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Full Length Research Paper

Differential transform method for systems of Volterra integral equations of the second kind and comparison with homotopy perturbation method

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In this article, differential transform method was employed to solve Volterra integral equations of the second kind. The procedure of the method is illustrated systematically. In this paper some results useful have been proved for the first time, and they have been used in the solution of these systems. To show the power of this method, some examples have been created (Examples 2 and 3) and solved. The results of them are compared with the results of homotopy perturbation method which has been used to solve the same systems (Biazar and Ghazvini, 2009) by plots. Results of the comparison reveal that, this method is more effective and promising than homotopy perturbation method for Volterra integral equations of the second kind.

Key words: Differential transform method, homotopy perturbation method, systems of Volterra integral equations of the second kind.

INTRODUCTION

The concept of the differential transform method was first proposed by Zhou (1986) and has been used to solve both linear and nonlinear initial value problems, in electric circuit analysis. It is different from the high-order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. Taylor series method is computationally tedious for high orders. Differential transform method leads to an iterative procedure for obtaining an analytic series solutions of functional equations. In recent years, applications of differential transform theory have appeared in many researches (Arikhoglu and Ozkol, 2007, 2006, 2005; Ayaz, 2003; Chen and Ho, 1996, 1999; Hassan and Abdel-Halim, 2008). A system of Volterra integral equations of the second kind can be presented as the following:

$$F(t) = G(t) + \int_0^t K(t,s,F(s))ds, \qquad (1)$$

Where

$F(t) = [f_1(t), f_2(t), \ldots, f_n(t)]^T$,

$G(t) = [g_1(t), g_2(t), \ldots, g_n(t)]^T$,

$K(t,s,F(t)) =$

$[k_1(t,s,F(s)), k_2(t,s,F(s)), \ldots, k_n(t,s,F(s))]^T$.

BASIC IDEAS OF DIFFERENTIAL TRANSFORM METHOD

The basic definitions and fundamental operations of the differential transform are defined as follows (Chen and Ho, 1996, 1999). The differential transform of the function

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$u(x)$ is defined as,

$$U(k)=\frac{1}{k!}\left[\frac{d^k u(x)}{dx^k}\right]_{x=x_0}, \tag{2}$$

Where $u(x)$ is the original function and $U(k)$ is the transformed function. Differential inverse transform of $U(k)$, is defined as

$$u(x)=\sum_{k=0}^{\infty} U(k)(x-x_0)^k. \tag{3}$$

When x_0 is taken as 0, the function $u(x)$ defined as (3), is expressed as the following

$$u(x)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{d^k u(x)}{dx^k}\right]x^k. \tag{4}$$

In real applications when the general term of the series cannot be recognized, a truncated series can be considered. Equation (4) implies that, the concept of one-dimensional differential transform is almost the same as one-dimensional Taylor series expansion. In this study use lower case letters to present the original functions and upper case letters stand for the transformed functions (T-functions). From definition (2 to 4), one can easily prove that the transformed functions comply with the following basic mathematical operations. The following theorems can be easily proved:

Theorem 1. If $u(x)=g(x)\pm h(x)$, then $U(k)=G(k)\pm H(k)$.

Theorem 2. If $u(x)=\lambda g(x)$, then $U(k)=\lambda G(k)$, where, λ is a constant.

Theorem 3. If $u(x)=\frac{d^n g(x)}{dx^n}$, then $U(k)=\frac{(k+n)!}{k!}G(k+n)$.

Theorem 4. If $u(x)=x^m$, then $U(k)=\delta(k-m)$, where $\delta(k-m)=\begin{cases}1, & k=m,\\ o & otherwise.\end{cases}$

Theorem 5. If $u(x)=g(x)h(x)$, then

$$U(k)=\sum_{k_1=0}^{k} G(k_1)H(k-k_1).$$

Theorem 6. If $u(x)=f_1(x)f_2(x)\ldots f_n(x)$, then

$$U(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\ldots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}F_1(k_1)F_2(k_2-k_1)\ldots F_n(k-k_{n-1}).$$

Theorem 7. If $u(x)=\int_0^x g(x)dx$, then $U(k)=\frac{G(k-1)}{k}$. Where $k\geq 1$ and $U(0)=0$.

Theorem 8. If $u(x)=\int_0^x f_1(t)f_2(t)\ldots f_n(t)dt$, then

$$U(k)=\frac{1}{k}\sum_{k_{n-1}=0}^{k-1}\sum_{k_{n-2}=0}^{k_{n-1}}\ldots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}F_1(k_1)F_2(k_2-k_1)\ldots F_n(k-k_{n-1}-1).$$

Where $k\geq 1$ and $U(0)=0$.

Theorem 9. If $u(x)=g_1(x)g_2(x)\ldots g_n(x)\int_0^x f_1(t)f_2(t)\ldots f_m(t)dt$, then

$$U(k)=\sum_{k_{m+n-1}=1}^{k-1}\sum_{k_{m+n-2}=1}^{k_{n+m-1}}\ldots\sum_{k_2=1}^{k_3}\sum_{k_1=1}^{k_2}\frac{1}{k_m}F_1(k_1)F_2(k_2-k_1)\ldots F_m(k_m-k_{m-1})\times G_1(k_{m+1}-k_m)G_2(k_{m+2}-k_{m+1})\ldots G_n(k-k_{n+m-1}).$$

Where $k\geq 1$ and $U(0)=0$.

NUMERICAL EXAMPLE

To illustrate the ability and reliability of the method four examples were presented, here examples 2 and 3 were created by authors.

Example 1. Consider the following linear system of Volterra integral equations of the second kind

$$\begin{cases} f_1(t)+\int_0^t e^{-(s-t)}f_1(s)ds+\int_0^t \cos(s-t)f_2(s)ds \\ =\cosh t+t\sin t, \\ f_2(t)+\int_0^t e^{s+t}f_1(s)ds+\int_0^t t\cos s f_2(s)ds \\ =2\sin t+t\sin^2 t+te^t, \end{cases} \tag{5}$$

with the exact solutions $f_1(t)=e^{-t}$, $f_2(t)=2\sin t$, (Biazar and Ghazvini, 2009). One can readily find the differential transform of (5), as follows:

$$F_1(k)=-\sum_{k_1=1}^{k}\frac{1}{k_1}\frac{1}{(k-k_1)!}F_1(k_1-1)$$
$$-\frac{1}{k}\sum_{k_1=0}^{k-1}\frac{(-1)^{k_1}}{k_1!}F_1(k-k_1-1)$$
$$-\sum_{k_2=1}^{k}\sum_{k_1=1}^{k_2}\frac{1}{k_2}\left(\frac{1}{(k-1)!}\cos\frac{(k_1-1)\pi}{2}\right)$$
$$\times F_2(k_2-k_1)\frac{1}{(k-k_2)!}\cos\frac{(k-k_2)\pi}{2}$$
$$-\sum_{k_2=1}^{k}\sum_{k_1=1}^{k_2}\frac{1}{k_2}\left(\frac{1}{(k-1)!}\sin\frac{(k_1-1)\pi}{2}\right)$$
$$\times F_2(k_2-k_1)\frac{1}{(k-k_2)!}\sin\frac{(k-k_2)\pi}{2}+\frac{1+(-1)^k}{2k!}$$
$$+\sum_{k_1=0}^{k}\delta(k_1-1)\frac{1}{(k-k_1)!}\sin(\frac{\pi(k-k_1)}{2}),$$

$$F_2(k)=-\sum_{k_1=1}^{k}\frac{1}{k_1}\frac{1}{(k-k_1)!}F_1(k_1-1)$$
$$-\frac{1}{k}\sum_{k_1=0}^{k-1}\frac{(-1)^{k_1}}{k_1!}F_1(k-k_1-1)$$
$$-\sum_{k_2=1}^{k}\sum_{k_1=1}^{k_2}\frac{1}{k_2}\left(\frac{1}{(k-1)!}\cos\frac{(k_1-1)\pi}{2}\right)F_2(k_2-k_1)$$
$$\times\delta(k-k_2-1)+\frac{2}{k!}\sin(\frac{k\pi}{2})$$
$$+\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\delta(k_1-1)\frac{1}{(k_2-k_1)!}\sin(\frac{\pi(k_2-k_1)}{2})$$
$$\times\frac{1}{(k-k_1)!}\sin\left(\frac{\pi(k-k_1)}{2}\right)+\sum_{k_1=0}^{k}\delta(k_1-1)\frac{1}{(k-k_1)!}.$$

Consequently,

$$F_1(0)=1,F_1(1)=-1,F_1(2)=\frac{1}{2},F_1(3)=-\frac{1}{6},\ldots$$

$$F_2(0)=0,F_2(1)=2,F_2(2)=0,F_2(3)=-\frac{1}{3},\ldots$$

Closed form of the solution in this example are as the following:

$$f_1(t)=\sum_{k=0}^{\infty}F(k)t^k=e^{-t},f_2(t)=\sum_{k=0}^{\infty}G(k)t^k=2\sin t.$$ In this example, we have derived an exact solution.

Example 2. Consider the following non-linear system of Volterra integral equations of the second kind.

$$\begin{cases} f_1(t)=\sin t-t+\int_0^t(f_1^2(s)+f_2^2(s))ds,\\ f_2(t)=\cos t-\frac{1}{2}\sin^2 t+\int_0^t f_1(s)f_2(s)ds, \end{cases}\qquad(6)$$

with the exact solutions, $f_1(t)=\sin t$, $f_2(t)=\cos t$. Taking the differential transform, leads to

$$F_1(k)=\frac{1}{k!}\sin\frac{k\pi}{2}-\delta(k-1)$$
$$+\frac{1}{k}\sum_{k_1=0}^{k-1}F_1(k_1)F_1(k-k_1-1)+\frac{1}{k}\sum_{k_1=0}^{k-1}F_2(k_1)F_2(k-k_1-1),$$
$$F_2(k)=\frac{1}{k!}\cos\frac{k\pi}{2}-\frac{1}{2}\sum_{k_1=0}^{k}\frac{1}{k!}\sin(\frac{k_1\pi}{2})\frac{1}{(k-k_1)!}\sin(\frac{(k-k_1)\pi}{2})$$
$$+\frac{1}{k}\sum_{k_1=0}^{k-1}F_1(k_1)F_2(k-k_1-1).$$

According to these equations, a few first coefficients are as follows:

$$F_1(0)=0,F_1(1)=1,F_1(2)=0,F_1(3)=-\frac{1}{6},\ldots$$
$$F_2(0)=1,F_2(1)=-\frac{1}{2},\ F_2(2)=0,F_2(3)=\frac{1}{4!},\ldots$$

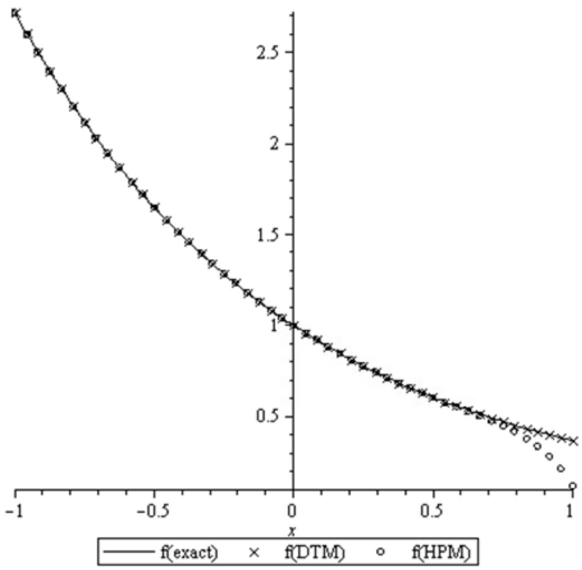
From (4), the solutions of the system of integral Equations (6), can be stated as follows

$$f_1(t)=\sum_{k=0}^{\infty}F(k)t^k=\sin t,f_2(t)=\sum_{k=0}^{\infty}G(k)t^k=\cos t.$$

which is an exact solution.

Example 3. Consider the following non-linear system of Volterra integral equations of the second kind

$$\begin{cases} 5f(t)+(\frac{1}{2}t^2+t)g(t)h(t)+\int_0^t(f(s)+tg(s)h(s))ds\\ =\frac{1}{8}t^5+\frac{19}{54}t^4+\frac{19}{18}t^3+\frac{7}{2}t^2+2t+5,\\ (\frac{1}{2}t^2+t)f(t)+3g(t)+\int_0^t(tf(s)+g(s)+\frac{1}{2}th(s))ds\\ =\frac{5}{24}t^4+\frac{35}{72}t^3+\frac{17}{6}t^2+\frac{5}{2}t+\frac{9}{2},\\ tf(t)g(t)-t^2g^2(t)-5h(t)-\int_0^t(sg^2(s))ds\\ =-\frac{7}{54}t^6+\frac{1}{12}t^5-\frac{5}{4}t^4+\frac{17}{24}t^3-\frac{27}{8}t^2-t-\frac{10}{3}. \end{cases}\qquad(7)$$



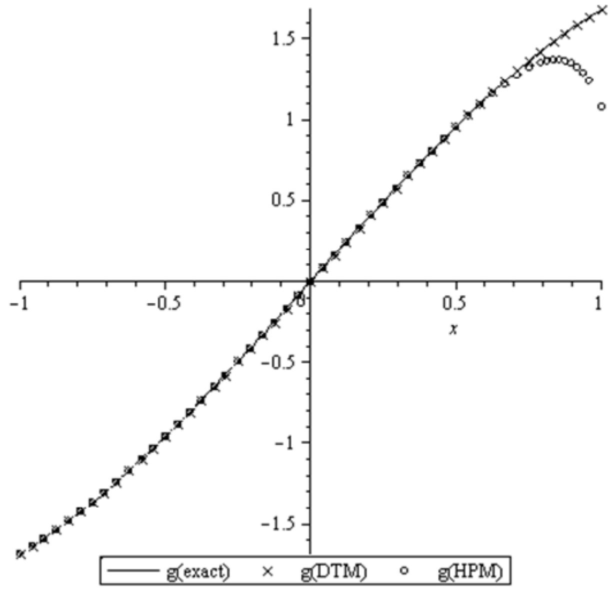


Figure 1. DTM, HPM, and exact solutions of Example 1.

with the exact solutions, $f(t)=\frac{1}{4}t^2+1$, $g(t)=\frac{1}{3}t^2+\frac{3}{2},\ and\ h(t)=\frac{1}{2}t+\frac{2}{3}$. Taking the differential transform, reads to

$$5F(k)+\frac{1}{2}\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\delta(k_1-2)G(k_2-k_1)H(k-k_2)$$
$$+\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\delta(k_1-1)G(k_2-k_1)H(k-k_2)+\frac{F(k-1)}{k}$$
$$+\sum_{k_2=1}^{k-1}\sum_{k_1=1}^{k_2}\frac{1}{k_2}G(k_1)H(k_2-k_1)\delta(k-k_2-1)$$
$$=\frac{1}{8}\delta(k-5)+\frac{19}{54}\delta(k-4)+\frac{19}{18}\delta(k-3)$$
$$+\frac{7}{2}\delta(k-2)+2\delta(k-1)+5\delta(k),$$

$$\frac{1}{2}\sum_{k_1=0}^{k}\delta(k_1-2)F(k-k_1)+\sum_{k_1=0}^{k}\delta(k_1-1)F(k-k_1)$$
$$+3G(k)+\sum_{k_1=1}^{k-1}\frac{1}{k_1}F(k_1)\delta(k-k_1-1)+\frac{G(k-1)}{k}$$
$$+\frac{1}{2}\sum_{k_1=1}^{k-1}\frac{1}{k_1}H(k_1)\delta(k-k_1-1)$$
$$=\frac{5}{24}\delta(k-4)+\frac{35}{72}\delta(k-3)+\frac{17}{6}\delta(k-2)$$
$$+\frac{5}{2}\delta(k-1)+\frac{9}{2}\delta(k),$$

$$\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\delta(k_1-1)F(k_2-k_1)G(k-k_2)$$
$$-\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\delta(k_1-2)G(k_2-k_1)G(k-k_2)-5H(k)$$
$$-\frac{1}{k}\sum_{k_2=0}^{k-1}\sum_{k_1=0}^{k_2}\delta(k_1-1)G(k_2-k_1)G(k-k_2-1)$$
$$=-\frac{7}{54}\delta(k-6)+\frac{1}{12}\delta(k-5)+\frac{5}{4}\delta(k-4)+\frac{17}{24}\delta(k-3)$$
$$-\frac{27}{8}\delta(k-2)-\delta(k-1)-\frac{10}{3}\delta(k).$$

The following results can be extracted from these equations

$$F(0)=1,\ F(1)=0,\ F(2)=\frac{1}{4},F(3)=0,\ldots$$

$$G(0)=\frac{3}{2},G(1)=0,G(2)=\frac{1}{3},\ G(3)=0,\ldots$$

$$H(0)=\frac{2}{3},\ H(1)=\frac{1}{2},\ H(2)=0,H(3)=0,\ldots$$

Considering (4), the solutions of the system of integral Equations (8), can be presented as follows:

$$f(t)=\sum_{k=0}^{\infty}F(k)t^k=\frac{1}{4}t^2+1,\ g(t)=\sum_{k=0}^{\infty}G(k)t^k=\frac{1}{3}t^2+\frac{3}{2},$$
$$h(t)=\sum_{k=0}^{\infty}H(k)t^k=\frac{1}{2}t+\frac{2}{3}.$$

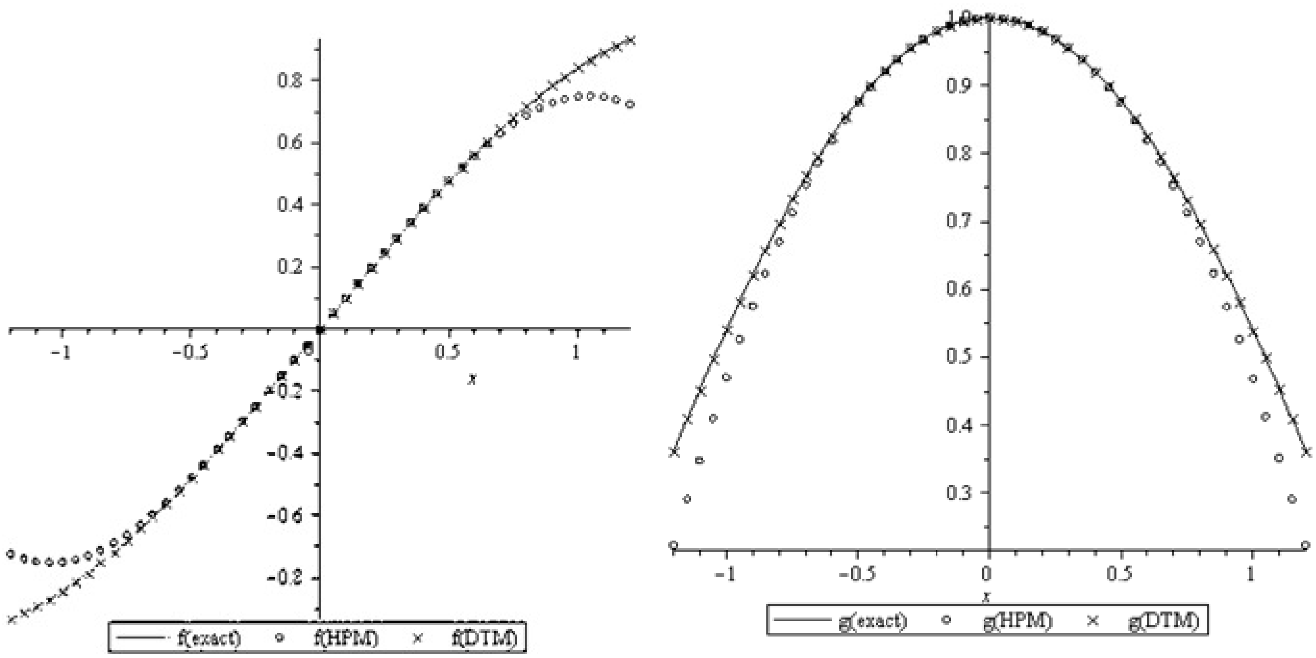


Figure 2. DTM, HPM, and exact solutions of Example 2.

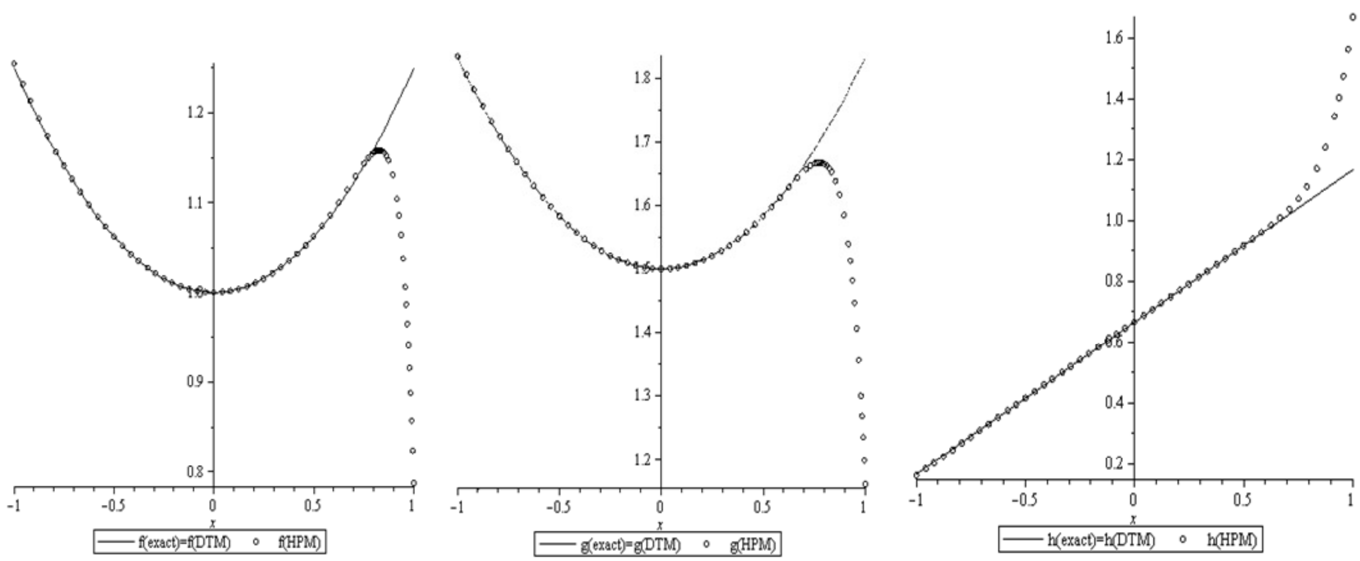


Figure 3. DTM, HPM, and exact solutions of Example 3.

In this example, an exact solution has been resulted.

CONCLUSION

In this paper, differential transform method has been used successfully for finding the solutions of linear and non-linear systems of Volterra integral equations of the second kind. To illustrate the method four examples, one linear and three others, non-linear have been presented. In these examples, DTM leads to exact solutions. But for more reasonable comparison DTM and HPM, five term approximations of both methods have been considered. These results are plotted in Figures 1 to 3, as well as the exact solutions. It can be concluded that, differential transform method is a very powerful and efficient technique, for finding exact solutions for a wide class of problems. The computations associated with the examples in this paper were performed using Maple 13.

REFERENCES

Arikhoglu A, Ozkol I (2005). Solution of boundary value problems for integro-differential equations by using differential transform method, Appl. Math. Comput., 168 :1145–1158.
Arikhoglu A, Ozkol I (2006). Solution of difference equations by using differential transform method, Appl. Math. Comput., 1216-1228.
Arikhoglu A, Ozkol I (2007). Solution of fractional differential equations by using differential transform method, Chaos Solitons Fractals. 34: 1473-1481.
Ayaz F (2003). On the two-dimensional differential transform method, Appl. Math. Comput., 143:361-374.
Biazar J, Ghazvini H (2009). He's homotopy perturbation method for solving systems of Volterra integral equations of the second kind, Chaos Solitons Fractals, 2: 770-777.
Chen CK, Ho SH (1996). Application of differential transformation to eigenvalue problems, Appl. Math. Comput., 79 : 173-18.
Chen CK, Ho SH (1999). Solving partial differential by two-dimensional differential transform, Appl. Math. Comput., 106 :171-179.
Hassan IH, Abdel-Halim (2008). Comparison differential transformation technique with Adomian decomposition method for linear and nonlinear initial value problems, Choas Solitons Fractals, 36(1): 53-65.
Zhou JK (1986). Differential Transformation and Its Application for Electrical Circuits, Huazhong University Press, Wuhan, China, (in Chinese).