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Full Length Research Paper

Exact and explicit solutions for the discrete Burgers equation by means of the Exp-function method

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In recent years, various direct methods were proposed to find exact solutions not only for nonlinear partial differential equations but also for nonlinear differential-difference equations (NDDEs). In this paper, the Exp-function method was extended to the discrete Burgers equation for searching traveling waves in closed forms. It was shown that the equation supported discrete solitary wave solutions. Our results, as test problems, might be of great importance for those who study in the field of numerical analysis. It can be concluded that the Exp-function approach provides a direct and efficient way to construct generic exponential wave solutions to NDDEs.

Key words: Exp-function method, discrete Burgers equation, nonlinear differential-difference equation.

INTRODUCTION

Expansion methods for finding traveling wave solutions to nonlinear evolution equations (NEEs) have received considerable attention over the past dozen years or so. For instance, the basic tanh-function method (Malfliet and Hereman, 1996) was introduced to find solitary wave solutions. Since then the method has been extended, generalized and adapted. Other related methods have also been developed and extended such as the (G'/G)-expansion method (Wang et al., 2008) and the first integral method (Feng, 2002). Recently, based on He and Wu (2006) pioneer work, the Exp-function method has found some popularity in a research community, and there have been a number of papers refining the initial idea. The Exp-function ''method" consists of trying rational combinations of exponential functions as an ''ansatz" to find exact solutions of the reduced ordinary differential equation (ODE) for traveling waves of the original equation. The method is powerful for it can take full advantage of computer algebra systems, the solution procedure is actually impossible if it is performed by hand.

On the other hand, nonlinear differential-difference equations (NDDEs) have also served as an essential tool for describing and analyzing problems in many scientific disciplines. For example, they occurred in condensed matter physics, mechanical engineering, biophysics, and in such fields as molecular crystals, atomic chains, currents in electrical networks, etc. (Scott and Macheil, 1983; Su et al., 1979; Davydov, 1973; Marquii et al., 1995). Their importance has motivated mathematicians and other scientists to develop many integrable NDDEs (Toda, 1989; Wadati, 1976; Ohta and Hirota, 1991; Ablowitz and Ladik, 1975). Contrary to difference equations which are being fully discretized, NDDEs are semi-discretized with some (or all) of their space variables discretized while time is usually kept continuous.

More recently, a start has been made on showing how methods of solution for NEEs can be adapted for application to NDDEs. Even though there are some difficulties which are usually encountered when we search for iterative relations from indices n to $n \pm i$, many well-known analytic methods primarily developed for solving NEEs are successfully extended to a considerable number of NDDEs (Hu and Ma, 2002; Liu et al., 2008; Xie et al., 2007; Yang et al., 2009; Zhu et al., 2009). But, no method is perfect and can promise finding all solutions to all types of NDDEs. The present paper is a worthwhile contribution to this effort as well. Our objective in this study is to demonstrate the applicability of the Exp-function method to NDDEs by examining the discrete Burgers equation for the first time.

METHODOLOGY

Let us consider a system of M polynomial NDDEs in the form

$$P\left(\mathbf{u}_{\mathbf{n}+\mathbf{p}_1}(\mathbf{x}),\ldots,\mathbf{u}_{\mathbf{n}+\mathbf{p}_k}(\mathbf{x}),\ldots,\mathbf{u}'_{\mathbf{n}+\mathbf{p}_1}(\mathbf{x}),\ldots,\mathbf{u}'_{\mathbf{n}+\mathbf{p}_k}(\mathbf{x}),\ldots,\mathbf{u}^{(r)}_{\mathbf{n}+\mathbf{p}_1}(\mathbf{x}),\ldots,\mathbf{u}^{(r)}_{\mathbf{n}+\mathbf{p}_k}(\mathbf{x})\right)=0\,, \quad (1)$$

Where the dependent variable $\mathbf{u}_{\mathbf{n}}$ have M components $u_{i,\mathbf{n}}$ and so do its shifts; the continuous variable $\mathbf{x}$ has N components x_i; the discrete variable $\mathbf{n}$ has Q components n_j; the k shift vectors $\mathbf{p}_i \in \mathbb{Z}^Q$; and $\mathbf{u}^{(r)}(\mathbf{x})$ denotes the collection of mixed derivative terms of order r.

Step 1

To find traveling wave solutions for Equation 1, we make the wave transformation

$$\mathbf{u}_{\mathbf{n}+\mathbf{p}_s}(\mathbf{x})=\mathbf{U}_{\mathbf{n}+\mathbf{p}_s}(\xi_{\mathbf{n}}),\ \xi_{\mathbf{n}}=\sum_{i=1}^{Q}d_i n_i+\sum_{j=1}^{N}c_j x_j+\zeta,\ \ (s=1,2,\ldots,k), \quad (2)$$

Where the coefficients $c_1,c_2,\ldots,c_N,d_1,d_2,\ldots,d_Q$ and the phase ζ are all constants. Then, Equation 1 becomes:

$$P\left(\mathbf{U}_{\mathbf{n}+\mathbf{p}_1}(\xi_{\mathbf{n}}),\ldots,\mathbf{U}_{\mathbf{n}+\mathbf{p}_k}(\xi_{\mathbf{n}}),\ldots,\mathbf{U}'_{\mathbf{n}+\mathbf{p}_1}(\xi_{\mathbf{n}}),\ldots,\mathbf{U}'_{\mathbf{n}+\mathbf{p}_k}(\xi_{\mathbf{n}}),\ldots,\mathbf{U}^{(r)}_{\mathbf{n}+\mathbf{p}_1}(\xi_{\mathbf{n}}),\ldots,\mathbf{U}^{(r)}_{\mathbf{n}+\mathbf{p}_k}(\xi_{\mathbf{n}})\right)=0\,. \quad (3)$$

Step 2

We suppose that the solution of Equation 3 can be expressed in the form

$$\mathbf{U}_{\mathbf{n}}(\xi_{\mathbf{n}})=\frac{a_f\exp(f\xi_{\mathbf{n}})+\cdots+a_{-g}\exp(-g\xi_{\mathbf{n}})}{b_p\exp(p\xi_{\mathbf{n}})+\cdots+b_{-q}\exp(-q\xi_{\mathbf{n}})}\,, \quad (4)$$

Where f, g, p and q are unknown positive integers to be determined, a_l's and b_m's are unknown constants.

Step 3

A simple and straightforward calculation leads to the identity

$$\xi_{\mathbf{n}+\mathbf{p}_s}=\xi_{\mathbf{n}}+\varphi_s,\quad \varphi_s=p_{s1}d_1+p_{s2}d_2+\cdots+p_{sQ}d_Q\,, \quad (5)$$

Where p_{sj} is the jth component of the shift vector $\mathbf{p}_s$. Thus,

$$\mathbf{U}_{\mathbf{n}+\mathbf{p}_s}(\xi_{\mathbf{n}})=\frac{a_f\exp\left(f(\xi_{\mathbf{n}}+\varphi_s)\right)+\cdots+a_{-g}\exp\left(-g(\xi_{\mathbf{n}}+\varphi_s)\right)}{b_p\exp\left(p(\xi_{\mathbf{n}}+\varphi_s)\right)+\cdots+b_{-q}\exp\left(-q(\xi_{\mathbf{n}}+\varphi_s)\right)}\,, \quad (6)$$

Step 4

In Equation 3, determine the highest order nonlinear term and the linear term of highest order in $\mathbf{U}_{\mathbf{n}}(\xi_{\mathbf{n}})$ as in the continuous case, and express them in terms of Equation 4. Then, in the resulting terms, balance the highest order Exp-function to determine f and p, and the lowest order Exp-function to determine g and q.

Step 5

Substitute Equations 4 and 6 into Equation 3 and equate the coefficients of $\exp(r\xi_{\mathbf{n}})$ $(r=0,1,2,\ldots)$ to zero, obtain a system of nonlinear algebraic equations for a_l, b_m, d_i and c_j. Then, to determine these constants, solve the system with the aid of a computer algebra system such as MATHEMATICA.

Step 6

Substitute the values obtained in Step 5 into Equation 4 and find traveling wave solutions for Equation 1. Then, it is necessary to substitute them into the original Equation 1 to assure the correctness of the solutions.

RESULTS AND DISCUSSION

The discrete Burgers equation (Hirota, 2004) reads

$$\frac{du_n(t)}{dt}=\left(1+u_n(t)\right)\left(u_{n+1}(t)-u_n(t)\right), \quad (7)$$

Where $u_n(t)=u(n,t)$, $n\in\mathbb{Z}$, is the displacement of the *n*th particle from the equilibrium position. Equation 7 is a soliton equation and has a solution structure distinct from that of the modified Volterra lattice equation, the Hybrid lattice equation and the mKdV lattice equation (Wadati, 1976; Ohta and Hirota, 1991; Ablowitz and Clarkson, 1991). Moreover, Equation 7 can be considered as a discrete analogue to the nonlinear partial differential equation

$$u_t+uu_x=u_{xx}\,,$$

which, first developed by Burgers (1948), represents a simple nonlinear model equation for diffusive waves in fluid dynamics. The existence of traveling waves for the continuous Burgers equation is well-known. Therefore, one might expect that the discrete Burgers equation also admits a traveling wave solution. Indeed, such a traveling wave solution does exist. For solving Equation 7, we make the traveling wave transformation

$$u_n(t) = U_n(\xi_n), \ \xi_n = dn + ct + \zeta \ , \tag{8}$$

Where d , c are parameters to be determined later, and ζ is an arbitrary phase constant. Then, Equation 7 reduces to

$$c\frac{dU_n(\xi_n)}{d\xi_n} = \left(1 + U_n(\xi_n)\right)\left(U_{n+1}(\xi_n) - U_n(\xi_n)\right). \tag{9}$$

Now, according to Exp-function method, we suppose Equation 9 has a solution in the form of Equation 4 and balance the terms $U_n'(\xi_n)$ and $U_n^2(\xi_n)$. By a simple calculation, we have

$$U_n'(\xi_n) = \frac{k_1 \exp[(p+f)\xi_n] + \cdots}{k_2 \exp[2p\xi_n] + \cdots} \tag{10}$$

and

$$U_n^2(\xi_n) = \frac{k_3 \exp[2f\xi_n] + \cdots}{k_4 \exp[2p\xi_n] + \cdots}, \tag{11}$$

Where k_i's are determined coefficients for simplicity. Balancing highest order of Exp-function in Equations 10 and 11, we have

$$p + f = 2f \ , \tag{12}$$

which leads to the result

$$p = f \ . \tag{13}$$

Similarly, from the ansatz Equation 4, we have

$$U_n'(\xi_n) = \frac{\cdots + h_1 \exp[-(g+q)\xi_n]}{\cdots + h_2 \exp[-2q\xi_n]} \tag{14}$$

and

$$U_n^2(\xi_n) = \frac{\cdots + h_3 \exp[-2g\xi_n]}{\cdots + h_4 \exp[-2g\xi_n]}, \tag{15}$$

Where h_i's are determined coefficients for simplicity. Balancing lowest order of Exp-function in Equations 14 and 15, we have

$$-(g+q) = -2g \ , \tag{16}$$

which leads to the result

$$q = g \ . \tag{17}$$

We can freely choose the values of f and g in general. However, the final solution does not strongly depend on the values of f and g (He and Abdou, 2007). For simplicity, we set $f = p = 1$ and $g = q = 1$. Then, from Equations 4 and 6, we get

$$U_n(\xi_n) = \frac{a_1 \exp(\xi_n) + a_0 + a_{-1}\exp(-\xi_n)}{b_1 \exp(\xi_n) + b_0 + b_{-1}\exp(-\xi_n)}, \tag{18}$$

$$U_{n+1}(\xi_n) = \frac{a_1 \exp(\xi_n + d) + a_0 + a_{-1}\exp(-\xi_n - d)}{b_1 \exp(\xi_n + d) + b_0 + b_{-1}\exp(-\xi_n - d)}, \tag{19}$$

Where a_1, a_0, a_{-1}, b_1, b_0, b_{-1}, d and c are constants to be determined later. Inserting Equations 18 and 19 into Equation 9 leads to the equation

$$\frac{1}{A}\left[C_1 \exp(\xi_n) + C_2 \exp(2\xi_n) + C_3 \exp(3\xi_n) + C_4 \exp(4\xi_n) + C_5 \exp(5\xi_n)\right] = 0 \, , \tag{20}$$

Where

$$A = \left(b_{-1} + b_0 \exp(\xi_n) + b_1 \exp(2\xi_n)\right)^2 \left(b_{-1} + b_0 \exp(d + \xi_n) + b_1 \exp\left(2(d + \xi_n)\right)\right)$$

and

$$\begin{aligned} C_1 = {} & a_{-1}a_0b_{-1} - a_{-1}a_0b_{-1}\exp(d) + a_0b_{-1}^2 - a_0b_{-1}^2\exp(d) + ca_0b_{-1}^2 - a_{-1}^2b_0 + a_{-1}^2b_0\exp(d) - a_{-1}b_{-1}b_0 \\ & + a_{-1}b_{-1}b_0\exp(d) - ca_{-1}b_{-1}b_0, \end{aligned}$$

$$C_2 = a_0^2 b_{-1} - a_0^2 b_{-1}\exp(d) + a_{-1}a_1 b_{-1} - a_{-1}a_1 b_{-1}\exp(2d) + a_1 b_{-1}^2 - a_1 b_{-1}^2\exp(2d) + 2ca_1 b_{-1}^2 - a_{-1}a_0 b_0$$
$$+ a_{-1}a_0 b_0 \exp(d) + a_0 b_{-1} b_0 - a_0 b_{-1} b_0 \exp(d) + ca_0 b_{-1} b_0 \exp(d) - a_{-1} b_0^2 + a_{-1} b_0^2 \exp(d)$$
$$- ca_{-1} b_0^2 \exp(d) - a_{-1}^2 b_1 + a_{-1}^2 b_1 \exp(2d) - a_{-1} b_{-1} b_1 + a_{-1} b_{-1} b_1 \exp(2d) - 2ca_{-1} b_{-1} b_1,$$

$$C_3 = 2a_0 a_1 b_{-1} - a_0 a_1 b_{-1}\exp(d) - a_0 a_1 b_{-1}\exp(2d) - a_{-1}a_1 b_0 + 2a_{-1}a_1 b_0 \exp(d) - a_{-1}a_1 b_0 \exp(2d)$$
$$+ a_1 b_{-1} b_0 + a_1 b_{-1} b_0 \exp(d) - 2a_1 b_{-1} b_0 \exp(2d) + ca_1 b_{-1} b_0 + 2ca_1 b_{-1} b_0 \exp(d) - a_{-1} a_0 b_1$$
$$- a_{-1} a_0 b_1 \exp(d) + 2a_{-1} a_0 b_1 \exp(2d) + a_0 b_{-1} b_1 - 2a_0 b_{-1} b_1 \exp(d) + a_0 b_{-1} b_1 \exp(2d) - ca_0 b_{-1} b_1$$
$$+ ca_0 b_{-1} b_1 \exp(2d) - 2a_{-1} b_0 b_1 + a_{-1} b_0 b_1 \exp(d) + a_{-1} b_0 b_1 \exp(2d) - 2ca_{-1} b_0 b_1 \exp(d)$$
$$- ca_{-1} b_0 b_1 \exp(2d),$$

$$C_4 = a_1^2 b_{-1} - a_1^2 b_{-1}\exp(2d) + a_0 a_1 b_0 \exp(d) - a_0 a_1 b_0 \exp(2d) + a_1 b_0^2 \exp(d) - a_1 b_0^2 \exp(2d)$$
$$+ ca_1 b_0^2 \exp(d) - a_0^2 b_1 \exp(d) + a_0^2 b_1 \exp(2d) - a_{-1} a_1 b_1 + a_{-1} a_1 b_1 \exp(2d) + a_1 b_{-1} b_1$$
$$- a_1 b_{-1} b_1 \exp(2d) + 2ca_1 b_{-1} b_1 \exp(2d) - a_0 b_0 b_1 \exp(d) + a_0 b_0 b_1 \exp(2d) - ca_0 b_0 b_1 \exp(d)$$
$$- a_{-1} b_1^2 + a_{-1} b_1^2 \exp(2d) - 2ca_{-1} b_1^2 \exp(2d),$$

$$C_5 = a_1^2 b_0 \exp(d) - bca_0 \exp(2d) - a_1^2 b_0 \exp(2d) - a_0 a_1 b_1 \exp(d) + a_0 a_1 b_1 \exp(2d) + a_1 b_0 b_1 \exp(d)$$
$$- a_1 b_0 b_1 \exp(2d) + ca_1 b_0 b_1 \exp(2d) - a_0 b_1^2 \exp(d) + a_0 b_1^2 \exp(2d).$$

Equating the coefficients of $\exp(r\zeta)$ $(1 \le r \le 5)$ to zero in Equation 20 and solving the resulting nonlinear algebraic system for a_1, a_0, a_{-1}, b_1, b_0, b_{-1}, d and c, we obtain the solution sets

$$\left\{ a_0 = \frac{1}{2}\left(\left(-2 + c\coth\left(\frac{d}{2}\right)\right) b_0 \mp c\sqrt{b_0^2 - 4b_{-1}b_1}\right),\ a_{\mp 1} = \frac{1}{2}\left(-2 \mp c + c\coth\left(\frac{d}{2}\right)\right) b_{\mp 1}, \right\}, \tag{21}$$

$$\left\{ a_0 = 0,\ a_{-1} = (-1 - c + c\coth(d)) b_{-1},\ a_1 = (-1 + c + c\coth(d)) b_1,\ b_0 = 0 \right\}. \tag{22}$$

Now, substituting Equation 21 into 18 yields more general exponential function solutions to Equation 7 as:

$$u_{n,1}^{\mp}(t) = \frac{\left(-2 - c + c\coth\left(\frac{d}{2}\right)\right) b_{-1} + \left(\left(-2 + c\coth\left(\frac{d}{2}\right)\right) b_0 \exp(\xi_n) + \left(-2 + c + c\coth\left(\frac{d}{2}\right)\right) b_1 \exp(\xi_n) \mp c\sqrt{b_0^2 - 4b_{-1}b_1}\right)}{2\left(b_{-1} + \left(b_0 + b_1 \exp(\xi_n)\right)\exp(\xi_n)\right)}, \tag{23}$$

Where $\xi_n = dn + ct + \zeta$, b_1, b_0, b_{-1}, c and d are non-zero real constants. Furthermore, if we take $b_1 = b_{-1} = 1$ and $b_0 = \mp 2$ in Equation 23, then we get the formal solitary wave solutions to Equation 7 as:

$$u_{n,2}(t) = -1 + \frac{c}{2}\left(\coth\left(\frac{d}{2}\right) + \tanh\left(\frac{1}{2}(dn + ct + \zeta)\right)\right), \tag{24}$$

$$u_{n,3}(t) = -1 + \frac{c}{2}\left(\coth\left(\frac{d}{2}\right) + \coth\left(\frac{1}{2}(dn + ct + \zeta)\right)\right), \tag{25}$$

Where c and d are non-zero real constants.

Next, substituting Equation 22 into Equation 18 leads to another more general exponential function solution to Equation 7 as:

$$u_{n,4}(t) = -1 + c\left(1 + \coth(d)\right) - \frac{2cb_{-1}}{b_{-1} + b_1 \exp\left(2(dn + ct + \zeta)\right)} \tag{26}$$

Where b_{-1}, b_0, c and d are non-zero real constants. Moreover, if we take $b_1 = \mp 1$ and $b_{-1} = 1$ in Equation 26, then we get other formal solitary wave solutions to Equation 7 as:

$$u_{n,5}(t) = -1 + c\left(\coth(d) + \tanh(dn + ct + \zeta)\right), \quad (27)$$

$$u_{n,6}(t) = -1 + c\left(\coth(d) + \coth(dn + ct + \zeta)\right), \quad (28)$$

Where c and d are non-zero real constants.

Remark

To the best of our knowledge, the discrete Burgers equation first appeared in Hirota (2004) and has not been studied by any other expansion methods so far. The obtained generalized solitary wave solutions with free parameters might imply some fascinating physical meanings hidden in Equation 7. Of course, we can set the parameters equal to other values, resulting in different solitary wave shapes. These free parameters might be related to the initial and/or the boundary data for the problem, as well.

Conclusion

We presented a successful demonstration of how the Exp-function method may be adapted to NDDEs by analyzing the discrete Burgers equation. It appears that new solutions are obtained. The correctness of the results is assured by putting them back into the original equation with the aid of MATHEMATICA. Of course, we are unable to give further details about the real physical meaning of our analytic solutions due to the lack of experimental and theoretical basis related to these solutions. The method is reliable and effective, as well as does not require a large amount of run-time.

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