International Journal of the Physical Sciences Vol. 6(1), pp. 93-97, 4 January, 2011
Available online at http://www.academicjournals.org/IJPS
DOI: 10.5897/IJPS10.594
ISSN 1992 - 1950 ©2011 Academic Journals

*Full Length Research Paper*

# Approximate Kerr–like interior and exterior solutions for a very slowly rotating star of perfect fluid

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Accepted 30 December, 2010.

**Approximate Kerr- like interior and exterior solutions for a very slowly rotating star of perfect fluid is presented here which is valid for a very slowly rotating and hence very slightly oblate star of perfect fluid rotating with a very small constant angular velocity. Corresponding approximate exterior metric represents approximate Kerr exterior metric for a perfect fluid. The aim here is to obtain Kerr-like approximate exterior and interior solutions for a very slowly rotating star of perfect fluid simply following standard procedure used to obtain Schwarzschild solutions.**

**Key words:** Slowly rotating oblate star, approximate Kerr- like metric, exterior interior solution.

## INTRODUCTION

The first solution of Einstein field equation for a single spherical and non-rotating mass has been derived by Schwarzschild (1916). The next solution for a rotating star has been derived by Kerr (1965). However the Schwarzschild metric is very nice, but it is describing only for, non-rotating and spherical objects. By Schwarzschild metric it is possible to match the Schwarzschild vacuum exterior to a Schwarzschild fluid interior and it is more general for static spherically symmetric perfect fluid solutions. But the problem of finding a rotating fluid interior which can be matched to Kerr exterior or to any asymptotically flat vacuum exterior solution has proven very difficult and it is a major unsolved problem in general relativity (Wiltshire, 2003; Lorenzo, 2004).

Wahlquist (1968) considered this problem and described a solution with a new metric. Bradley et al. (2000) contradicted Wahliquist's metric. Here we are trying to derive an approximate Kerr-like interior and exterior solution for a very slowly rotating star and slightly oblate of perfect fluid with a different line element.

As we know most of stars are not exactly in spherical form but rather slightly oblate in shape. This essentially involves obtaining the interior solution to Einstein's equations Einstein (1916).

$$R^i_j - \frac{1}{2} g^i_j R = -8\pi G T^i_j \qquad (1)$$

The quantities in the right-hand side of the above equations vanish where no matter is present. In this purpose, we start by describing the situation (Landau and Lifshitz, 1987) for a very slow rotating and hence very slight oblate star in the following spheroid coordinate system. Here we supposed $r >> a$ and $c = 1$ (velocity of light).

$$x=(r^2+a^2)^{1/2} Sin\theta Cos\varphi \quad y=(r^2+a^2)^{1/2} Sin\theta Sin\varphi \quad z=rCos\theta \quad t=t \qquad (2)$$

In this surfaces $r =$ constant, are oblate ellipsoids of rotation. In these coordinate systems, the rotating flat-space metric assumes the following form (Nikouravan, 2001).

$$ds^2 = -\left[\frac{r^2+a^2Cos^2\theta}{r^2+a^2}\right]dr^2 - (r^2+a^2Cos^2\theta)d\theta^2 - (r^2+a^2)Sin^2\theta d\varphi^2 - 2\omega(r^2+a^2)\,Sin^2\theta d\varphi dt + \left[1-\omega^2(r^2+a^2)Sin^2\theta\right]dt^2 \qquad (3)$$

Therefore in the presence of matter, the metric (3) takes the form:

$$ds^2=-e^{\lambda}\left[\frac{r^2+a^2Cos^2\theta}{r^2+a^2}\right]dr^2-(r^2+a^2Cos^2\theta)d\theta^2-(r^2+a^2)Sin^2\theta d\varphi^2-2\omega(r^2+a^2)\,Sin^2\theta d\varphi dt+e^{\upsilon}\left[1-\omega^2(r^2+a^2)Sin^2\theta\right]dt^2 \quad (4)$$

Here supposed $\lambda=\lambda(r,\theta)$ and $\upsilon=\upsilon(r,\theta)$ not only depends on $r$ but also depends on $\theta$, and is a constant angular velocity.

The assumption used here enables us to study the behavior of the object not only in the radial direction but also along the angle $\theta$ simultaneously. However, if the basis of this metric is in the spherical coordinate system $(r,\theta,\varphi)$, but basically the metric (4) explains the shape of any stars and/or galaxies in more complete form so that the spherical form is only a part of this coordinate system. Otherwise, instead of working in terms of $(\xi,\eta,\varphi)$ with difficulty in calculation we can transform the coordinate system in term of $(r,\theta,\varphi)$ as an easier way.

## APPROXIMATE KERR-LIKE INTERIOR SOLUTION

To arrive at the solution let us start with the metric (4). When we assume that the star is rotating very slowly and hence $\omega$ being very small, the terms involving $\omega^2$ in the metric (4) may be neglected as compared with the other higher order terms. It can even be shown that the terms involving $\omega^2$ in $R_{ij}$'s are smaller by many orders of magnitude in comparison with other terms at the surface of the Sun or planets. Therefore, it is justifiable to neglect the terms involving $\omega^2$ compared to other higher order terms. The metric (4) therefore assumes the form:

$$ds^2=-e^{\lambda}\left[\frac{r^2+a^2Cos^2\theta}{r^2+a^2}\right]dr^2-(r^2+a^2Cos^2\theta)d\theta^2-(r^2+a^2)Sin^2\theta\,d\varphi^2+2\omega\,(r^2+a^2)\,Sin^2\theta\,d\varphi\,dt+e^{\upsilon}dt^2 \quad (5)$$

Let us write $\rho^2\equiv r^2+a^2\,Cos^2\theta$ and $\Delta\equiv r^2+a^2$. Then the above metric can be written as,

$$ds^2=-e^{\lambda}\left[\frac{\rho^2}{\Delta}\right]dr^2-\rho^2d\theta^2-\Delta\,Sin^2\theta\,d\varphi^2-2\omega\Delta\,Sin^2\theta\,d\varphi\,dt+e^{v}dt^2 \quad (6)$$

Therefore, $g_{ij}$ are given by,

$$g_{ij}=\begin{pmatrix}-e^{\lambda} & 0 & 0 & 0\\ 0 & -\rho^2 & 0 & 0\\ 0 & 0 & -\Delta Sin^2\theta & -\omega\,\Delta Sin^2\theta\\ 0 & 0 & -\omega\,\Delta Sin^2\theta & e^{\upsilon}\end{pmatrix} \quad (7)$$

and,

$$g=\left\|g_{ij}\right\|=-e^{\lambda+\upsilon}\rho^4\,Sin^2\theta \quad (8)$$

The Non-zero Christoffel symbols of second kind are given as follows:

$$\Gamma^1_{11}=\frac{\lambda'}{2}+\left(\frac{ra^2\,Sin^2\theta}{\rho^2\Delta}\right),\ \Gamma^1_{12}=\frac{\dot{\lambda}}{2}-\left(\frac{a^2\,Sin2\theta}{2\rho^2}\right),\ \Gamma^1_{22}=-\left(\frac{re^{-\lambda}\Delta}{\rho^2}\right),\ \Gamma^1_{33}=-\left(\frac{re^{-\lambda}\Delta Sin^2\theta}{\rho^2}\right)$$

$$\Gamma^1_{34}=-\left(\frac{r\omega e^{-\lambda}\Delta Sin^2\theta}{\rho^2}\right),\ \Gamma^1_{44}=-\left(\frac{\upsilon' e^{\upsilon-\lambda}\Delta}{2\rho^2}\right),\ \Gamma^2_{11}=-\left(\frac{\dot{\lambda}e^{\lambda}}{2\Delta}\right)+\left(\frac{a^2\,e^{\lambda}Sin2\theta}{2\rho^2\Delta}\right),\ \Gamma^2_{12}=\frac{r}{\rho^2},$$

$$\Gamma^2_{22}=-\left(\frac{a^2\,Sin2\theta}{2\rho^2}\right),\ \Gamma^2_{33}=-\left(\frac{\Delta Sin2\theta}{2\rho^2}\right),\ \Gamma^2_{34}=-\left(\frac{\omega\Delta Sin2\theta}{2\rho^2}\right),\ \Gamma^2_{44}=\left(\frac{\dot{\upsilon}e^{\upsilon}}{2\rho^2}\right),\ \Gamma^2_{11}=\left(\frac{r}{\Delta}\right),$$

$$\Gamma^3_{14}=-\left(\frac{\omega r}{\Delta}\right)+\left(\frac{\omega\upsilon'}{2}\right),\ \Gamma^3_{23}=-Cot\theta,\ \Gamma^3_{24}=-\omega Cot\theta+\left(\frac{\omega\upsilon'}{2}\right),\ \Gamma^4_{14}=\frac{\upsilon'}{2},\ \Gamma^2_{24}=\frac{\dot{\upsilon}}{2} \quad (9)$$

For the calculation of all Christoffel symbols of first and second kind and also the Ricci tensors which are very tedious, it is possible to use the software Mathematica (Nikouravan, 2009). Here a prime denotes derivative with respect to $r$, ($\lambda'=\frac{\partial\lambda}{\partial r}$) and the dot denotes derivative respect to $t$ ($\dot{\lambda}=\frac{\partial\lambda}{\partial t}$). Then $R_{ij}$'s are given as follows:

$$R_{11}=\frac{\upsilon''}{2}+\frac{\upsilon'^2}{4}-\frac{\lambda'\upsilon'}{4}-\frac{\lambda'}{4}+\left(\frac{e^{\lambda}}{2r^2}\right)\left[\ddot{\lambda}+\frac{\dot{\lambda}^2}{2}+\frac{\dot{\lambda}\dot{\upsilon}}{2}+\dot{\lambda}Cot\theta\right]+\left[small\ terms\ \omega^2, a^3\ or\ a^4\right]$$

$$R_{22}=-1+e^{-\lambda}+re^{-\lambda}\frac{(\upsilon'-\lambda')}{2}+\frac{\ddot{\lambda}}{2}+\frac{\ddot{\upsilon}}{2}+\frac{(\dot{\lambda}^2+\dot{\upsilon}^2)}{4}+\left[small\ terms\ \omega^2, a^3\ or\ a^4\right]$$

$$R_{33}=Sin^2\theta\left[-1+e^{-\lambda}+re^{-\lambda}\frac{(\upsilon'-\lambda')}{2}+\frac{(\dot{\lambda}+\dot{\upsilon})}{2}\,Cot\theta\right]$$

$$R_{44}=-e^{(\nu-\lambda)}\left[\frac{\upsilon''}{2}+\frac{\upsilon'^2}{4}-\frac{\lambda'\upsilon'}{4}+\frac{\upsilon'}{r}\right]-\left(\frac{e^{\upsilon}}{2r^2}\right)\left[\ddot{\upsilon}+\frac{\dot{\upsilon}^2}{2}+\frac{\dot{\lambda}\dot{\upsilon}}{2}+\dot{\upsilon}Cot\theta\right]$$

$$R_{34}=R_{43}=-\omega Sin^2\theta\left[-1+e^{-\lambda}+re^{-\lambda}\frac{(\upsilon'-\lambda')}{2}+\frac{(\dot{\lambda}+\dot{\upsilon})}{2}\,Cot\theta\right]$$

$$R_{12}=R_{21}=-(\dot{\lambda}+\dot{\nu})\left[\frac{1}{2r}+\frac{\upsilon'}{4}\right]+\frac{\dot{\upsilon}}{2} \quad (10)$$

In $R_{ij}$'s, we can neglect the terms containing $\omega^2$, $a^2$, $a^3$, $a^4$ as they are very small. Similar way the terms involving $\dot{\lambda}^2, \dot{\upsilon}^2$ and $\dot{\lambda}\dot{\upsilon}$ can also be neglected.

For zeroth approximation we may even neglect terms involving $\dot{\lambda}$ and $\dot{\upsilon}$. Terms involving $\dot{\lambda}$ and $\dot{\upsilon}$ are not zero but are very small so that we can approximate $\dot{\lambda}\approx 0$ and $\dot{\upsilon}\approx 0$. This is rather stringent approximation but to obtain zeroth order solution we make this approximation. Nevertheless, we do consider that the star rotates extremely slow and hence have a very small bulge. Under this approximation scheme, we get $R^i_j$ from $R_{ij}$ and obtained Einstein's field equations as follows:

$$e^{-\lambda}\left(\frac{\upsilon'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}=8\pi\rho$$

$$e^{-\lambda}\left(\frac{\upsilon''}{2}+\frac{\upsilon'^2}{4}-\frac{\lambda'\upsilon'}{4}-\frac{\lambda'}{2r}+\frac{\upsilon'}{2r}\right)=8\pi\rho$$

$$e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}=8\pi\rho \tag{11}$$

These are same as those in Schwarzschild static interior solution but with a difference that while integrating them the constants of integration $A$ and $B$ turn out to be the function of $\theta$ and not just pure constants as they are in the Schwarzschild static case. This is because, although in stringent approximation $\dot{\lambda}\approx 0$ and $\dot{\upsilon}\approx 0$, but $\lambda$ and $\upsilon$ are functions of $\theta$. The relationship between $A(\theta)$ and $B(\theta)$ is,

$$A(\theta)=B(\theta)\left(1-\frac{r_1^2}{R^2}\right) \tag{12}$$

Where,

$$R^2=\frac{3}{8\pi\rho} \tag{13}$$

and $r_1$ is the equatorial radius of the spheroid. One can therefore arrive at the metric for a very slowly rotating and very slightly oblate star of perfect fluid of constant density $\rho$ and rotating with a constant velocity $\omega$ to be,

$$ds^2=-\left(\frac{1}{1-\frac{r^2}{R^2}}\right)\left[\frac{r^2+a^2Cos^2\theta}{r^2+a^2}\right]dr^2-(r^2+a^2Cos^2\theta)d\theta^2-(r^2+a^2)Sin^2\theta d\varphi^2+ \tag{14}$$
$$2\omega(r^2+a^2)\,Sin^2\theta d\varphi dt+\left[A(\theta)-B(\theta)\left(\frac{1}{1-\frac{r^2}{R^2}}\right)\right]^2dt^2$$

where relation between $A(\theta)$ and $B(\theta)$ is given by Equation (12). Now it remains to determine $B(\theta)$. Ramsey (1961) and MacMillan (1958) have given the guidelines to find the expression for a Newtonian potential for a rotating oblate sphere to be,

$$V=\frac{M}{r}+\frac{r_1^3}{r^3}\left[\frac{\omega^2r_1^2}{2}-\frac{\varepsilon \mathrm{M}}{\mathrm{r_1}}\right]\left(Cos^2\theta-\frac{1}{3}\right) \tag{15}$$

where,

$$\omega^2=\frac{16\pi\rho\varepsilon}{15} \tag{16}$$

And $\rho$ being the constant density of an oblate sphere of equatorial radius $r_1$, $\varepsilon$ the oblateness and $M$ is the mass of the oblate sphere contained within the equatorial radius $r_1\ (M=M(r_1))$.

Taking $\rho=\frac{3M}{4\pi r_1^3}$ for a slightly oblate sphere and expressing $V$ in terms of $M$ and $\varepsilon$ at the equator $r=r_1$, we get the potential with a negative sign as,

$$V=-\frac{M}{r_1}\left(1+\frac{\varepsilon}{5}\right)\left(1-3\,Cos^2\theta\right) \tag{17}$$

To determine $B(\theta)$ and consequently $A(\theta)$, we use the well known result, that is, approximation,

$$g_{44}=1+2V \tag{18}$$

where, $V$ is the internal Newtonian potential of a rotating and slightly oblate sphere at $r_1$, as given by Equation (15) and get $B(\theta)$ to be,

$$B(\theta)=\frac{1}{2}\left[\frac{1-\frac{r_1^2}{R^2}\left(1+(\frac{\varepsilon}{5})\,(1-3\,Cos^2\theta)\right)}{\left(1-\frac{r_1^2}{R^2}\right)}\right] \tag{19}$$

Note that $\varepsilon=\frac{15\omega^2}{16\pi\rho}$ hence, even for a small $\omega$, $\rho$ may be such that the ratio $\frac{\omega^2}{\rho}$ has small but significant value and hence may not be neglected and hence in this situation $\varepsilon$ will have a small but still significant value. In this situation, (14) along with (12) and (19), represents the gravitational field of a slowly rotating and slightly oblate sphere. Corresponding exterior metric is,

$$ds^2=-\left(\frac{1}{1-\frac{2M}{r}}\right)\left[\frac{r^2+a^2Cos^2\theta}{r^2+a^2}\right]dr^2-(r^2+a^2Cos^2\theta)\,d\theta^2-(r^2+a^2)Sin^2\theta d\varphi^2+$$
$$2\omega(r^2+a^2)\,Sin^2\theta d\varphi dt+\frac{1}{4}\left[\frac{\left(1-(\frac{r_1^2}{R^2})\right)\left(1+\frac{\varepsilon}{5}\left(1-3Cos^2\theta\right)\right)}{\left(1-\frac{r_1^2}{R^2}\right)}\right]\left[3\left(1-\frac{r_1^2}{R^2}\right)^{\frac{1}{2}}-\left(1-\frac{r^2}{R^2}\right)^{\frac{1}{2}}\right]^2dt^2 \tag{20}$$

**RELATION WITH KERR EXTERIOR METRIC**

As $a<<r$, the terms involving $\frac{1}{(r^2+a^2)}$, $\frac{1}{(r^2+a^2\,Cos^2\theta)}$, $\frac{(r^2+a^2\,Cos^2\theta)}{(r^2+a^2)}$, $\frac{(r^2+a^2)}{(r^2+a^2\,Cos^2\theta)}$ can be expanded in powers of $\left(\frac{a^2}{r^2}\right)$. The Equation (14) for inside the body and with

neglecting terms $\left(\frac{a^4}{r^4}\right)$ and other higher order terms, in the following form,

$$ds^2 = -\left(\frac{1}{1-\frac{r_1^2}{R^2}}\right)\left[1-\frac{a^2 Sin^2\theta}{r^2}\right]dr^2 - (r^2+a^2Cos^2\theta)d\theta^2 - (r^2+a^2)Sin^2\theta d\varphi^2 +$$
$$2\omega(r^2+a^2)\,Sin^2\theta d\varphi dt + \frac{1}{4}\left[\frac{\left(1-(\frac{r_1^2}{R^2})\right)\left(1+\frac{\varepsilon}{5}(1-3Cos^2\theta)\right)}{\left(1-\frac{r_1^2}{R^2}\right)}\right]\left[3\left(1-\frac{r_1^2}{R^2}\right)^{\frac{1}{2}} - \left(1-\frac{r^2}{R^2}\right)^{\frac{1}{2}}\right]^2 dt^2 \quad (21)$$

The above metric, for outside of the body assumes in the following form,

$$ds^2 = -\left(\frac{1}{1-\frac{2M}{r}}\right)\left[1-\frac{a^2 Sin^2\theta}{r^2}\right]dr^2 - (r^2+a^2Cos^2\theta)d\theta^2 - (r^2+a^2)Sin^2\theta d\varphi^2 +$$
$$+\frac{1}{4}\left[\frac{\left(1-\frac{2M}{r_1}\right)\left(1+\frac{\varepsilon}{5}(1-3Cos^2\theta)\right)}{\left(1-\frac{2M}{r_1}\right)}\right]\left[3\left(1-\frac{2M}{r_1}\right)^{\frac{1}{2}} - \left(1-\frac{2M}{r}\right)^{\frac{1}{2}}\right]^2 dt^2 + 2\omega(r^2+a^2)\,Sin^2\theta d\varphi dt \quad (22)$$

Moreover the Kerr exterior metric, in Boyer and Lindquist coordinates system is given by, Landau and Lifshitz (1987),

$$ds^2 = -\left[\frac{r^2+a^2Cos^2\theta}{r^2-2Mr+a^2}\right]dr^2 - (r^2+a^2Cos^2\theta)d\theta^2 - (r^2+a^2+\frac{2Mra^2}{r^2+a^2Cos^2\theta})Sin^2\theta d\varphi^2 +$$
$$\left[\frac{4Mra^2}{r^2+a^2Cos^2\theta}\right]Sin^2\theta d\varphi dt + \left[1-\frac{2Mr}{r^2+a^2Cos^2\theta}\right]dt^2 \quad (23)$$

In approximation as, $a << r$, then the term $\frac{1}{(r^2 + a^2 Cos^2\theta)}$ can be expanded in powers of $\left(\frac{a^2}{r^2}\right)$. By neglecting terms containing $\left(\frac{a^4}{r^4}\right)$ and higher order terms and similarly approximating the term $\frac{1}{(r^2 - 2Mr + a^2)}$, we can express the Kerr exterior metric (Equation 23) in the following form,

$$ds^2 = -\frac{1}{\left(1-\frac{2M}{r}\right)}\left[1-\frac{a^2 Sin^2\theta}{r^2}\right]dr^2 - (r^2+a^2Cos^2\theta)d\theta^2 - (r^2+a^2)Sin^2\theta d\varphi^2 +$$
$$+\left(\frac{4Ma^2}{r}\right)Sin^2\theta d\varphi dt + \left(1-\frac{2M}{r}\right)dt^2 \quad (24)$$

## RESULTS AND DISCUSSION

By comparing approximate metrics (Equations 24 and 22), we get $\frac{2aJ}{r} = \omega(r^2 + a^2)$. In the same level of approximation, this condition becomes, $\frac{2aJ}{r^3} = \omega$. This condition is approximately valid because $\frac{2a}{r^3} = \omega$, where $J = aM$. The discussion here shows that in approximation, the exterior metric (Equation 22) to the interior metric (Equation 21) obtained in this paper, really corresponds in approximation to the approximate exterior Kerr metric (Equation 24) of exterior metric (Equation 23). Thus we can conclude that the metrics (Equation 14), along with Equations (12) and (19), represents Kerr interior metric.

## CONCLUSION

The aim of this work was to obtain Kerr-like interior and exterior solutions for a slowly rotating star or galaxy with small in angular velocity. This approximate solution has been obtained by the standard procedure which is used in getting the standard Schwarzschild solutions. The advantage of the procedure is that the exterior Kerr solution has been obtained without going into detail, the procedure adopted in obtaining the actual exterior Kerr solution. The objective of the work has been achieved successfully.

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