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Full Length Research Paper

Proximal algorithms for solving mixed bifunction variational inequalities

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In this paper, we suggest and analyze some proximal methods for solving mixed bifunction variational inequalities using the auxiliary principle technique. Convergence of these methods is considered under some mild suitable conditions on the bifunction and the operators. Several cases are also discussed. Results in this paper include some new and known results as special cases.

Key words: Bifunction, variational inequality, auxiliary principle technique, proximal point algorithms, convergence.

INTRODUCTION

Variational inequalities are being used to study a wide class of diverse unrelated problems arising in various branches of pure and applied sciences in a unified framework. It is worth mentioning that the variational inequalities are natural extension of the variational principles, the origin of which can be traced back to Fermat, Newton, Leibniz, Bernoullie, Euler and Largrange. Various generalizations and extensions of variational inequalities have been considered in different directions using novel and innovative techniques. It is well known that the optimality conditions of the minimum of the differentiable convex function can be characterized by the variational inequalities. However, if the convex function is directionally (Gateux) differentiable, then the minimum of the convex function is characterized by a class of variational inequalities which is called the bifunction variationa inequalities, which are quite different from the classical variational inequalities. The formulation and numerical methods for bifunction variational inequalities is given in Crepsi et al. (2004, 2005, 2006, 2008), Fang et al. (2007), Lalitha et al.(2005), Noor (2006), and Noor et al. (2011, 2011a, b).It is natural to study these problems in a unified way since these classes of the variational inequalities are different from each other. Inspired and motivated by the research going on in this field, we consider and introduce a new class of variational inequalities, which is called mixed bifunction variataional inequalities. There are a substantial number of numerical methods including projection technique and its variant forms, Wiener-Hopf equations, auxiliary principle, and resolvent equations methods for solving variational inequalities. However, it is known that projection, Wiener-Hopf equations, and resolvent equations techniques cannot be extended and generalized to suggest and analyze similar iterative methods for solving mixed bifunction variational inequalities due to the nature of the problem. This fact motivated Noor et al. (1993, 2011, a, b) to use the auxiliary principle technique of Glowinski et al (1983). In this paper, we use the auxiliary principle technique to suggest and analyze some proximal type iterative methods for solving the mixed bifunction variational inequalities (Alogrithms 1 to 3). We also consider the convergence criterion of the proposed methods (Algorithm 1) under suitable mild conditions, which are the main results (Theorems 1 and 2) of this paper. Several special cases of our main results are also considered. Results obtained in this paper may be viewed as an improvement and refinement of the previously known results.

PRELIMINARIES

Let H *be* a real Hilbert space, whose inner product and norm are

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denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. Let $F(.,.): \mathrm{H}\times\mathrm{H}\rightarrow R$ be a bifunction and $T: \mathrm{H} \rightarrow H$ be any operator. First of all, we recall the following well-known results and concepts.

For a given nonlinear operator $T: H \rightarrow H$ and the bifunction $F(.,.)$, we consider the problem of finding $u\in H$ such that

$$F(u,v-u)+\langle Tu,v-u\rangle+\phi(v)-\phi(u)\geq 0,\ \forall v\in H, \tag{1}$$

where $\varphi(.)$ is an arbitrary continuous function. An inequality of the type (1) is called the mixed bifunction variational inequality, which is quite different from the existing bifunction variational inequalities. One can easily show that the minimum of the sum of the directionally differentiable convex function and differentiable convex function can be characterized by the mixed bifunction variational inequalities of the form (1).

Special cases

1. If $F(u,v-u)=0$, then problem (1) is equivalent to finding $u\in H$ such that

$$\langle Tu,v-u\rangle+\phi(v)-\phi(u)\geq 0,\ \forall v\in H, \tag{2}$$

which is known as a mixed variational inequality. A wide class of problems arising in elasticity, fluid flow through porous media and optimization can be studied in the general framework of problems (2) (Giannessi et al., 2001; Glowinski et al., 1983; Noor, 1975, 2004, a, c, 2006a; Noor et al., 1993, 2004, 2009, 2011a, b).

2. If $\langle Tu,v-u\rangle=0$, then problem (1) reduces to the problem of finding $u\in H$ such that

$$F(u,v-u)+\phi(v)-\phi(u)\geq 0,\ \forall v\in H, \tag{3}$$

which is known as mixed bifunction variational inequality, considered by Noor (2006) and Noor et al. (2011, 2011a).

3. If $\phi(.)$ is an indicator function on the closed convex set in H, then problem (1) is equivalent to finding $u\in K$ satisfying

$$F(u,v-u)+\langle Tu,v-u\rangle\geq 0,\quad \forall v\in K, \tag{4}$$

which is known as bifunction variational inequality and appears to be new one.

4. If $\varphi(.)$ is an indicator function, then (2) is equivalent to finding $u\in K$ such that

$$\langle Tu,v-u\rangle\geq 0,\quad \forall v\in K, \tag{5}$$

Which is classical variational inequality introduced and studied by Stampacchia (1964).

5. If $\varphi(.)$ is an indicator function, then the problem (3) is equivalent to finding $u\in K$ such that:

$$F(u,v-u)\geq 0,\quad \forall v\in K, \tag{6}$$

which is known as classical bifunction variational inequalitiy (Crepsi et al., 2004, 2005, 2006, 2008; Fang et al., 2007; Lalitha et al., 2005; Noor, 2006).For appropriate and suitable choice of the operators and spaces, one can obtain several new and previously known classes of the mixed bifunction variational inequalities and related optimization problems (Noor, 1975, 1988, 2000, 2003, 2004; Noor et al., 2004, 2011, a, b).

Definition 1

An operator $T: H \rightarrow H$ is said to be:

1. Monotone, if and only if,

$$\langle Tu-Tv,u-v\rangle\geq 0,\quad \forall u,v\in H.$$

2. Partially relaxed strongly -monotone, if there exists a constant $\sigma>0$ such that $\langle Tu-Tv,z-v\rangle\geq -\sigma\|z-u\|^2,\quad \forall u,v,z\in H.$

Note that for $z=u$, partially relaxed strong monotonicity reduces to monotonicity of the operator *T*.

Definition 2

A bifunction $F: H\times H \rightarrow R$ is said to be:

1.Monotone, if and only if,

$$F(u,v-u)+F(v,u-v)\leq 0,\quad \forall\ u,v\in H.$$

2. Partially relaxed strongly -monotone, if there exists a constant $\alpha>0$ such that

$$F(u,v-u)+F(v,z-v)\leq \mu\|z-u\|^2,\quad \forall u,v,z\in H.$$

MAIN RESULTS

Here, we use the auxiliary principle technique to suggest and analyze some implicit iterative methods for solving mixed bifunction variational inequalities (1). This technique is due to Glowinski et al. (1981). We consider three different types of implicit iterative schemes for solving the mixed equilibrium-variational inequalities and this is the main motivation of this paper.

For a given $u\in H$, consider the problem of finding $w\in H$ such that

$$\rho F(w,v-w)+\langle \rho Tw,v-w\rangle+\langle w-u,v-w\rangle\geq \rho\phi(w)-\rho\phi(v),\ \forall v\in H. \tag{7}$$

Note that, if $w=u$, then $w\in K$ is a solution of (1). This observation enables us to suggest and analyze the following iterative method for solving mixed equilibrium-variational inequalities (1).

Algorithm 1

For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme

$$\rho F(u_{n+1}, v-u_{n+1}) + \langle \rho T u_{n+1}, v-u_{n+1}\rangle + \langle u_{n+1}-u_n, v-u_{n+1}\rangle \geq \rho\phi(u_{n+1}) - \rho\phi(v), \quad \forall v \in H \tag{8}$$

For a given $u \in H$, consider the problem of finding a $w \in H$ such that F(u,v) ====> F(u,v-u)

$$\rho F(u,v) + \langle \rho T w, v-w\rangle + \langle w-u, v-w\rangle \geq \rho\phi(w) - \rho\phi(v), \ \forall v \in H.$$

Note that if $w = u$, then $w \in K$ is a solution of (1). This observation enables us to suggest and analyze the following proximal iterative method for solving mixed bifunction variational inequalities (1).

Algorithm 2

For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme

$$\rho F(u_n, v-u_{n+1}) + \langle \rho T u_{n+1}, v-u_{n+1}\rangle + \langle u_{n+1}-u_n, v-u_{n+1}\rangle \geq \rho\phi(u_{n+1}) - \rho\phi(v), \quad \forall v \in H$$

For a given $u_0 \in H$, consider the problem of finding a $w \in H$ such that

$$\rho F(w, v-w) + \langle \rho T u, v-w\rangle + \langle w-u, v-w\rangle \geq \rho\phi(w) - \rho\phi(v), \forall v \in H,$$

Note that if $w = u$, then $w \in K$ is a solution of (1). This observation enables us to suggest and analyze the following iterative method for solving mixed bifunction variational inequalities.

Algorithm 3

For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme

$$\rho F(u_{n+1}, v-u_{n+1}) + \langle \rho T u_n, v-u_{n+1}\rangle + \langle u_{n+1}-u_n, v-u_{n+1}\rangle \geq \rho\phi(u_{n+1}) - \rho\phi(v), \quad \forall v \in H.$$

Some special cases of these algorithms are as:

1. If $F(u, v-u) = 0$, then Algorithm 1 reduces to the following scheme for mixed variational inequlities given as (2).

Algorithm 4

For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme:

$$\langle \rho T u_{n+1}, v-u_{n+1}\rangle + \langle u_{n+1}-u_n, v-u_{n+1}\rangle \geq \rho\phi(u_{n+1}) - \rho\phi(v), \quad \forall v \in H.$$

If $\langle Tu, v-u\rangle = 0,$ then Algorithm 1 reduces to: the following method for solving the mixed bifunction variational inequality (3).

Algorithm 5

For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme

$$F(u_{n+1}, v-u_{n+1}) + \langle u_{n+1}-u_n, v-u_{n+1}\rangle \geq +\rho\phi(u_{n+1}) - \rho\phi(v), \quad \forall v \in H,$$

which is used for finding the solution of mixed equilibrium problems given as (3).If $\phi(.)$ is an indicator function on the closed convex set K in H, then Algorithm 1 can be used to find the approximate solution of problem (4) and which appears to be new one.

Algorithm 6

For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme:

$$\rho F(u_{n+1}, v-u_{n+1}) + \langle \rho T u_{n+1}, v-u_{n+1}\rangle + \langle u_{n+1}-u_n, v-u_{n+1}\rangle \geq 0, \quad \forall v \in H.$$

For suitable and appropriate choice of $F(.,.), T, \varphi(.)$ and spaces, one can define iterative algorithms as special cases of Algorithms 2 and 3 for finding the solution of different classes of the bifunction variational inequalities. We now study the convergence analysis of Algorithm 1. Convergence analysis of other algorithms can be proved using the same technique.

Theorem 1

Let $u \in H$ be a solution of (1) and $u_{n+1} \in H$ be an approximate solution obtained from Algorithm 1. If bifunction $F(.,.)$ and T are monotone, then:

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_{n+1} - u_n\|^2. \tag{9}$$

Proof

Let $u \in H$ be a solution of (1).Then, replacing v by u_{n+1} in (1), we have

$$\rho F(u,u_{n+1}-u)+\rho\langle Tu,u_{n+1}-u\rangle+\rho\phi(u_{n+1})-\rho\phi(u)\geq 0. \quad (10)$$

Let $u_{n+1} \in H$ be the approximate solution obtained from Algorithm 1. Taking $v=u$ in (8), we have

$$\rho F(u_{n+1},u-u_{n+1})+\langle\rho Tu_{n+1},u-u_{n+1}\rangle+\rho\phi(u)-\rho\phi(u_{n+1})+\langle u_{n+1}-u_n,u-u_{n+1}\rangle\geq 0. \quad (11)$$

Adding (10) and (11), we have

F(u, u_(n+1)-u)+ F(u_n, u-u_n)

which implies that

F(u_n, u-u_n)+ F(u, u_{n+1}-u) (12)

where we have used the monotonicity of the operator T and bifunction $F(.,.)$.

Using the relation

$$2\langle u,v\rangle=\|u+v\|^2-\|u\|^2-\|v\|^2, \qquad \forall u,v\in H \quad \text{in (12),}$$

we have

$$\|u-u_{n+1}\|^2\leq\|u-u_n\|^2-\|u_{n+1}-u_n\|^2,$$

which is the required result (9). □

Theorem 2

Let H be a finite dimensional space. If u_{n+1} is the approximate solution obtained from Algorithm 1 and $u\in H$ is a solution of problem (1), then $\lim_{n\to\infty}u_n=u$.

Proof

Let $u\in H$ be a solution of (1). Then, we see that the sequence $\{\|u-u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Also from (9), we have $\sum_{n=0}^{\infty}\|u_{n+1}-u_n\|^2\leq\|u-u_0\|^2$, which implies that

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=0. \quad (13)$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the sub sequence $\{u_{n_j}\}$ of this sequence converges to $\hat{u}\in H$. Replacing u_n by u_{n_j} in (8) and taking the limit $n_j\to\infty$ and using (13), we have

$$F(\hat{u},v-\hat{u})+\langle T\hat{u},v-\hat{u}\rangle+\phi(v)-\phi(\hat{u})\geq 0,\ \forall v\in H,$$

which shows $\hat{u}$ solves the mixed bifunction variational inequality (1) and $\|u_{n+1}-\hat{u}\|^2\leq\|u_n-\hat{u}\|^2$

Thus, it follows from the aforementioned inequality that the sequence $\{u_n\}$ has exactly one cluster point and $\lim_{n\to\infty}u_n=\hat{u}$, the required result.

Conclusion

In this paper, we have used the auxiliary principle technique to suggest some proximal iterative methods for solving the mixed bifunction variational inequalities. We have analyzed the convergence criteria of these new proximal methods under some very mild conditions on the involved operators. Some special cases are also discussed. These results can be extended for nonconvex bifunction variational inequalities .The technique and ideas used in obtaining the results of this paper inspire further research in this field and related areas.

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