International Journal of the Physical Sciences Vol. 6(25), pp. 6072-6075, 23 October, 2011
Available online at http://www.academicjournals.org/IJPS
DOI: 10.5897/IJPS11.948
ISSN 1992 - 1950 ©2011 Academic Journals

Full Length Research Paper

An augmented Lagrangian method of optimization problems with cone constraints

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Accepted 3 October, 2011

A deterministic global optimization algorithm based on augmented Lagrangian is proposed for solving the nonlinear programming with cone constraints, which is often encountered in technical design and operational research. At each outer iteration k, the method requires the ε_k-global minimization of the augmented Lagrangian of problems with cone constraints, where $\varepsilon_k \to \varepsilon$. The convergence to an ε-global minimizer of the original problem is proved.

Key words: Global optimization, augmented Lagrangians, cone optimization, algorithms.

INTRODUCTION

Nowadays, global optimization has ubiquitous applications in all branches of engineering sciences and applied sciences (Floudas, 2000; Lu and Quan, 2011). During the last decade, several textbooks addressed different facets of global optimization theory and applications.

We often construct augmented Lagrangian framework and use this method to solve nonlinear programming problems (Andreani et al., 2007, 2008). We hope the augmented Lagrangian algorithm we used is global convergent, but generally, global convergence is difficult to achieve. In Birgin et al. (2010), Birgin EG has proposed a weaker global convergence for nonlinear programming problems with linear constraints and box constraints, that is, he has proposed an ε-global minimization. At each outer iteration k, we can get the corresponding ε_k-global minimization, which are convergent to the ε-global minimization, so the convergence of the algorithm is proved. In this paper, we choose the augmented Lagrangian method in study of Al-Khayyal and Sherali (2000), and amend it, then we discuss the ε-global convergence of augmented Lagrangian algorithm for nonlinear programming problem with cone constraints (Anca, 2010; Gu et al., 2009; Liu and Zhang 2007).

The algorithm

The problem to be addressed is:

$$\begin{aligned} &\min imize \quad f(x) \\ &subjected\ to \quad G(x) \in K \end{aligned},$$

where $f: R^n \to R, G: R^n \to R$ are continuous and K is convex.

Assumption

From now on, we will assume that there exists a global minimizer z of the problem. We define the following augmented Lagrangian function:

$$L_\rho(x,\lambda) = f(x) + \rho[dist(G(x) + \frac{\lambda}{2\rho}, K)]^2 - \frac{\|\lambda\|^2}{4\rho}, \tag{1}$$

where $\rho > 0, \lambda \in \alpha B, \alpha > 0$, B is a bounded set, $dist(G(x) + \frac{\lambda}{2\rho}, K)$ is the distance between K and $G(x) + \frac{\lambda}{2\rho}$.

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Algorithm

Let $\gamma > 1, 0 < \tau < 1, \{\varepsilon_k\}$ be a sequence of nonnegative numbers such that $\lim_{k\to\infty} \varepsilon_k = \varepsilon \geq 0.$ Let $\lambda^1 \in \alpha B, \alpha > 0$ and $\rho_1 > 0.$ Initialize $k \leftarrow 1.$

Step 1: Let $P_k \subset R^n$ be a closed set such that a global minimizer z (the same for all k) belongs to P_k. Find an ε_k-global minimizer x^k of the problem $\min L_{\rho_k}(x, \lambda^k)$ subject to $x \in P_k$.

That is $x^k \in P_k$ satisfying:

$$L_{\rho_k}(x^k, \lambda^k) \leq L_{\rho_k}(x, \lambda^k) + \varepsilon_k, \tag{2}$$

for all $x \in P_k$. The ε_k-global minimum can be obtained using a deterministic global optimization approach, such as αBB method (Adjiman et al., 1996, 1998a, b).

Step 2: Define

$$V_k = dist(G(x^k), K) + dist(\lambda^k, N_k(G(x^k))), \tag{3}$$

where $N_k(G(x^k))$ is the normal cone of $G(x^k)$.

If $k = 1$ or

$$dist(G(x^k), K) \leq \tau dist(G(x^{k-1}), K), \tag{4}$$

define $\rho_{k+1} = \rho_k$, otherwise, define $\rho_{k+1} = \gamma\rho_k$.

Step 3: Compute $\lambda^{k+1} \in N_k(G(x^k)) \cap \alpha B, \alpha > 0.$ Set $k \leftarrow k+1$ and go to Step 1.

Convergence to an ε-global minimum

In the theorems that follow, we assume that the sequence $\{x^k\}$ is well defined. In other words, the ε_k-global minimizer of the Augmented Lagrangian can always be found.

Theorem 1

Assume that the sequence $\{x^k\}$ is well defined and admits a limit point x^*. Then, x^* is feasible.

Proof: We consider two cases: $\{\rho_k\}$ bounded and $\rho_k \to \infty$. If $\{\rho_k\}$ is bounded, there exists k_0 such that $\rho_k = \rho_{k_0}$ for all $k \geq k_0$. Therefore, for all $k \geq k_0$, (4) holds. So, the limit point is feasible.

Now assuming that $\rho_k \to \infty$. Let z be as in Step 1, therefore, z is feasible. So, $dist(G(z), K) = 0$. Suppose, by contradiction, that x^* is not feasible. Therefore,

$$dist(G(x^*), K) > dist(G(z), K). \tag{5}$$

Let $\hat{K}$ be an infinite sequence of indices such that $\lim_{k\in\hat{K}} x^k = x^*$. Since λ_k is bounded and $\rho_k \to \infty$, there exists $c > 0$ such that $k \in \hat{K}$ large enough:

$$[dist(G(x^k) + \frac{\lambda^k}{2\rho_k}, K)]^2 > [dist(G(z) + \frac{\lambda^k}{2\rho_k}, K)]^2 + c. \tag{6}$$

Therefore,

$$\begin{aligned} & f(x^k) + \rho_k[dist(G(x^k) + \frac{\lambda^k}{2\rho_k}, K)]^2 - \frac{\|\lambda^k\|^2}{4\rho_k} \\ & > f(z) + \rho_k[dist(G(z) + \frac{\lambda^k}{2\rho_k}, K)]^2 - \frac{\|\lambda^k\|^2}{4\rho_k} + \rho_k c + f(x^k) - f(z). \end{aligned} \tag{7}$$

Since $\lim_{k\in\hat{K}} x^k = x^*$ and f is continuous, for $k \in \hat{K}$ large enough.

$$\rho_k c + f(x^k) - f(z) > \varepsilon_k. \tag{8}$$

Therefore,

$$\begin{aligned} & f(x^k) + \rho_k[dist(G(x^k) + \frac{\lambda^k}{2\rho_k}, K)]^2 - \frac{\|\lambda^k\|^2}{4\rho_k} \\ & > f(z) + \rho_k[dist(G(z) + \frac{\lambda^k}{2\rho_k}, K)]^2 - \frac{\|\lambda^k\|^2}{4\rho_k} + \varepsilon_k. \end{aligned} \tag{9}$$

Now, since z is a global minimizer, we have that $z \in P_k$ for all k. Therefore, the inequality aforementioned contradicts the definition of x^k.

Theorem 2

Under the same assumptions of Theorem 1, every limit point x^* of a sequence $\{x^k\}$ generated by Algorithm previously given is an ε-global minimizer of the problem.

Proof: Let $\hat{K} \underset{\infty}{\subset} N$ be such that $\lim_{k \in \hat{K}} x^k = x^*$. By Theorem 1, x^* is feasible. Let z as in Step 1. Then, $z \in P_k$ for all k. We consider two cases: $\rho_k \to \infty$ and $\{\rho_k\}$ bounded.

Case 1($\rho_k \to \infty$): By the definition of the algorithm:

$$f(x^k) + \rho_k [dist(G(x^k) + \frac{\lambda^k}{2\rho_k}, K)]^2 - \frac{\|\lambda^k\|^2}{4\rho_k}$$
$$\leq f(z) + \rho_k [dist(G(z) + \frac{\lambda^k}{2\rho_k}, K)]^2 - \frac{\|\lambda^k\|^2}{4\rho_k} + \varepsilon_k, \tag{10}$$

for all $k \in N$.

Since $dist(G(z), K) = 0$, we have:

$$[dist(G(z) + \frac{\lambda^k}{2\rho_k}, K)]^2 \leq \frac{\|\lambda^k\|^2}{4\rho_k}. \tag{11}$$

Therefore, by (10),

$$f(x^k) - \frac{\|\lambda^k\|^2}{4\rho_k} \leq f(x^k) + \rho_k [dist(G(z) + \frac{\lambda^k}{2\rho_k}, K)]^2 - \frac{\|\lambda^k\|^2}{4\rho_k}$$
$$\leq f(z) + \varepsilon_k. \tag{12}$$

Taking limits for $k \in \hat{K}$ and using $\lim_{k \in \hat{K}} \frac{\|\lambda^k\|}{\rho_k} = 0$ and $\lim_{k \in \hat{K}} x^k = x^*$, by the continuity of f and the convergence of x^k, we get:

$$f(x^*) \leq f(z) + \varepsilon. \tag{13}$$

Since z is a global minimizer, it turns out that x^* is an ε-global minimizer, as we wanted to prove.

Case 2 ($\{\rho_k\}$ bounded): In this case, we have that $\rho_k = \rho_{k_0}$ for all $k \geq k_0$. Therefore, by the definition of Algorithm, we have:

$$f(x^k) + \rho_k [dist(G(x^k) + \frac{\lambda^k}{2\rho_k}, K)]^2 - \frac{\|\lambda^k\|^2}{4\rho_k}$$
$$\leq f(z) + \rho_k [dist(G(z) + \frac{\lambda^k}{2\rho_k}, K)]^2 - \frac{\|\lambda^k\|^2}{4\rho_k} + \varepsilon_k \tag{14}$$

for all $k \geq k_0$. Since $dist(G(z), K) = 0$, we have:

$$f(x^k) + \rho_{k_0} [dist(G(x^k) + \frac{\lambda^k}{2\rho_{k_0}}, K)]^2 - \frac{\|\lambda^k\|^2}{4\rho_{k_0}}$$
$$\leq f(z) + \varepsilon_k, \tag{15}$$

for all $k \geq k_0$.

By the feasibility of x^*, taking limits in the inequality aforementioned for $k \in \hat{K}$, we get:

$$f(x^*) + \rho_{k_0} [dist(G(x^*) + \frac{\lambda^k}{2\rho_{k_0}}, K)]^2 - \frac{\|\lambda^k\|^2}{4\rho_{k_0}}$$
$$\leq f(z) + \varepsilon. \tag{16}$$

Since $\frac{\|\lambda^k\|^2}{4\rho_{k_0}} \geq 0$, and

$$dist(G(x^*) + \frac{\lambda^k}{2\rho_{k_0}}, K) = 0, \tag{17}$$

We have that

$$f(x^*) \leq f(z) + \varepsilon. \tag{18}$$

Since z is a global minimizer, the proof is complete.

CONCLUSION

One of the advantages of the augmented Lagrangian approach for solving nonlinear programming problems is its intrinsic adaptability to the global optimization problem. But generally, global convergence is difficult to achieve.

For optimization problems with cone constraints, an ε-global minimization has been proposed in this paper. At each outer iteration k, we can get the corresponding ε_k-global minimization, which are convergent to the ε-global minimization, so the convergence of the algorithm is proved. There are lots of unconstrained and constrained global optimization problems in Engineering, Physics, Chemistry, Economy and other areas that are not solvable with the present computer facilities. The field for improvement is all concerned in the global optimazation. So much research is expected in the following years in order to be able to efficiently solve more challenging practical problems. We will do more research on the global convergence of nonlinear programming problems at more general case, and strive to get better results.

ACKNOWLEDGEMENT

This work was supported financially by the National Natural Science Foundation of China Grants (10971118) and the Youth Science Foundation of China Grants (11101248), the Youth Foundation of Shandong Province, China (ZR2010EQ014), Shandong Province Natural Science Foundation (ZR2010AQ026) and National Natural Science Foundation of China (11026047).

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