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Full Length Research Paper

Input-to-state stable (ISS) and exponential chaotic lag synchronization

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In this paper, the lag synchronization problem is considered for chaotic dynamical systems with external disturbance via input-to-state stability (ISS). A sufficient condition for determining the lag synchronization between the drive and corresponding response systems is derived based on Lyapunov theory. The proposed ISS lag synchronization controller guarantees the exponential lag synchronization and achieves the bounded lag synchronization error for any bounded disturbance. The effectiveness of the proposed schemes is verified via numerical simulation.

Key words: Input-to-state stability (ISS), chaotic systems, lag synchronization, linear matrix inequality (LMI), Lyapunov theory.

INTRODUCTION

Chaos synchronization is an interesting subject, which has attracted much attention particularly since it was suggested in Pecora and Carroll (1996) that it could be applied to secure communication. Originally, chaos synchronization refers to the state in which two chaotic systems will have identical state trajectories for $t \to \infty$, which is now called complete synchronization or identical synchronization. It has been widely explored in a variety of fields including physical, chemical and ecological systems (Chen and Dong, 1998; Ahn, 2009).

Input-to-state stability (ISS) is an interesting concept first introduced in Sontag (1989) to nonlinear control systems. The ISS property is concern with the continuity of state trajectories on the initial states and the inputs. Roughly speaking, a system is ISS if every state trajectory corresponding to a bounded control remains bounded, and the trajectory eventually becomes small if the input signal is small no matter what the initial state is. It had been widely accepted as an important concept in control engineering and many research results have been reported (Sontag, 1990; Jiang et al., 1994; Sontag and Wang, 1995; Christofides and Teel, 1996; Sontag, 1998; Angeli and Nesic, 2001; Ahn, 2010b).

On the other hand, experimental results have been shown that the complete synchronization of chaos is practically impossible for the finite transmission speed of signals (Shahverdiev et al., 2002; Taherion and Lai, 1999; Barsella and Lepers, 2002). Different from complete synchronization, chaotic lag synchronization appears as a coincidence of shift-in-time states of interactive systems. It is just synchronization lag that makes lag synchronization practically available (Huang et al., 2001). Thus, knowledge of the lag synchronization is of considerable practical importance (Li, 2009; Ahn et al., 2009 ; Ahn, 2010a). To the best of our knowledge, however, for the ISS based lag synchronization of chaotic systems with external disturbance, there is no result in the literature so far. This situation motivates our investigation.

In this paper, we present a new control scheme for the ISS lag synchronization of chaotic dynamical systems with external disturbance. By the presented scheme, the lag synchronization error converges exponentially and remains bounded for any bounded disturbance. Based on linear matrix inequality (LMI) formulation, a sufficient condition for the ISS lag synchronization is represented in terms of the LMI, which can be solved efficiently by using recently developed convex optimization algorithms (Boyd

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et al., 1994). Furthermore, we show that the proposed lag synchronization sheme can guarantee the integral ISS lag synchronization (Angeli et al., 2000).

PROBLEM FORMULATION

Throughout this paper, we will use the following definitions:

Definition 1

A function $\gamma : R_{\geq 0} \to R_{\geq 0}$ is a K function if it is continuous, strictly increasing and $\gamma(0) = 0$.

Definition 2

A function $\gamma : R_{\geq 0} \to R_{\geq 0}$ is a K_∞ function if it is a K function and also $\gamma(s) \to \infty$ as $s \to \infty$.

Definition 3

A function $\beta : R_{\geq 0} \times R_{\geq 0} \to R_{\geq 0}$ is a KL function if, for each fixed $t \geq 0$, the function $\beta(\cdot, t)$ is a K function, and for each fixed $s \geq 0$, the function $\beta(s, \cdot)$ is decreasing and $\beta(s,t) \to 0$ as $t \to \infty$.

Consider a class of chaotic systems:

$$\dot{x}(t) = Ax(t) + Bf(x(t)) \tag{1}$$

where $x(t) \in R^n$ is the state vector, $f(x(t)) \in R^n$ is a nonlinear function vector, $A \in R^{n \times n}$ and $B \in R^{n \times n}$ are known constant matrices. The system (1) is considered as a drive system. Now we design a response system as

$$\dot{z}(t) = Az(t) + Bf(z(t)) + u_l(t) + u_n(t) + Gd(t) \tag{2}$$

where $z(t) \in R^n$ is the state vector of the response system, $G \in R^{n \times k}$ is a known constant matrix of the controlled response system. $u_l(t) \in R^m$, $u_n(t) \in R^n$, and $d(t) \in L_\infty^k$ are the linear control input, the nonlinear control input, and the disturbance input, respectively.
Defining the lag synchronization error $e(t) = z(t) - x(t - \tau)$, where $\tau > 0$ is the synchronization lag. Then we obtain the lag synchronization error system

$$\dot{e}(t) = Ae(t) + B\big(f(z(t)) - f(x(t-\tau))\big) + u_l(t) + u_n(t) + Gd(t) \tag{3}$$

Definition 4 (Exponential lag synchronization)

The error system (3) is exponentially lag-synchronized if the lag synchronization error $e(t)$ satisfies

$$\|e(t)\| < Me^{-Nt} \tag{4}$$

where M and N are positive scalars.

Definition 5 (ISS lag synchronization)

The error system (3) is ISS lag-synchronized if there exist a K function $\gamma(s)$ and a $\mathrm{K}L$ function $\beta(s,t)$, such that, for each disturbance input $d(t) \in L_\infty^k$ and each initial lag synchronization error $e(0) \in R^n$, it holds that

$$\|e(t)\| \leq \beta(\|e(0)\|, t) + \gamma(\|d(t)\|) \tag{5}$$

for each $t \geq 0$.

Subsequently, we design the controllers $u_l(t)$ and $u_n(t)$ guaranteeing the ISS lag synchronization if there exists the disturbance input $d(t)$. It will be shown that these controllers $u_l(t)$ and $u_n(t)$ can guarantee the exponential lag synchronization when the disturbance input $d(t)$ disappears.

ISS and exponential lag synchronization

Now we present the LMI feasibility problem for achieving the ISS lag synchronization in the following theorem.

Theorem 1

If there exist $P = P^T > 0$, $Q = Q^T > 0$, $R = R^T > 0$, *and* Y *such that*

$$\begin{bmatrix} A^T P + PA + Y + Y^T + Q & PG \\ G^T P & -R \end{bmatrix} < 0, \tag{6}$$

then the ISS lag synchronization is achieved and the controllers are given by $u_l(t) = P^{-1} Y(z(t) - x(t-\tau))$ *and* $u_n(t) = -B(f(z(t)) - f(x(t-\tau)))$.

Proof: If we apply the linear control input $u_l(t) = Ke(t)$

to the error system (3), we obtain

$$\dot{e}(t) = (A+K)e(t) + B(f(z(t)) - f(x(t-\tau))) + Gd(t) + u_n(t) \quad (7)$$

If we select the nonlinear control input $u_n(t) = -B(f(z(t)) - f(x(t-\tau)))$, we have

$$\dot{e}(t) = (A+K)e(t) + Gd(t). \quad (8)$$

Consider a quadratic Lyapunov function $V(e(t)) = e^T(t)Pe(t)$. Note that $V(e(t))$ satisfies the following Rayleigh inequality (Strang, 1986):

$$\lambda_{\min}(P)\|e(t)\|^2 \le V(e(t)) \le \lambda_{\max}(P)\|e(t)\|^2 \quad (9)$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ are the maximum and minimum eigenvalues of the matrix. The time derivative of $V(e(t))$ is

$$\begin{aligned}\dot{V}(e(t)) &= e^T(t)[A^TP + PA + PK + K^TP]e(t) + e(t)^T PGd(t) + d^T(t)G^T Pe(t) \\ &= \begin{bmatrix} e(t) \\ d(t) \end{bmatrix}^T \begin{bmatrix} A^TP + PA + PK + K^TP + Q & PG \\ G^TP & -R \end{bmatrix} \begin{bmatrix} e(t) \\ d(t) \end{bmatrix} \\ &\quad - e^T(t)Qe(t) + d^T(t)Rd(t). \end{aligned} \quad (10)$$

If the following matrix inequality is satisfied

$$\begin{bmatrix} A^TP + PA + PK + K^TP + Q & PG \\ G^TP & -R \end{bmatrix} < 0, \quad (11)$$

we have

$$\dot{V}(e(t)) < -e^T(t)Qe(t) + d^T(t)Rd(t) \quad (12)$$

$$\le -\lambda_{\min}(Q)\|e(t)\|^2 + \lambda_{\max}(R)\|d(t)\|^2 \quad (13)$$

Define functions $\alpha_1(r)$, $\alpha_2(r)$, $\alpha_3(r)$, and $\alpha_4(r)$ as

$$\alpha_1(r) \cong \lambda_{\min}(P)r^2, \quad (14)$$

$$\alpha_2(r) \cong \lambda_{\max}(P)r^2, \quad (15)$$

$$\alpha_3(r) \cong \lambda_{\min}(Q)r^2, \quad (16)$$

$$\alpha_4(r) \cong \lambda_{\max}(R)r^2. \quad (17)$$

Note that $\alpha_1(r)$, $\alpha_2(r)$, $\alpha_3(r)$, and $\alpha_4(r)$ are K_∞ functions. From (9) and (13), we can obtain

$$\alpha_1(\|e(t)\|) \le V(e(t)) \le \alpha_2(\|e(t)\|) \quad (18)$$

$$\dot{V}(e(t)) \le -\alpha_3(\|e(t)\|) + \alpha_4(\|d(t)\|) \quad (19)$$

According to Sontag and Wang (1995), it is concluded that $V(e(t))$ is an ISS-Lyapunov function and the ISS lag synchronization is achieved. If we let $Y = PK$, (11) is equivalently changed into the LMI (6). Then the gain matrix of the linear control input $u_l(t)$ is given by $K = P^{-1}Y$.

Corollary 1

If there is no the disturbance input, the control inputs $u_l(t)$ and $u_n(t)$ proposed in Theorem 1 can guarantee the exponential lag synchronization.

Proof: When $d(t) = 0$, we obtain

$$\begin{aligned}\dot{V}(e(t)) &< -\lambda_{\min}(Q)\|e(t)\|^2 \\ &\le -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} V(e(t)) \end{aligned} \quad (20)$$

This implies that

$$V(e(t)) < V(e(0))\exp\left(-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}t\right). \quad (21)$$

Using (9), we have

$$\begin{aligned}\|e(t)\| &< \sqrt{\frac{V(e(0))}{\lambda_{\min}(P)}\exp\left(-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}t\right)} \\ &= \sqrt{\frac{e^T(0)Pe(0)}{\lambda_{\min}(P)}}\exp\left(-\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}t\right) \end{aligned} \quad (22)$$

This guarantees the exponential lag synchronization.

INTEGRAL ISS LAG SYNCHRONIZATION

First, we introduce the following definitions:

Definition 6

A function $\gamma : R_{\geq 0} \rightarrow R_{\geq 0}$ is a positive definite function if it is zero at 0 and positive otherwise.

Definition 7 (Integral ISS lag synchronization)

The error system (3) is integral ISS lag-synchronized if there exist a K_{∞} function $\alpha(\cdot)$, a $\mathrm{K}\Lambda$ function $\beta(\cdot,\cdot)$, and a K function $\gamma(\cdot)$ such that

$$\alpha(\|e(t)\|) \leq \beta(\|e(0)\|, t) + \int_0^t \gamma(\|d(\tau)\|)d\tau \tag{23}$$

Next, we show that the controllers in Theorem 1 guarantee the integral ISS lag synchronization.

Corollary 2

The control inputs $u_l(t)$ and $u_n(t)$ proposed in Theorem 1 can guarantee the integral ISS lag synchronization.

Proof

We use the proof of Theorem 1. The function $\alpha_3(r)$ in (16) is a positive definite function. In addition, the function $\alpha_4(r)$ in (17) is regarded as a K function. Thus, according to Angeli et al. (2000), it is concluded that $V(e(t))$ is an integral ISS-Lyapunov function and the integral ISS lag synchronization is achieved.

APPLICATION TO LORENZ SYSTEM

Considering the following Lorenz chaotic system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -x_1(t)x_3(t) \\ x_1(t)x_2(t) \end{bmatrix}. \tag{24}$$

Here, we let $G = 1\ 1\ 1^{T}$ and $\tau = 0.5$. If we apply Theorem 1 to the system (23), we obtain the following feasible solution:

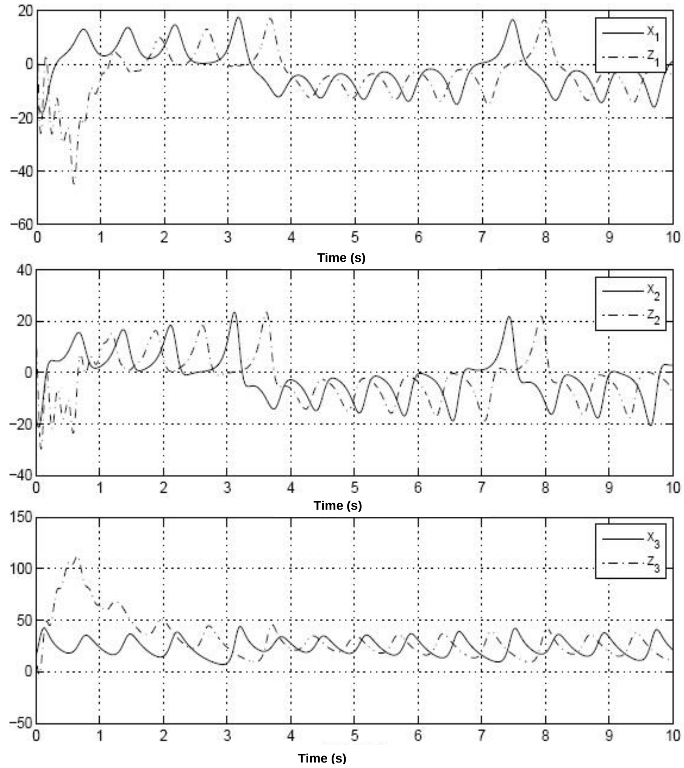

Figure 1. State trajectories.

$$P = \begin{bmatrix} 0.8973 & -0.3589 & -0.3589 \\ -0.3589 & 0.8973 & -0.3589 \\ -0.3589 & -0.3589 & 0.8973 \end{bmatrix}, \quad Q = \begin{bmatrix} 1.2562 & 0.0000 & 0.0000 \\ 0.0000 & 1.2562 & 0.0000 \\ 0.0000 & 0.0000 & 1.2562 \end{bmatrix}, \tag{25}$$

$$R = 1.2562,\ Y = \begin{bmatrix} 17.7656 & -27.2707 & -0.9669 \\ -10.7727 & 3.2301 & 3.8583 \\ 6.4700 & -1.5853 & 1.1365 \end{bmatrix}. \tag{26}$$

Figure 1 shows state trajectories for drive and response systems when the initial conditions are given by

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} -15.8 \\ -17.48 \\ 15.64 \end{bmatrix}, \quad \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \end{bmatrix} = \begin{bmatrix} 10.8 \\ 12 \\ 12 \end{bmatrix}, \tag{27}$$

and the disturbance input $d(t)$ is given by a Gaussian noise with mean 0 and variance 100. This figure shows the lag synchronization between drive system and response system with a synchronization lag $\tau = 0.5$. We can see from Figure 2 that the lag synchronization is achieved and the lag synchronization error $e(t)$ is bounded for the disturbance input $d(t)$.

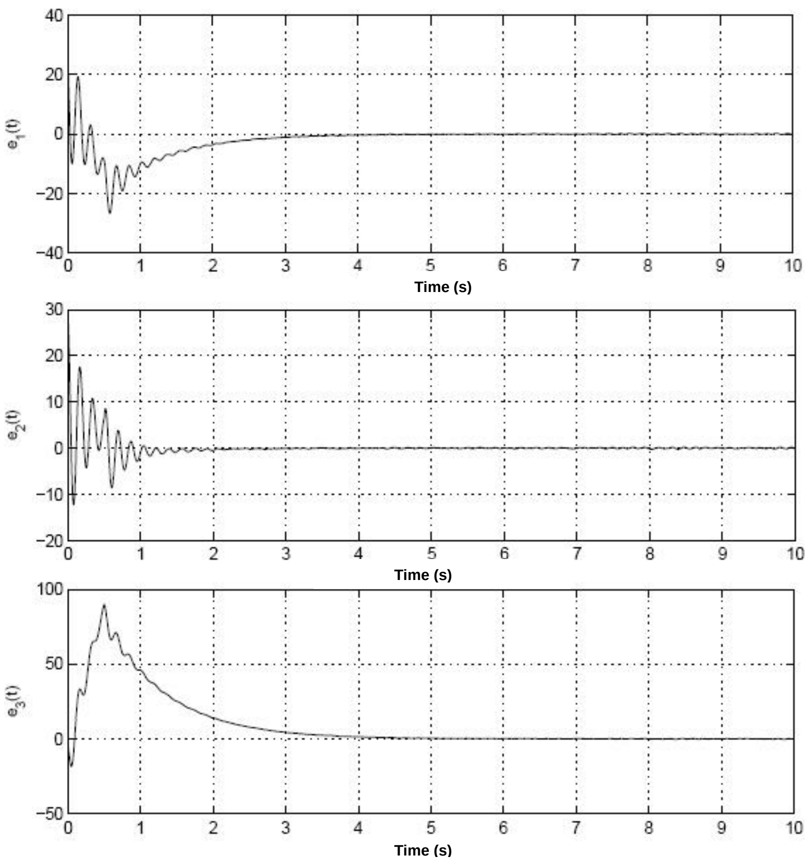

Figure 2. Lag synchronization errors.

Conclusion

In the paper, the exponential lag synchronization has been investigated based on ISS approach. An LMI condition for the ISS lag synchronization of chaotic systems with disturbance input was derived. This condition also guaranteed the exponential lag synchronization when there is no disturbance input. In addition, the integral ISS lag synchronization was shown to be possible under this LMI condition. A numerical simulation was given to show the effectiveness and feasibility of the proposed method.

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