International Journal of Physical Sciences Vol. 7(43), pp. 5723-5728, 16 November, 2012
Available online at http://www.academicjournals.org/IJPS
DOI: 10.5897/IJPS12.378
ISSN 1992 -1950 ©2012 Academic Journals

Full Length Research Paper

Chebyshev-Tau method for the linear Klein-Gordon equation

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Accepted 17 July, 2012

In this paper, we study a numerical method based on polynomial approximation, using the shifted Chebyshev polynomial, to construct the approximate solutions of the one dimensional linear Klein-Gordon equation with constant coefficients. Also, we give general forms of the operational matrices of integral and derivative. We solve two illustrative examples to test this method known as the shifted Chebyshev Tau method. By using this method, the problem is reduced to a set of linear algebraic equations by the operational matrices of integral and derivative. Then, solving the systems, we obtain the exact solutions to these problems. It is shown that the method produces accurate results.

Key words: Chebyshev polynomials, Tau method, shifted Chebyshev series, Klein-Gordon equation.

INTRODUCTION

The linear or nonlinear partial differential equations arising in physics and engineering play an important role in mathematical modelling. The hyperbolic partial differential equations model the vibrations of structures. So, searching the numerical and exact solutions to these linear or nonlinear models gains importance. For this reason, many methods were developed for solving differential equations in literature (Dehghan et al., 2009; Dehghan and Shokri, 2009; Dehghan and Ghesmati, 2010; Lakestani and Dehghan, 2010; Shakeri and Dehghan, 2008). In this research, we use shifted Chebyshev-Tau method defined in the paper (Saadatmandi and Dehghan, 2010), for the linear Klein-Gordon equation. Also, several applications for linear partial differential equations were considered by Bülbül and Sezer (2011a, b), Canuto et al. (1988), Horng and Chou (1985) and Saadatmandi and Dehghan (2008, 2010).

In this paper, we take into account the following problem

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+f(x,t)=0,\quad 0<x<\ell,\ 0<t<\tau, \tag{1}$$

with initial conditions given by

$$u(x,0)=g_1(x),\quad 0<x<\ell, \tag{2}$$

$$u_t(x,0)=g_2(x),\quad 0<x<\ell, \tag{3}$$

and Dirichlet boundary conditions

$$u(0,t)=h_1(t),\quad 0<t\le\tau, \tag{4}$$

$$u(l,t)=h_2(t),\quad 0<t\le\tau, \tag{5}$$

where f, g_1, g_2, h_1 and h_2 are known functions and u is unknown function.

The linear Klein-Gordon equation has been recently studied by some authors. In their study, they used the double integral transform method to solve the hyperbolic

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partial differential equation or the linear Klein-Gordon equation (Kiliçman and Eltayeb, 2010). Also, Bülbül and Sezer (2011b) applied the Taylor polynomial approximation method to the Klein-Gordon equation. Other studies by Dehghan et al. (2009), Dehghan and Shokri (2009), Dehghan and Ghesmati (2010) and Lakestani and Dehghan (2010) developed various numerical methods for the solution of the nonlinear Klein-Gordon equations.

OPERATIONAL PROPERTIES OF SHIFTED CHEBYSHEV POLYNOMIALS

The Chebyshev polynomial $T_n(\bar{x})$ is defined as follows (Szego, 1975):

$$T_n(\bar{x}) = \cos\left(n\cos^{-1}(\bar{x})\right), \quad -1 \le \bar{x} \le 1, \tag{6}$$

where the independent variable $\bar{x}$ is defined between −1 and 1. To solve the problem (1) to (5), the domain must be transformed into values between 0 and h. Let

$$x = \frac{h}{2}(1+\bar{x}). \tag{7}$$

Then, the shifted Chebyshev polynomials of x can be defined as follows:

$$T_0^h(x) = 1, \quad T_1^h(x) = \frac{2x}{h} - 1,$$
$$T_{i+1}^h(x) = \left(2 - \frac{4x}{h}\right)T_i^h(x) - T_{i-1}^h(x), \quad i = 1, 2, \ldots \tag{8}$$

The orthogonality condition of the shifted Chebyshev polynomials is given by

$$\int_0^h \frac{T_i^h(x)T_j^h(x)}{\sqrt{hx-x^2}}dx = \begin{cases} 0, & i \ne j \\ \dfrac{\pi}{2}, & i = j \ne 0 \\ \pi, & i = j = 0. \end{cases} \tag{9}$$

A function $y(x)$ can be approximated by shifted Chebyshev polynomials as

$$y_m(x) = \sum_{j=0}^{m-1} c_j T_j^h(x) = C^T \Phi_{m,h}(x), \tag{10}$$

where the superscript T denotes transpose, C is shifted Chebyshev coefficients vector and $\Phi_{m,h}(x)$ is the shifted Chebyshev polynomials vectors. These vectors are given as:

$$C = [c_0, c_1, \ldots, c_{m-1}]^T, \tag{11}$$

and

$$\Phi_{m,h}(x) = \left[T_0^h(x), T_1^h(x), \ldots, T_{m-1}^h(x)\right]^T. \tag{12}$$

On the other hand, a function $u(x,t)$ defined for $0 < x < \ell$ and $0 < t \le \tau$ can be expanded in terms of double shifted Chebyshev polynomials as follows:

$$u_{n,m}(x,t) = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} a_{ij} T_i^\tau(t) T_j^\ell(x) = \Phi_{n,\tau}^T(t) A \Phi_{m,\ell}(x) \tag{13}$$

where the shifted Chebyshev vectors $\Phi_{m,\ell}(x)$ and $\Phi_{n,\tau}(t)$ are defined similarly to Equation (12). Also the shifted Chebyshev coefficient matrix **A** can be given by

$$A = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0,m-1} \\ a_{10} & a_{11} & \cdots & a_{1,m-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,m-1} \end{pmatrix}. \tag{14}$$

The exact definition of operational matrix was first defined by Eslahchi and Dehghan (2011). Also, the general operational matrix of integration for the shifted Chebyshev polynomials can be written as (Saadatmandi and Dehghan, 2010; Horng and Chou, 1985)

$$\int_0^t\int_0^t \cdots \int_0^t \Phi_{n,\tau}(t)(dt)^k = P^k \Phi_{n,\tau}(t). \tag{15}$$

where **P** is the *n* × *n* operational matrix of integration given by

$$P = \tau\begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & 0 & \ldots & 0 & 0 & 0 \\ \frac{1}{8} & 0 & \frac{-1}{8} & \ldots & 0 & 0 & 0 \\ \frac{-1}{6} & \frac{1}{4} & 0 & \ldots & 0 & 0 & 0 \\ \frac{-1}{16} & 0 & \frac{1}{8} & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{-1}{2(n-1)(n-3)} & 0 & 0 & \ldots & \frac{1}{4(n-3)} & 0 & \frac{-1}{4(n-1)} \\ \frac{-1}{2n(n-2)} & 0 & 0 & \ldots & 0 & \frac{1}{4(n-2)} & 0 \end{pmatrix}. \tag{16}$$

Some properties of the shifted Chebyshev polynomial can be given as (Saadatmandi and Dehghan, 2010; Horng and Chou, 1985)

$$\frac{dT_{2k+1}^\ell(x)}{dx} = \left(-\frac{4}{\ell}\right)(2k+1)\left[T_{2k}^\ell(x) + T_{2k-2}^\ell(x) + \ldots + T_2^\ell(x) + \frac{T_0^\ell(x)}{2}\right], \tag{17}$$

and

$$\frac{dT_{2k}^{\ell}(x)}{dx}=\left(-\frac{8k}{\ell}\right)\left[T_{2k-1}^{\ell}(x)+T_{2k-3}^{\ell}(x)+...+T_{1}^{\ell}(x)\right], \quad (18)$$

where k is an integer. From Equations (17) and (18), we obtain the general derivative of the vector $\Phi_{m,\ell}(x)$ as follows:

$$\frac{d^k\Phi_{m,\ell}(x)}{dx^k}=\boldsymbol{D}^k\Phi_{m,\ell}(x), \quad (19)$$

where **D** is the *m* × *m* operational matrix of differentiation given as:

$$\boldsymbol{D}=\frac{1}{\ell}\begin{pmatrix} 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ -2 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & -8 & 0 & 0 & \dots & \dots & 0 & 0 \\ -6 & 0 & -12 & 0 & \dots & \dots & 0 & 0 \\ 0 & -16 & 0 & -16 & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ (m-2)\left[(-1)^{m-2}-1\right] & -2(m-2)\left[(-1)^{m-2}+1\right] & 2(m-2)\left[(-1)^{m-2}-1\right] & -2(m-2)\left[(-1)^{m-2}+1\right] & \dots & \dots & 0 & 0 \\ (m-2)\left[(-1)^{m-1}-1\right] & -2(m-1)\left[(-1)^{m-1}+1\right] & 2(m-1)\left[(-1)^{m-1}-1\right] & -2(m-1)\left[(-1)^{m-1}+1\right] & \dots & \dots & -4(m-1) & 0 \end{pmatrix}.$$

Lemma 1

Let $g_1(x)$ and $g_2(x)$ be approximated by shifted Chebyshev polynomials as

$$g_{1m}(x)=\sum_{i=0}^{m-1}g_{1i}T_i^{\ell}(x), \quad g_{2m}(x)=\sum_{i=0}^{m-1}g_{2i}T_i^{\ell}(x). \quad (20)$$

Then $g_1(x)$ and $tg_2(x)$ for $0\le x\le \ell$ and $0\le t\le \tau$ can be approximated as

$$g_{1m}(x)=\Phi_{n,\tau}^{T}(t)E\Phi_{m,\ell}(x), \quad tg_{2m}(x)=\Phi_{n,\tau}^{T}(t)H\Phi_{m,\ell}(x), \quad (21)$$

where

$$E=\begin{pmatrix} g_{10} & g_{11} & \cdots & g_{1,m-1} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad (22)$$

and

$$H=\frac{\tau}{2}\begin{pmatrix} g_{20} & g_{21} & \cdots & g_{2,m-1} \\ -g_{20} & -g_{21} & \dots & -g_{2,m-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}. \quad (23)$$

Proof

We first approximate $g_1(x)$ and $g_2(x)$ given in Equations (2) and (3) by the shifted Chebyshev polynomials. Then $1.g_1(x)$ and $tg_2(x)$ can be written in the matrix form. From the point of view, 1 and t can be easily expanded as

$$1=T_0^{\tau}(t), \quad t=\frac{\tau}{2}T_0^{\tau}(t)-\frac{\tau}{2}T_1^{\tau}(t), \quad (24)$$

which leads to

$$g_{1m}(x)=T_0^{\tau}(t)\sum_{i=0}^{m-1}g_{1i}T_i^{\ell}(x), \quad (25)$$

and

$$tg_{2m}(x)=\left(\frac{\tau}{2}T_0^{\tau}(t)-\frac{\tau}{2}T_1^{\tau}(t)\right)\sum_{i=0}^{m-1}g_{2i}T_i^{\ell}(x). \quad (26)$$

We suppose that Equations (25) and (26) can be written in the form

$$g_{1m}(x)=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}e_{ij}T_i^{\tau}(t)T_j^{\ell}(x),$$
$$tg_{2m}(x)=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}h_{ij}T_i^{\tau}(t)T_j^{\ell}(x). \quad (27)$$

Equating Equations (25) and (26) to Equation (27), we find

$e_{0j}=g_{1j}$, $e_{ij}=0$, $i=1,2,...,n-1$, $j=0,1,...,m-1$,
and
$h_{0j}=\frac{\tau}{2}g_{2j}$, $h_{1j}=-\frac{\tau}{2}g_{2j}$, $h_{ij}=0$, $i=2,3,...,n-1$, $j=0,1,...,m-1$.

Using these results, we obtain Equations (22) and (23). Therefore, the Proof is completed.

PROCEDURE OF SOLUTION

Integrating Equation (1) from 0 to t and using Equations (2) and (3), we obtain:

$$u(x,t)-g_1(x)-tg_2(x)-\int_0^t\int_0^t u_{xx}(x,t)\,dtdt+\int_0^t\int_0^t f(x,t)\,dtdt=0 \quad (28)$$

Similar to Equation (13), we approximate $f(x,t)$ as:

$$f_{n,m}(x,t)=\Phi^T_{n,\tau}(t)F\Phi_{m,\ell}(x) \quad (29)$$

where **F** is a *n* × *m* known matrix. From Equations (13) and (15) we get:

$$\int_0^t\int_0^t\cdots\int_0^t u_{n,m}(x,t)(dt)^k=\Phi^T_{n,\tau}(t)\left(P^T\right)^k A\Phi_{m,\ell}(x) \quad (30)$$

Similar to Equation (30), we obtain the general integration form of $f(x,t)$ as:

$$\int_0^t\int_0^t\ldots\int_0^t f_{n,m}(x,t)(dt)^k=\Phi^T_{n,\tau}(t)\left(P^T\right)^k F\Phi_{m,\ell}(x) \quad (31)$$

Also using Equations (13), (15) and (19) we obtain

$$\int_0^t\int_0^t\cdots\int_0^t \frac{\partial^p u_{n,m}(x,t)}{\partial x^p}(dt)^k=\Phi^T_{n,\tau}(t)\left(P^T\right)^k AD^p\Phi_{m,\ell}(x). \quad (32)$$

Applying (13), (21), (29)-(32) the residual $R_{n,m}(x,t)$ for Equation (28) can be given by:

$$R_{n,m}(x,t)=\Phi^T_{n,\tau}(t)\left[A-E-H-\alpha\left(P^T\right)^2AD^2+\left(P^T\right)^2F\right]\Phi_{m,\ell}(x)$$
$$=\Phi^T_{n,\tau}(t)Q\,\Phi_{m,\ell}(x),$$

where

$$Q=A-E-H-\left(P^T\right)^2AD^2+\left(P^T\right)^2F$$

According to standard Tau method (Canuto et al., 1988), we generate *n* × *(m* − 2*)* linear algebraic equations by using the following system:

$$Q_{i,j}=0,\quad i=0,1,\ldots,n-1,\quad j=0,1,\ldots,m-3 \quad (33)$$

Substituting Equation (13) in Equations (4) and (5), we have

$$\Phi^T_{n,\tau}(t)A\Phi_{m,\ell}(0)=h_1(t) \quad (34)$$

$$\Phi^T_{n,\tau}(t)A\Phi_{m,\ell}(\ell)=h_2(t) \quad (35)$$

Equations (34) and (35) are collocated at *n* points. We use the shifted Chebyshev roots t_i, $i=1,\cdots,n$ of $T^\tau_n(t)$ for these collocation points. The number of the unknown coefficients a_{ij} equals to $n\times m$ and can be computed from Equations (33) to (35). Hence, $u_{n,m}(x,t)$ given in Equation (13) can be determined.

NUMERICAL EXAMPLES

In this section, we apply the method described in the previous section and give some computational results of numerical experiments.

Example 1

Consider (1) to (5) with $\ell=1$, $\tau=1$ (Bülbül and Sezer, 2011b)

$g_1(x)=0$, $g_2(x)=0$, $f(x,t)=t^2-x^2$, $h_1(t)=0$, $h_2(t)=\frac{1}{2}t^2$.

Applying the method with $m=n=3$, we obtain the approximation for this problem as follows:

$$u_{3,3}(x,t)=\sum_{i=0}^{2}\sum_{j=0}^{2}a_{ij}T^1_i(t)T^1_j(x)=\Phi^T_{3,1}(t)A\Phi_{3,1}(x)$$

From here, we can write:

$$A=\begin{pmatrix}a_{00}&a_{01}&a_{02}\\a_{10}&a_{11}&a_{12}\\a_{20}&a_{21}&a_{22}\end{pmatrix},\ P=\begin{pmatrix}\frac{1}{2}&\frac{-1}{2}&0\\\frac{1}{8}&0&\frac{-1}{8}\\\frac{-1}{6}&\frac{1}{4}&0\end{pmatrix},\ D=\begin{pmatrix}0&0&0\\-2&0&0\\0&-8&0\end{pmatrix},$$

$$E=H=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},\ F=\begin{pmatrix}0&\frac{1}{2}&\frac{-1}{8}\\\frac{-1}{2}&0&0\\\frac{1}{8}&0&0\end{pmatrix}.$$

Using Equations (33) to (35), we find nine equations as follows:

$$6a_{00}-18a_{02}-8a_{12}+5a_{22}-\frac{37}{128}=0,$$

$$6a_{10}+24a_{02}+9a_{12}-8a_{22}+\frac{11}{32}=0,$$

$$2a_{20}-2a_{02}+a_{22}-\frac{1}{128}=0,$$

$$a_{00}+a_{01}+a_{02}+a_{10}+a_{11}+a_{12}+a_{20}+a_{21}+a_{22}=0,$$

$$a_{00}-a_{01}+a_{02}+a_{10}-a_{11}+a_{12}+a_{20}-a_{21}+a_{22}=0,$$

$$a_{10}+a_{11}+a_{12}+4a_{20}+4a_{21}+4a_{22}=0,$$

$$a_{10}-a_{11}+a_{12}+4a_{20}-4a_{21}+4a_{22}=0,$$

$$a_{20}+a_{21}+a_{22}=0,$$

$$a_{20}-a_{21}+a_{22}-\frac{1}{16}=0.$$

Solving this system of algebraic equations, we get:

$$A=\begin{pmatrix}\frac{9}{128} & \frac{-3}{32} & \frac{3}{128}\\ \frac{-3}{32} & \frac{1}{8} & \frac{-1}{32}\\ \frac{3}{128} & \frac{-1}{32} & \frac{1}{128}\end{pmatrix}.$$

Therefore, we obtain the exact solution:

$$u_{3,3}(x,t)=\begin{pmatrix}1 & -2t+1 & 8t^2-8t+1\end{pmatrix}\begin{pmatrix}\frac{9}{128} & \frac{-3}{32} & \frac{3}{128}\\ \frac{-3}{32} & \frac{1}{8} & \frac{-1}{32}\\ \frac{3}{128} & \frac{-1}{32} & \frac{1}{128}\end{pmatrix}\begin{pmatrix}1\\ -2x+1\\ 8x^2-8x+1\end{pmatrix}=\frac{1}{2}x^2t^2.$$

Example 2

Consider (1) to (5) with $\ell=1$, $\tau=1$ (Bülbül and Sezer, 2011b)

$$g_1(x)=x^2,\quad g_2(x)=4x^3,\quad f(x,t)=0,\quad h_1(t)=t^2,$$
$$h_2(t)=4t^3+t^2+4t+1.$$

Applying the method with $m=n=4$, we obtain the approximation for this problem as follows:

$$u_{4,4}(x,t)=\sum_{i=0}^{3}\sum_{j=0}^{3}a_{ij}T_i^1(t)T_j^1(x)=\Phi_{4,1}^T(t)A\Phi_{4,1}(x).$$

From here, we can write:

$$A=\begin{pmatrix}a_{00} & a_{01} & a_{02} & a_{03}\\ a_{10} & a_{11} & a_{12} & a_{13}\\ a_{20} & a_{21} & a_{22} & a_{23}\\ a_{30} & a_{31} & a_{32} & a_{33}\end{pmatrix},\ P=\begin{pmatrix}\frac{1}{2} & \frac{-1}{2} & 0 & 0\\ \frac{1}{8} & 0 & \frac{-1}{8} & 0\\ \frac{-1}{6} & \frac{1}{4} & 0 & \frac{-1}{12}\\ \frac{-1}{16} & 0 & \frac{1}{8} & 0\end{pmatrix},\ D=\begin{pmatrix}0 & 0 & 0 & 0\\ -2 & 0 & 0 & 0\\ 0 & -8 & 0 & 0\\ -6 & 0 & -12 & 0\end{pmatrix},$$

$$E=\begin{pmatrix}\frac{3}{8} & \frac{-1}{2} & \frac{1}{8} & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\ H=\begin{pmatrix}\frac{5}{8} & \frac{-15}{16} & \frac{3}{8} & \frac{-1}{16}\\ \frac{-5}{8} & \frac{15}{16} & \frac{-3}{8} & \frac{1}{16}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\ F=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Using Equations (33) to (35), we obtain the system of algebraic equations as follows:

$$a_{00}+a_{01}+a_{02}+a_{03}+a_{10}+a_{11}+a_{12}+a_{13}+a_{20}+a_{21}+a_{22}+a_{23}+a_{30}+a_{31}+a_{32}+a_{33}=0,$$

$$a_{10}+a_{11}+a_{12}+a_{13}+4a_{20}+4a_{21}+4a_{22}+4a_{23}+9a_{30}+9a_{31}+9a_{32}+9a_{33}=0,$$

$$a_{20}+a_{21}+a_{22}+a_{23}+6a_{30}+6a_{31}+6a_{32}+6a_{33}=\frac{1}{8},$$

$$a_{30}+a_{31}+a_{32}+a_{33}=0,$$

$$a_{00}-a_{01}+a_{02}-a_{03}+a_{10}-a_{11}+a_{12}-a_{13}+a_{20}-a_{21}+a_{22}-a_{23}+a_{30}-a_{31}+a_{32}-a_{33}=1,$$

$$a_{10}-a_{11}+a_{12}-a_{13}+4a_{20}-4a_{21}+4a_{22}-4a_{23}+9a_{30}-9a_{31}+9a_{32}-9a_{33}=-2,$$

$$a_{20}-a_{21}+a_{22}-a_{23}+6a_{30}-6a_{31}+6a_{32}-6a_{33}=\frac{1}{8},$$

$$a_{30}-a_{31}+a_{32}-a_{33}=\frac{-1}{8},$$

$$a_{00}-3a_{02}-\frac{4}{3}a_{12}+\frac{3}{4}a_{22}+\frac{5}{6}a_{32}=1,$$

$$a_{10}+4a_{02}+\frac{3}{2}a_{12}-\frac{4}{3}a_{22}-a_{32}=\frac{-5}{8},$$

$$a_{01}-18a_{03}-8a_{13}+\frac{9}{2}a_{23}+5a_{33}=\frac{-23}{16},$$

$$a_{11}+24a_{03}+9a_{13}-8a_{23}-6a_{33}=\frac{15}{16},$$

$$a_{20}-a_{02}+\frac{2}{3}a_{22}=0,$$

$$a_{30}-\frac{1}{6}a_{12}+\frac{1}{6}a_{32}=0,$$

$a_{21} - 6a_{03} + 4a_{23} = 0$,

$a_{31} - a_{13} + a_{33} = 0$.

Solving this system, we find

$$A = \begin{pmatrix} 2 & \frac{-33}{16} & \frac{1}{2} & \frac{-1}{16} \\ \frac{-33}{16} & \frac{15}{8} & \frac{-3}{8} & \frac{1}{16} \\ \frac{1}{2} & \frac{-3}{8} & 0 & 0 \\ \frac{-1}{16} & \frac{1}{16} & 0 & 0 \end{pmatrix}.$$

Therefore, we obtain the exact solution

$$u_{4,4}(x,t) = \begin{pmatrix} 1 \\ -2t+1 \\ 8t^2-8t+1 \\ -32t^3+48t^2-18t+1 \end{pmatrix}^T \begin{pmatrix} 2 & \frac{-33}{16} & \frac{1}{2} & \frac{-1}{16} \\ \frac{-33}{16} & \frac{15}{8} & \frac{-3}{8} & \frac{1}{16} \\ \frac{1}{2} & \frac{-3}{8} & 0 & 0 \\ \frac{-1}{16} & \frac{1}{16} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2x+1 \\ 8x^2-8x+1 \\ -32x^3+48x^2-18x+1 \end{pmatrix} = x^2+t^2+4x^3t+4xt^3.$$

Conclusıon

In this research, we apply the Chebyshev-Tau method for solving one-dimensional linear Klein-Gordon equation with constant coefficients. These problems are difficult to solve analytically. So, Chebyshev-Tau method is used for analytical solutions or approximate solutions. We can say that this method is simple and suitable for computer programming and for providing solution to the problem. If the known functions in the problem and the solution are polynomials, then this method gives the exact solution. Moreover, Chebyshev-Tau method is applicable for the approximate solution of other type partial differential equations. This method can also be extended to nonlinear partial differential equations.

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