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Minimal and maximal bi-ideals in ordered ternary semigroups

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In this paper, some characterizations of minimal and maximal bi-ideals of ordered ternary semigroups were given. Moreover, the results of minimal and maximal bi-ideals of ordered ternary semigroups to ordered n-ary semigroups were generalized.

Key words: Bi-ideals, minimal bi-ideals, maximal bi-ideals, ordered ternary semigroups, ordered n-ary semigroups.

INTRODUCTION AND PRELIMINARIES

Algebraic structures play a prominent role in mathematics with wide applications in many fields, such as physics, engineering, computer sciences, information sciences, etc. Semigroups are universal algebra with one associative binary operation. In some respects, the theory of semigroups is similar to group theory and ring theory. Ideal theory in semigroups was widely studied. The notion of a bi-ideal of semigroups was first introduced by Good and Hughes (1952). Let S be a semigroup. A non-empty subset A of S is called a subsemigroup of S if $A^2 \subseteq A$. A subsemigroup B of S is called a bi-ideal of S if $BSB \subseteq B$. Bi-ideals in semigroups have been widely studied. Later, bi-ideals of ordered semigroups were studied by some authors (Kehayopulu, 1995; Kehayopulu et al., 2002; Xu and Ma, 2003; Iampan, 2007). Let $(S,\leq)$ be an partially ordered set. For any non-empty subset A of S, we let $(A] := \{t \in S \mid t \leq h$ for some $h \in A\}$. For any non-empty subsets A and B of S, we have $(A] \subseteq (B]$, if $A \subseteq B$ and $(A \cup B] = (A] \cup (B]$. A partially ordered semigroup $(S,\leq)$ is called an ordered semigroup if for all $a,b,x \in S$, $a \leq b \Rightarrow ax \leq bx$ and $xa \leq xb$.

Let S be an ordered semigroup. A non-empty subset A of S is called an ordered subsemigroup of S if $(A] \subseteq A$ and $A^2 \subseteq A$. An ordered subsemigroup B of S is called a bi-ideal of S if $(B] \subseteq B$ and $BSB \subseteq B$.

The theory of ternary algebraic systems was introduced by Lehmer (1932) who investigated certain algebraic systems called triplexes which turn out to be commutative ternary groups. But earlier such structure was studied by Kasner (1904) who give the idea of n-ary algebras. The notion of ternary semigroups was known to Banach (cf. Los, 1955). A non-empty set T with a ternary operation $T \times T \times T \rightarrow T$, written as $(x_1, x_2, x_3) \mapsto x_1x_2x_3$ is called a ternary semigroup if it satisfies the following associative law holds:

$$(x_1x_2x_3)x_4x_5 = x_1(x_2x_3x_4)x_5 = x_1x_2(x_3x_4x_5) \quad \text{for any } x_1, x_2, x_3, x_4, x_5 \in T.$$

Any semigroup can be reduced to a ternary semigroup. Banach showed that a ternary semigroup does not necessarily reduce to a semigroup by the following example.

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Example 1

$T=\{-i,0,i\}$ is a ternary semigroup, but T is not a semigroup under the multiplication. The following example is also an example of ternary semigroup but not semigroup.

Example 2

Z^- is a ternary semigroup, but T is not a semigroup under the multiplication where Z^- denote the set of all negative integers. However, Los (1955) proved that every ternary semigroup can be embedded in a semigroup.

Let T be a ternary semigroup. A non-empty subset S of T is called a ternary subsemigroup if $SSS \subseteq S$. Sioson (1965) studied the ideal theory in ternary semigroups, while, Dixit and Dewen (1995) studied quasi-ideals and bi-ideals ternary semigroup. A ternary subsemigroup B of T is called a bi-ideal of T if $BTBTB \subseteq B$. A ternary subsemigroup Q of T is called a quasi-ideal of T if $QTT \cap (TQT \cup TTQTT) \cap TTQ \subseteq Q$. Dixit and Dewan (1995) proved that every quasi-ideal of T is a bi-ideal of T, but the converse is not true in general by giving example.

A partially ordered ternary semigroup $(T,\leq)$ is called an ordered ternary semigroup if for all $a,b,x_1,x_2 \in T, a\leq b$ implies $ax_1x_2 \leq ax_1x_2$, $x_1ax_2 \leq x_1ax_2$ and $x_1x_2a \leq x_1x_2a$.

Example 3

(1) $\{-i,0,i\}$ is an ordered ternary semigroup under the multiplication and the partial order $\leq:=\{(0,0),(0,i),(0,-i),(-i,-i),(i,i)\}$.

(2) $(Z^-,\cdot,\leq)$ is an ordered ternary semigroup where $\cdot$ is the multiplication and the partial order $\leq$ is the less than or equal relation.

A non-empty subset S of T is called an ordered ternary subsemigroup of T if $(S] \subseteq S$ and $S^3 \subseteq S$. An ordered ternary subsemigroup B of T is called a *bi-ideal* of T if $(B] \subseteq B$ and $BTBTB \subseteq B$. An element z of T is called a *zero* if $zxy = xzy = xyz = z$ for all $x,y \in T$ and is denoted by 0.

Xu and Ma (2003) gave some results of minimal bi-ideals of ordered semigroups. Iampan (2007) gave some characterization of minimal and maximal (ordered) bi-ideals of ordered semigroups analogous to the characterizations of minimal and maximal left ideals of ordered semigroups considered by Cao and Xu (2000).

Subsequently, some characterizations of minimal and maximal bi-ideals in ordered ternary semigroups analogous to the characterizations of minimal and maximal bi-ideals in ordered semigroups were given (Iampan, 2007; Xu and Ma, 2003). Then, the results of minimal and maximal bi-ideals of ordered ternary semigroups to ordered n-ary semigroups were generalized.

Before the characterizations of minimal and maximal bi-ideals for the main theorem, some results were given, which are necessary in what follows.

Proposition 1

The intersection of arbitrary set of bi-ideals of an ordered ternary semigroup T is either empty or a bi-ideal of T.

Let T be an ordered ternary semigroup and A a non-empty subset of T. A bi-ideal $(A)_b$ denote the bi-ideal of T generated by A, that is $(A)_b$ is the smallest bi-ideal of T containing A. The following proposition holds.

Proposition 2

Let T be an ordered ternary semigroup and A a non-empty subset of T. Then, $(A)_b = (A \cup A^3 \cup ATATA]$.

So, for any $a \in T$, $(a)_b = (\{a,a^3\} \cup aTaTa]$.

Theorem 1

Let B be a bi-deal of an ordered ternary semigroup T. Then $(uBvBw]$ is a bi-ideal of T for every $u,v,w \in T$. In particular, $(uTvTw]$ is a bi-ideal of T for every $u,v,w \in T$.

Proof

Let $x,y,z \in (uBvBw]$ and $s,t \in T$. Then $x \leq c, y \leq d$ and $z \leq e$. For some $c,d,e \in uBvBw$. Then, there exist $a,b,m,n,k,l \in B$ such that $c = uavbw, d = umvnw$ and $e = ukvlw$. Then, $avb(wumvnwu)k \in BTBTB \subseteq B$ and $xyz \leq cde = (uavbw)(umvnw)(ukv/w) = u(avb(wumvnwu)k)v/w$. Thus, $xyz \in (uBvBw]$. This show that $(uBvBw]$ is an ordered ternary subsemigroup of T. Next, we have:

$$\begin{aligned} xsytz \leq csdte &= (uavbw)s(umvnw)t(ukvlw) \\ &= u(avb(wsu)m)v(n(wtu)kvl)w \end{aligned}$$

and

$avb(wsu)m, n(wtu)kvl \in BTBTB \subseteq B$, so, $xsytz \in (uBvBw]$. Thus, $(uBvBw]T(uBvBw]T(uBvBw] \subseteq (uBvBw]$. Therefore, $(uBvBw]$ is a bi-ideal of T, and consequently $(uTvTw]$ is a bi-ideal of T because T is a bi-ideal of itself.

Corollary 1

Let T be an ordered ternary semigroup, B a bi-ideal of T

and A a non-empty subset of T. Then, $(ABABA]$ is a bi-ideal of T. An ordered ternary semigroup T is called B-simple if T has no proper bi-ideals.

Proposition 3

Let T be an ordered ternary semigroup. The following statements are equivalent.

(1) T is B-simple.
(2) $(aTaTa] = T$ for all $a \in T$.
(3) $(a)_b = T$ for all $a \in T$.

Proof

Assume T is B-simple and let $a \in T$. Since by Theorem 1, $(aTaTa]$ is a bi-ideal of T, $(aTaTa] = T$. Hence, statement 1 implies statement 2. Since $(aTaTa] \subseteq (a)_b$, thus, statement 2 implies statement 3. Next, assume statement 3 holds. Let B be a bi-ideal of T and $a \in B$. Therefore, $T = (a)_b \subseteq B$, so $B = T$. Thus statement 3 implies statement 1.

Let T be an ordered ternary semigroup with zero $0, T^3 \neq \{0\}$ and $|T| > 1$. T is called *0-B-simple* if T has no nonzero proper bi-ideals. Note that if 0 is a minimum element of T, we have $\{0\}$ to be a bi-ideal of T.

Proposition 4

Let T be an ordered ternary semigroup with zero 0, $T^3 \neq \{0\}$ and $|T| > 1$. If 0 is a minimum element of T, then the following statements are equivalent.

(1) T is 0-B-simple.
(2) $(aTaTa] = T$ for all $a \in T \setminus \{0\}$.
(3) $(a)_b = T$ for all $a \in T \setminus \{0\}$.

Proof

Assume T is 0-B-simple and let $B = \{x \in T \mid (xTxTx] = \{0\}\}$. Thus, B is a bi-ideal of T. So $B = \{0\}$ or $B = T$. But if $B = T$, then $T^3 = \{0\}$, a contradiction. Hence, B = $\{0\}$. This implies $(aTaTa] \neq \{0\}$ for all $a \in T \setminus \{0\}$. Since by Theorem 1, $(aTaTa]$ is a bi-ideal of T, $(aTaTa] = T$ for all $a \in T \setminus \{0\}$. Hence statement 1 implies statement 2. Since $(aTaTa] \subseteq (a)_b$, thus statement 2 implies statement 3. Next, assume statement 3 holds. Let B be a nonzero bi-ideal of T and $a \in B \setminus \{0\}$. Hence, $T = (a)_b \subseteq B$, so $B = T$. Thus statement 3 implies statement 1.

Example 4

(1) The ordered ternary semigroup $\{-i, i\}$ under the multiplication and the partial order $\leq = \{(-i,-i),(i,i)\}$ is B-simple.
(2) The ordered ternary semigroup $\{0,-i,i\}$ under the multiplication and the partial order $\leq = \{(0,0),(0,i),(0,-i),(-i,-i),(i,i)\}$ is 0-B-simple.

MINIMAL BI-IDEALS IN ORDERED TERNARY SEMIGROUPS

The aim of minimal bi-ideals in ordered ternary semigroups is to give some characterizations of minimal bi-ideals and 0-minimal bi-ideals in ordered ternary semigroups. A bi-ideal B of an ordered ternary semigroup T is called a *minimal bi-ideal* of T if B does not properly contain any bi-ideal of T.

Theorem 2

Let T be an ordered ternary semigroup and B a bi-ideal of T. B is a minimal bi-ideal of T if and only if B is B-simple.

Proof

Assume B is a minimal bi-ideal of T and let A be a bi-ideal of B. Thus, $(ABABA] \subseteq A$. By Corollary 1, $(ABABA]$ is a bi-ideal of T. Since B is minimal and $(ABABA] \subseteq A \subseteq B$, $(ABABA] = A = B$. Hence, B is B-simple. Conversely, assume B is B-simple. Let A be a bi-ideal of T such that $A \subseteq B$. Then, A is a bi-ideal of B, this implies $A = B$. Hence, B is a minimal bi-ideal of T.

A nonzero bi-ideal B of an ordered ternary semigroup T with zero is called a 0-minimal bi-ideal of T if B does not properly contain any nonzero bi-ideal of T.

Theorem 3

Let T be an ordered ternary semigroup with zero 0 and B a nonzero bi-ideal of T such that $B^3 \neq \{0\}$. The following statements hold.

(1) If B is 0-B-simple, then B is a 0-minimal bi-ideal of T.
(2) If B is a 0-minimal bi-ideal of T and $(ABABA] \neq \{0\}$ for all nonzero bi-ideal A of B, then B is 0-B-simple.

Proof

1) Assume B is 0-B-simple. Let A be a nonzero bi-ideal of T such that $A \subseteq B$. So A is a nonzero bi-ideal of B, this

(implies $A = B$. Hence, B is a 0-minimal bi-ideal of T.
(2) Assume B is a 0-minimal bi-ideal of T and let A be a nonzero bi-ideal of B. Thus, $(ABABA] \subseteq A$. Using the same proof of Theorem 2, B is 0-B-simple.

Example 5

(1) In Z_{30}, consider the ordered ternary semigroup $T = \{\overline{1}, \overline{5}, \overline{25}\}$ under the usual multiplication and the partial order $\leq$ = $\{(\overline{1},\overline{1}),(\overline{5},\overline{1}),(\overline{5},\overline{5}),(\overline{25},\overline{1}),(\overline{25},\overline{5}),(\overline{25},\overline{25})\}$. It is easy to see that $\{\overline{5},\overline{25}\}$ is a minimal bi-ideal of T. By Theorem 2, the ternary semigroup $\{\overline{5},\overline{25}\}$ is B-simple.
(2) In Z_{10}, consider the ordered ternary semigroup $T = \{\overline{0},\overline{1},\overline{5}\}$ under the usual multiplication and the partial order $\leq = \{(\overline{0},\overline{0}),(\overline{0},\overline{1}),(\overline{0},\overline{5}),(\overline{1},\overline{1}),(\overline{5},\overline{1}),(\overline{5},\overline{5})\}$. It is easy to see that $B = \{\overline{0},\overline{5}\}$ is 0-B-simple and B is a bi-ideal of T, by Theorem 3 (1), B is a 0-minimal bi-ideal of T.

Theorem 4

Let M be a minimal bi-ideal of an ordered ternary semigroup T and B a bi-ideal of S. Then, $M = (uBvBw]$ for every $u,v,w \in M$.

Proof

By Theorem 1, $(uBvBw]$ is a bi-ideal of T. Since M is minimal and
$(uBvBw] \subseteq (MBMBM] \subseteq (MTMTM] \subseteq (M] \subseteq M$, $M = (uBvBw]$. Hence, the theorem is proved.

Theorem 5

Let T be an ordered ternary semigroup having proper bi-ideals. Then, every proper bi-ideal of T is minimal if and only if the intersection of any two distinct proper bi-ideals is empty.

Proof

Let B_1 and B_2 be two distinct proper bi-ideals of T. Then, B_1 and B_2 are minimal. If $B_1 \cap B_2 \neq \varnothing$, then by Proposition 1, $B_1 \cap B_2$ is a bi-ideal of T. Since $B_1 \cap B_2$ is a proper subset of B_1, there is a contradiction. Hence, $B_1 \cap B_2 = \varnothing$.
The converse is obvious. Using the same proof of Theorem 5, the next theorem holds.

Theorem 6

Let T be an ordered ternary semigroup with zero having nonzero proper bi-ideals. If 0 is a minimum element of T, then every nonzero proper bi-ideal of T is minimal if and only if the intersection of any two distinct proper bi-ideals is {0}.

MAXIMAL BI-IDEALS OF ORDERED TERNARY SEMIGROUPS

Here, the maximal bi-ideals of ordered ternary semigroups are characterized.

Theorem 7

Let B be a bi-ideal of an ordered ternary semigroup T. If either

(1) $T \setminus B = \{a\}$ for some $a \in T$, or
(2) $T \setminus B \subseteq (bTbTb]$ for all $b \in T \setminus B$,

then B is a maximal bi-ideal of T.

Proof

Let A be a bi-ideal of T such that B is a proper subset of A.

Case 1: $T \setminus B = \{a\}$ for some $a \in T$. Since B is a proper subset of A, $A \setminus B \subseteq T \setminus B = \{a\}$. Then $A \setminus B = \{a\}$. Thus $A = B \cup \{a\} = T$.

Case 2: $T \setminus B \subseteq (bTbTb]$ for all $b \in T \setminus B$. Consider $b \in T \setminus B$. So, $T \setminus B \subseteq (bTbTb] \subseteq (ATATA] \subseteq A$. Thus $T = B \cup (T \setminus B) \subseteq B \cup A = A \subseteq T$, so $A = T$. Therefore, B is a maximal bi-ideal of T.

Example 6

(1) Consider an ordered ternary semigroup Z^- under the multiplication and the partial order $\leq = \{(a, a) \mid a \in Z^-\} \cup \{(a,-1) \mid a \in Z^-\}$. Let $B = Z^- \setminus \{-1\}$. It is easy to verify that B is a bi-ideal of Z^-. By Theorem 7(1), B is a maximal bi-ideal of Z^-.

(2) Consider an ordered ternary semigroup $T = \{0,-i,i\}$ under the multiplication and the partial order $\leq = \{(0,0),(0,-i),(0,i),(-i,-i),(i,i)\}$. Let $B = \{0\}$. Clearly, B is a bi-ideal of T. Since $(iTiTi] = T$ and $((-i)T(-i)T(-i)] = T$, by Theorem 7(2), B is a maximal bi-ideal of T.

Theorem 8

If B is a maximal bi-ideal of an ordered ternary semigroup T and $B \cup (a)_b$ is a bi-ideal of T for all $a \in T \setminus B$, then either

(1) $T \setminus B = (a,a^3]$ and $a^5 \in B$ for some $a \in T \setminus B$ and $(bTbTb] \subseteq B$ for all $b \in T \setminus B$, or
(2) $T \setminus B \subseteq (a)_b$ for all $a \in T \setminus B$.

Proof

Let B be a maximal bi-ideal of T and assume $B \cup (a)_b$ is a bi-ideal of T for all $a \in T \setminus B$.

Case 1: $(aTaTa] \subseteq B$ for some $a \in T \setminus B$. Then $a^5 \in (aTaTa] \subseteq B$, so $a^5 \in B$. Since,

$$B \cup (a,a^3] = (B \cup (aTaTa]) \cup (a,a^3]$$
$$= B \cup ((a,a^3] \cup (aTaTa]) = B \cup (a)_b,$$

by assumption, $B \cup (a,a^3]$ is a bi-ideal of T. Since $a \in T \setminus B$, B is a proper subset of $B \cup (a,a^3]$. Thus, $B \cup (a,a^3] = T$, because B is a maximal bi-ideal of T. Thus, $T \setminus B \subseteq (a,a^3]$. Let $b \in T \setminus B$. So, $b \le a$ or $b \le a^3$. If $b \le a$, then $(bTbTb] \subseteq (aTaTa] \subseteq B$. If $b \le a^3$, then

$$(bTbTb] = (a^3Ta^3Ta^3] = (a(a^2Ta^2)a(Ta^2)a]$$
$$\subseteq (aTaTa] \subseteq B.$$

Hence, $(bTbTb] \subseteq B$.

Case 2: $(aTaTa] \subset B$ for all $a \in T \setminus B$. Let $a \in T \setminus B$. Then, B is a proper subset of $B \cup (a)_b$. By assumption and maximality of B, $B \cup (a)_b = T$. Hence, $T \setminus B \subseteq (a)_b$.

Example 7

(1) In Z_{16}, consider the ordered ternary semigroup $T = \{\bar{0},\bar{2},\bar{8}\}$ under the usual multiplication and the partial order $\le = \{(\bar{0},\bar{0}),(\bar{0},\bar{2}),(\bar{0},\bar{8}),(\bar{2},\bar{2}),(\bar{8},\bar{2}),(\bar{8},\bar{8})\}$. Let $B = \{\bar{0},\bar{8}\}$. It is easy to verify that B satisfies the condition (3) of Theorem 9.
(2) Consider an ordered ternary semigroup Z^- under the multiplication and the partial order $\le = \{(a,a) \mid a \in Z^-\} \cup \{(a,-1) \mid a \in Z^-\}$ and a maximal bi-ideal $B = Z^- \setminus \{-1\}$ of Z^-. It is easy to verify that B satisfies condition (2) of Theorem 8. The converse of Theorem 8 is not true in general. We can see the following example.

Example 8

In Z_{16}, consider an ordered ternary semigroup $T = \{\bar{0},\bar{2},\bar{8}\}$ under the usual multiplication and the partial order $\le = \{(\bar{0},\bar{0}),(\bar{0},\bar{2}),(\bar{0},\bar{8}),(\bar{2},\bar{2}),(\bar{8},\bar{2}),(\bar{8},\bar{8})\}$. Let $B = \{\bar{0}\}$. Thus, B is a bi-ideal of T; $T \setminus B = \{\bar{2},\bar{2}^3\} = \{\bar{2},\bar{8}\}$; $\bar{2}^5 = \bar{0} \in B$; $\bar{2}T\bar{2}T\bar{2} = \bar{8}T\bar{8}T\bar{8} = \{\bar{0}\}$ and $B \cup (\bar{2})_b = T$, and $B \cup (\bar{8})_b = \{\bar{0},\bar{8}\}$ are bi-ideal of T. So B satisfies (2) of Theorem 3.2, but B is not maximal because $\{\bar{0},\bar{8}\}$ is a proper bi-ideal of T containing B.
For an ordered ternary semigroup T, let S denote the union of all proper bi-ideals of T. Then, the following lemma holds.

Lemma 1

$T = \mathrm{S}$ if and only if $(a)_b \ne T$ for all $a \in T$.

Theorem 9

Let T be an ordered ternary semigroup. One of the following four conditions is satisfied.

(1) S is not a bi-ideal of T.
(2) $(a)_b \ne T$ for all $a \in T$.
(3) There exists $a \in T$ such that $(a)_b = T$, $(a,a^3] \not\subset$ (aTaTa] and $a^5 \in \mathrm{S}$., T is not B-simple, $T \setminus \mathrm{S} = \{x \in T \mid (x)_b = T\}$ and S is the unique maximal bi-ideal of T.
(4) $T \setminus \mathrm{S} \subseteq (a)_b$ for all $a \in T \setminus \mathrm{S}$, T is not B-simple, $T \setminus \mathrm{S} = \{x \in T \mid (x)_b = T\}$ and S is the unique maximal bi-ideal of T.

Proof

Assume that S is a bi-ideal of T. Now, we consider the following two cases.

Case 1: $\mathrm{S} = T$. In this case, by Lemma 1, condition (2) holds.
Case 2: $\mathrm{S} \ne T$. So, T is not B-simple. To show that S is the unique maximal bi-ideal of T, let B be a bi-ideal of T such that S is a proper subset of B. If $B \ne T$, then $B \subseteq \mathrm{S}$, this is a contradiction. Hence, S is a maximal ideal of T. Next, let B' be a maximal bi-ideal of T. Thus, $B' \subseteq \mathrm{S}$, this implies that $B' = \mathrm{S}$. Hence, S is the unique maximal bi-

ideal of T. Since $S \neq T$, clearly, $(a)_b = T$ for all $b \in T \setminus S$. So, $T \setminus S = \{x \in T \mid (x)_b = T\}$. Then, for all $x \in T \setminus S$, we have $S \cup (x)_b = T$ to be a bi-ideal of T. By Theorem 8, we have the following two cases.

(i) $T \setminus S = (a, a^3]$ and $a^5 \in S$ for some $a \in T \setminus S$, and $(bTbTb] \subseteq S$ for all $b \in T \setminus S$.
(ii) $T \setminus S \subseteq (a)_b$ for all $a \in T \setminus S$.

Assume that (i) holds. If $(a,a^3] \subseteq (aTaTa]$, then $T = (a)_b = (aTaTa]$. By assumption, $T = (aTaTa] \subseteq S$ and so $S = T$, this is a contradiction. Hence, $(a,a^3] \not\subseteq (aTaTa]$. In this case, the condition (3) holds. It is easy to see that case (ii) is the condition (4).

Example 9

(1) Consider the ordered ternary semigroup $T = \{-i, i\}$ under the multiplication and the partial order $\leq$ = $\{(-i,-i),(i,i)\}$. Then, T is B-simple, this implies that $S = \emptyset$. So, T satisfies condition (1) of Theorem 9.
(2) Consider the ordered ternary semigroup $T = Z^- \setminus \{-1\}$ under the multiplication and the partial order $\leq$ = $\{(a,a) \mid a \in T\}$. It is easy to verify that $(x)_b \neq T$ for all $x \in T$. Hence, T satisfies condition (2) of Theorem 9.
(3) In Z_{16}, consider the ordered ternary semigroup $T = \{\bar{0}, \bar{2}, \bar{8}\}$ under the usual multiplication and the partial order $\leq = \{(\bar{0},\bar{0}),(\bar{0},\bar{2}),(\bar{0},\bar{8}),(\bar{2},\bar{2}),(\bar{8},\bar{2}),(\bar{8},\bar{8})\}$. Thus, $S = \{\bar{0}, \bar{8}\}$. It is easy to verify that T satisfies condition (3) of Theorem 9.
(4) Consider the ordered ternary semigroup $T = Z^-$ under the multiplication and the partial order $\leq = \{(a,a) \mid a \in T\} \cup \{(a,-1) \mid a \in T\}$. Thus, $S = Z^- \setminus \{-1\}$. It is easy to verify that T satisfies condition (4) of Theorem 9.

Next, let T be an ordered ternary semigroup with zero. Let S_{00} denote the union of all nonzero proper bi-ideal of T. Using the same proof of Theorem 9, the following theorem holds.

Theorem 10

Let T be an ordered ternary semigroup with zero 0. If 0 is a minimum element of T, then one of the following four conditions is satisfied.

(1) S_{00} is not a bi-ideal of T.
(2) $(a)_b \neq T$ for all $a \in T$.
(3) There exists $a \in T$ such that $(a)_b = T$, $(a,a^3] \not\subseteq (aTaTa]$ and $a^5 \in S_{00}$, T is not 0-B-simple, $T \setminus S_{00} = \{x \in T \mid (x)_b = T\}$ and S_{00} is the unique maximal bi-ideal of T.
(4) $T \setminus S_{00} \subseteq (a)_b$ for all $a \in S_{00}$, T is not 0-B-simple, $T \setminus S_{00} = \{x \in T \mid (x)_b = T\}$ and S_{00} is the unique maximal bi-ideal of T.

MINIMAL AND MAXIMAL BI-IDEALS IN ORDERED N-ARY SEMIGROUPS

The idea of n-ary algebras was given by Kasner at the 53rd annual meeting of the American Association of the Advancement of Science in 1904. n-ary semigroups are universal algebras with one associative n-ary operation. Here, we generalize results of minimal and maximal bi-ideals in ordered ternary semigroups previously discussed to ordered n-ary semigroups.

A non-empty set G with an n-ary operation $f : G^n \to G$ is called an n-ary groupoid and will be denoted by $(G;f)$. An n-ary groupoid $(G;f)$ is called an n-ary semigroup (Dudek, 2001) if the following associative law holds: $f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$ for all $i, j \in \{1,2,\ldots,n\}$ and $x_1, x_2, \ldots, x_{2n-1} \in G$ where x_j^i denote the sequence $x_i, x_{i+1}, \ldots, x_j$.

Here, we denote $f(x_1, x_2, \ldots, x_n)$ by $x_1x_2\cdots x_n$. A partial ordered n-ary semigroup G is called an *ordered n-ary semigroup* if for all $a, b, x_1, x_2, \ldots, x_{n-1} \in G$,

$$a \leq b \Rightarrow ax_1^{n-1} \leq bx_1^{n-1}, x_1ax_2^{n-1} \leq x_1bx_2^{n-1}, \ldots, x_1^{n-1}a \leq x_1^{n-1}b.$$

Let G be an ordered n-ary semigroup. A non-empty subset A of G is called an *ordered n-ary subsemigroup* if $(A] \subseteq A$ and $A^n \subseteq A$. In this study, a bi-ideal of an ordered n-ary semigroup mean that an ordered n-ary subsemigroup B of G satisfies $\underbrace{BGBG\cdots BGB}_{2n-1 \text{ term}} \subseteq B$, that is, for all $b_1, b_3, \ldots, b_{2n-1} \in B$ and $a_2, a_4, \ldots, a_{2n-2} \in G$, $b_1a_2b_3a_4\cdots b_{2n-3}a_{2n-2}b_{2n-1} \in B$.

Note that this definition generalizes a bi-ideal of ordered semigroup and ordered ternary semigroup. Now, results of minimal and maximal bi-ideals of ordered ternary semigroups to ordered n-ary semigroups are generalize. The following generalizes some results shown previously.

Let G be an ordered n-ary semigroup and A a non-empty subset of G. A bi-ideal $(A)_b$ denote the bi-ideal of G generated by A, that is, $(A)_b$ is the smallest bi-ideal of G containing A. The following holds.

Proposition 5

Let G be an ordered n-ary semigroup and A a non-empty subset of G. Then,

$$(A)_b = A \cup A^n \cup \underbrace{AGAG\cdots AGA}_{2n-1 \text{ terms}}.$$

So, for any $a \in G$,

$$(a)_b = \{a, a^n\} \cup \underbrace{aGaG\cdots aGa}_{2n-1 \text{ terms}}.$$

An ordered n-ary semigroup G is called *B-simple* if G has no proper bi-ideals. If G have zero, $G^n \neq \{0\}$ and $|G| > 1$. G is called *0-B-simple* if G has no nonzero proper bi-ideals.

Proposition 6

Let G be an ordered n-ary semigroup. The following statements are equivalent.

(1) G is B-simple.
(2) $\underbrace{aGaG\cdots aGa}_{2n-1 \text{ terms}} = G$ for all $a \in G$.
(3) $(a)_b = G$ for all $a \in G$.

Proposition 7

Let G be an ordered n-ary semigroup with zero, $G^n \neq \{0\}$ and $|G| > 1$. If 0 is a minimum element of G, then the following statements are equivalent.

(1) G is 0-B-simple.
(2) $\underbrace{aGaG\cdots aGa}_{2n-1 \text{ terms}} = G$ for all $a \in G \setminus \{0\}$.
(3) $(a)_b = G$ for all $a \in G \setminus \{0\}$.

A bi-ideal B of an ordered n-ary semigroup G is called a minimal bi-ideal of G if B does not properly contain any bi-ideal of G.

Theorem 11

Let G be an ordered n-ary semigroup and B a bi-ideal of G. B is a minimal bi-ideal of G if and only if B is B-simple. A nonzero bi-ideal B of an ordered n-ary semigroup G with zero is called a *0-minimal bi-ideal* of G if B does not properly contain any nonzero bi-ideal of G.

Theorem 12

Let G be an ordered n-ary semigroup with zero and B a nonzero bi-ideal of G such that $B^n \neq \{0\}$. The following statements hold.

(1) If B is 0-B-simple, then B is a 0-minimal bi-ideal of G.
(2) If B is a 0-minimal bi-ideal of G and $\underbrace{ABAB\cdots ABA}_{2n-1 \text{ terms}} \neq \{0\}$ for all a nonzero bi-ideal A of B, then B is 0-B-simple.

Theorem 13

Let G be an ordered n-ary semigroup having proper bi-ideals. Then, every proper bi-ideal of G is minimal if and only if the intersection of any two distinct proper bi-ideals is empty.

Theorem 14

Let G be an ordered n-ary semigroup with zero having nonzero proper bi-ideals. If 0 is a minimum element of G, then every nonzero proper bi-ideal of G is minimal if and only if the intersection of any two distinct proper bi-ideals is {0}.

Theorem 15

Let B be a bi-ideal of an ordered n-ary semigroup G. If either

(1) $G \setminus B = \{a\}$ for some $a \in G$, or
(2) $G \setminus B = \underbrace{bGbG\cdots bGb}_{2n-1 \text{ terms}}$ for all $b \in G \setminus B$,

then B is a maximal bi-ideal of G.

Theorem 16

If B is a maximal bi-ideal of an ordered n-ary semigroup G and $B \cup (a)_b$ is a bi-ideal of G for all $a \in G \setminus B$, then either

(1) $G \setminus B = \{a, a^n\}$ and $a^{2n-1} \in B$ for some $a \in G \setminus B$, and $\underbrace{bGbG\cdots bGb}_{2n-1 \text{ terms}} \subseteq B$ for all $b \in G \setminus B$, or
(2) $G \setminus B \subseteq (a)_b$ for all $a \in G \setminus B$.

For an ordered n-ary semigroup G, let S denote the union of all proper bi-ideals of G.

Theorem 17

Let G be an ordered n-ary semigroup. One of the following four conditions is satisfied.

(1) S is not a bi-ideal of G.
(2) $(a)_b \neq G$ for all $a \in G$.
(3) There exists $a \in G$ such that $(a)_b = G$, $(a,a^n] \not\subseteq \underbrace{bGbG\cdots bGb}_{2n-1 \text{ terms}}$ and $a^{2n-1} \in S$, G is not B-simple, $G \setminus S = \{x \in G \mid (x)_b = G\}$ and S is the unique maximal bi-ideal of G.
(4) $G \setminus S \subseteq (a)_b$ for all $a \in S$, G is not B-simple, $G \setminus S = \{x \in G \mid (x)_b = G\}$ and S is the unique maximal bi-ideal of G.

Next, let G be an ordered n-ary semigroup with zero. Let S_{00} denote the union of all nonzero proper bi-ideal of G.

Theorem 18

Let G be an ordered n-ary semigroup with zero 0. If 0 is a minimum element of G, then one of the following four conditions is satisfied.

(1) S_{00} is not a bi-ideal of G.
(2) $(a)_b \neq G$ for all $a \in G$.
(3) There exists $a \in G$ such that $(a)_b = G$, $(a,a^n] \not\subseteq \underbrace{bGbG\cdots bGb}_{2n-1 \text{ terms}}$ and $a^{2n-1} \in S_{00}$, G is not 0-B-simple, $G \setminus S_{00} = \{x \in G \mid (x)_b = G\}$ and S_{00} is the unique maximal bi-ideal of G.
(4) $G \setminus S_{00} \subseteq (a)_b$ for all $a \in S_{00}$, G is not 0-B-simple, $G \setminus S_{00} = \{x \in G \mid (x)_b = G\}$ and S_{00} is the unique maximal bi-ideal of G.

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