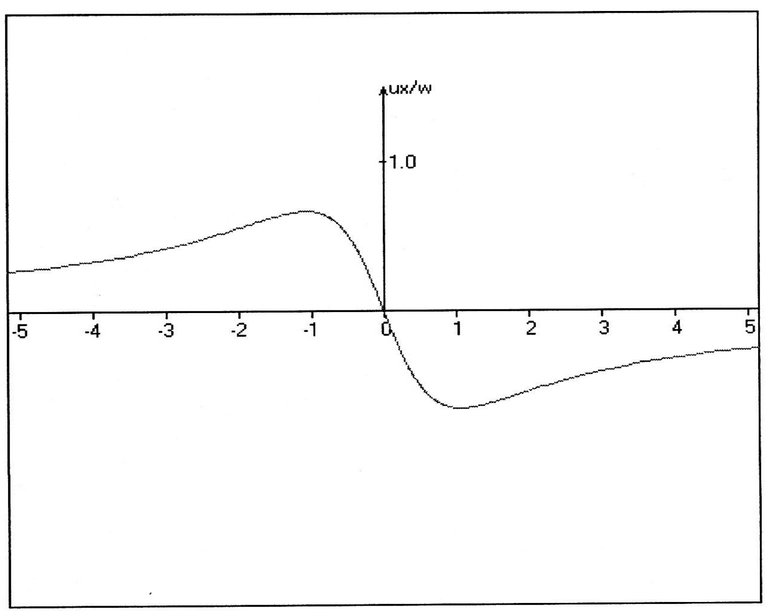

Figure 12. Upper boundary horizontal displacement u_x/w.

cavity boundary is drawn. It can be seen that the horizontal displacement is large along the side walls while a small displacement takes place at the tunnel's crown and invert.

Figure 12 shows the variation of horizontal displacement u_x/w along the upper boundary of the problem. It can be noticed that above the center of the tunnel, no horizontal movement occurs, while at left and right sides, opposed horizontal displacements can be observed. A uniform distribution of displacement takes place and symmetry about the center line of the tunnel is noticed.

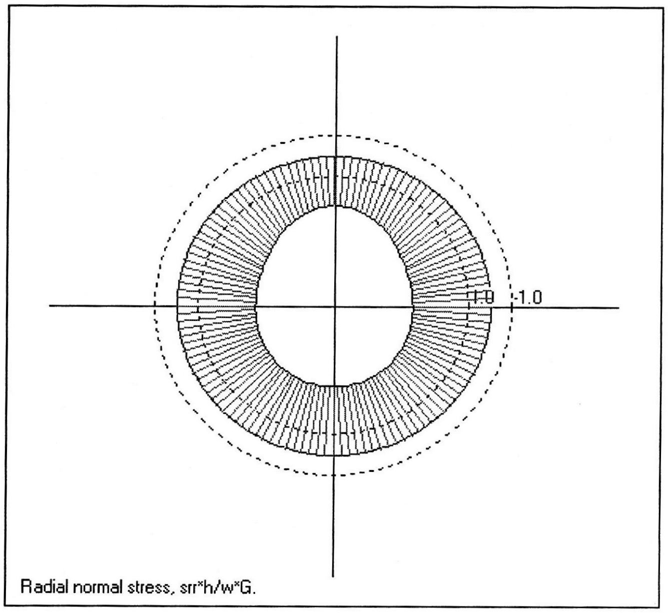

Figure 13. Cavity boundary radial normal stress σr.h/w.G.

In Figure 13, the normalized radial normal stress $\sigma_r h / w.G$ is drawn along the cavity boundary. It can be seen that the distribution of radial normal stress vectors are distributed uniformly along the boundary and approximately the same values of the radial normal stress are obtained at all points at the boundary(Figures 8 to 14).

Comparison with the boundary element method

The boundary element method has been applied to a wide variety of problems in stress analysis, including plasticity fracture mechanics, viscoelasticity and many others. Stress analysis problems in geomechanics are ideally suited to boundary elements, as this technique usually requires a very small number of nodes by comparison to finite elements. As only the surface of the continuum needs to be discretized, problems extending to infinity can be described by a very small number of elements on the soil surface or around the tunnel or excavation .In addition, the boundary conditions of the infinite domain are properly defined using boundary elements, as the technique is based on fundamental solutions valid for unbounded domains.

Influence of depth below the ground surface - case of a single cavity

Al-Adthami (2003) used the boundary element method (BEM) to study the effect of many factors that are mainly affecting the stresses and deformations around tunnels and cavities. The soil was assumed to be homogeneous, isotropic and linear elastic medium containing one cavity. Figure 14 shows a schematic representation of the problem studied for 6 values of depth/diameter ratios (Z_o/D = 1, 1.5, 2, 2.5, 3 and ∞).

Figures 15 and 16 shows the vertical and horizontal displacements along the ground surface calculated by the boundary element method, while Figures 17 and 18 present the surface vertical and horizontal displacements as calculated by the complex variable method.

It can be noticed that the vertical displacements predicted by the complex variable method are smaller than those predicted by the BEM. The difference increases as the depth of the cavity (Zo/D) increases. A better convergence was found between the surface horizontal displacements calculated by the two methods.

It can be noticed from these figures that as ($Z_o/D > 3$), the disturbing influence on the ground surface does not exceed 5% from the case of no-cavity condition.

Conclusion

It was investigated whether certain problems of stresses and deformations caused by deformation of a tunnel in an elastic half plane could be solved by the complex variable method. For this purpose, two elementary problems were considered in detail. These include the problem of a half plane with a circular cavity loaded by a uniform radial stress and the problem in which a uniform radial displacement was imposed on the cavity boundary (this is

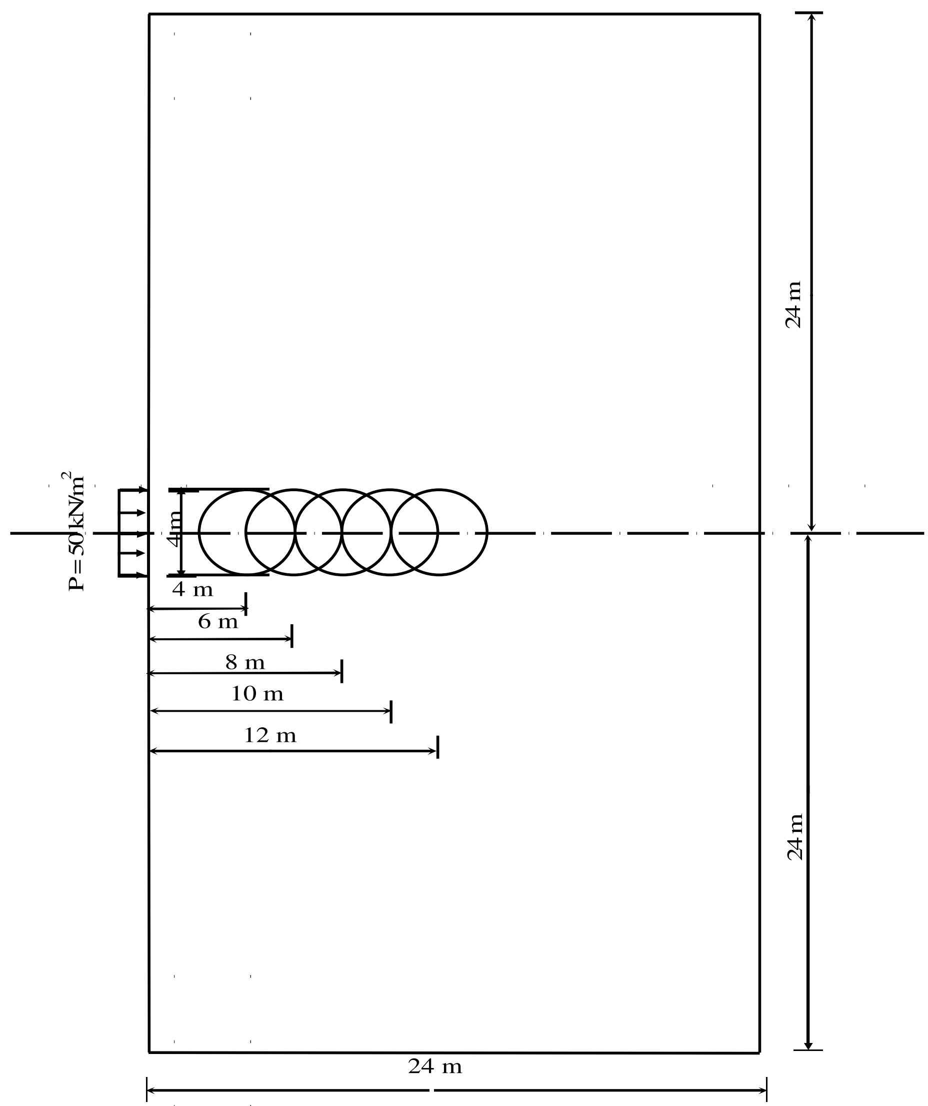

Figure 14. Schematic views of surface load-soil-cavities system (after Al-Adthami, 2003).

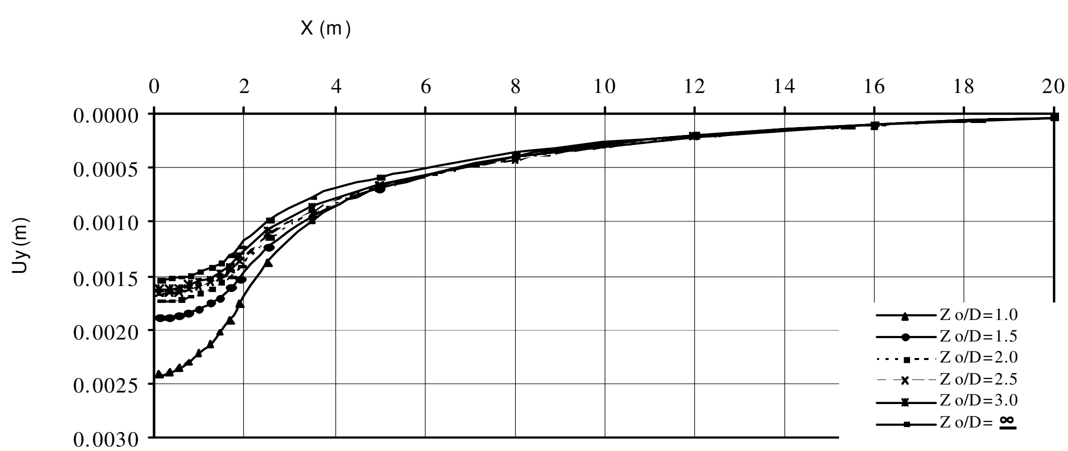

Figure 15. Vertical displacements on the surface predicted by the BEM (after Al-Adthami, 2003).

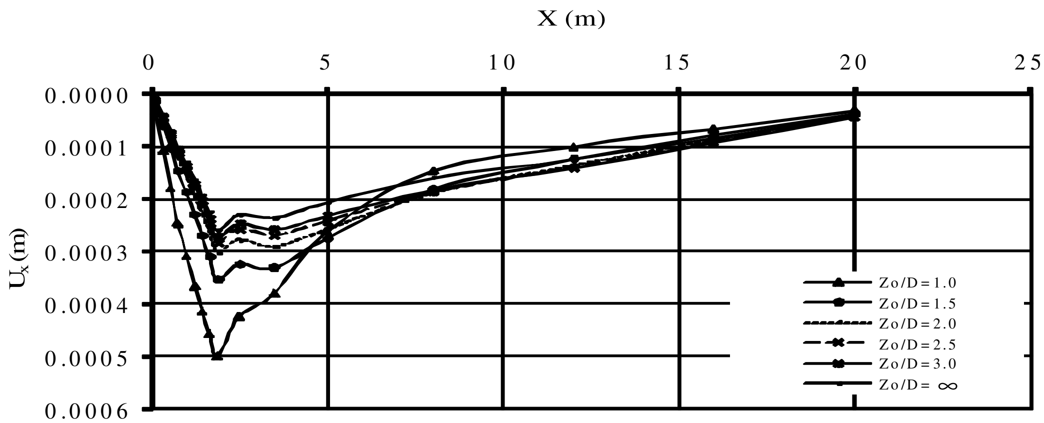

Figure 16. Horizontal displacements on the surface predicted by the BEM (after Al-Adthami, 2003).

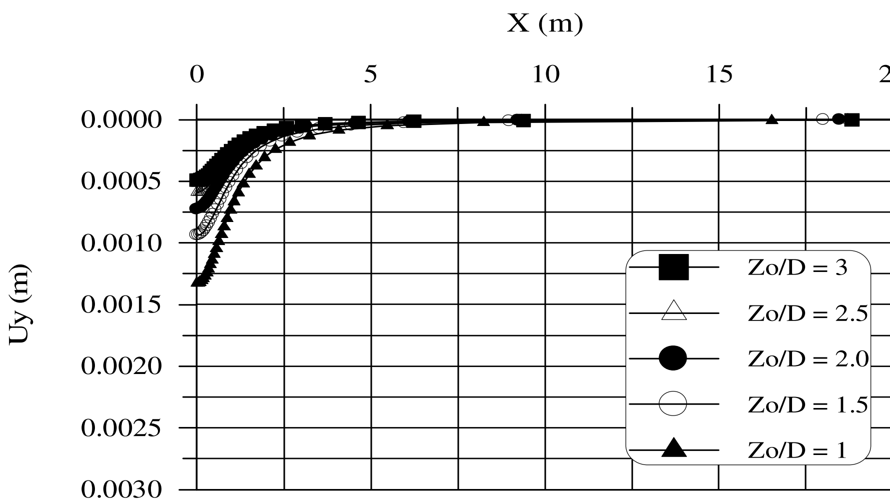

Figure 17. Vertical surface settlements calculated by the complex variable method.

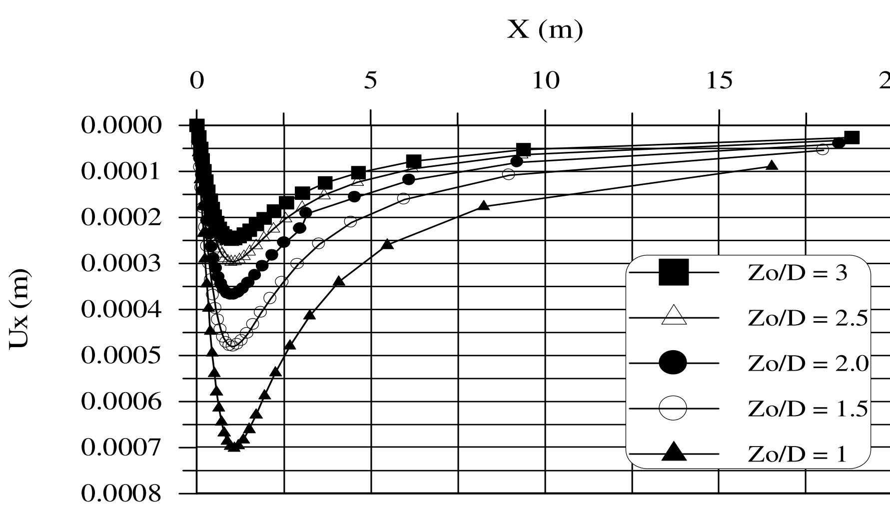

Figure 18. Horizontal surface settlements calculated by the complex variable method.

usually called the ground loss problem). From the analysis of the boundary value problems carried out by the computer program (Ground Loss), the following conclusions are obtained:

(1) The vertical displacement of the surface above the circular tunnel increased when Poisson's ratio ν decreased from 0.5 to 0.0. For small values of r/h (r is the radius of the cavity and h is the depth of tunnel's centerline), the displacement of the tunnel was practically zero. This case corresponds to the case of a tunnel in an infinite medium in which there is indeed no average displacement. For larger values of r/h (or in other words, closer to the soil surface), there was a marked vertical displacement of the tunnel. Its value was negative indicating a downward displacement. For certain combinations of ν and r/h, the displacement might even be larger than twice the imposed radial displacement.
(2) The displacement of the bottom of the tunnel was always smaller than the value U_o (the displacement of the cavity). For large values of r/h, the displacement might even be negative, that is, downward. The displacement of the bottom was always equal to the average displacement of the tunnel plus a constant value M_o which was the imposed radial displacement.
(3) Above the center of the tunnel, a very small horizontal movement occured while at the left and right sides, opposite horizontal displacements could be observed. A uniform distribution of displacement and symmetry about the center line of the tunnel could be obtained.
(4) The vertical displacement vectors of the tunnel's crown were several times larger than those at its bottom. There were points at the tunnel's bottom at which no displacement took place.
(5) The vertical displacements predicted by the complex variable method were smaller than those predicted by the boundary element method. The difference increases as the depth of the cavity (Zo/D) increased. A better convergence was found between the surface horizontal displacements calculated by the two methods.

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