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Dynamics of a higher - order nonlinear rational difference equation

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In this paper, we consider the higher-order nonlinear rational difference equation $x_{n+1} = (\alpha+\beta x_n + \gamma x_{n-k})/(A+Bx_n)$, $n = 0,1,...$, where the parameters $\alpha, \beta, \gamma, A, B \in (0,\infty)$ and the initial conditions $x_{-k},..., x_0 \in (0,\infty)$, $k \geq 1$. We investigate the periodic character, the invariant intervals and the global asymptotic stability of all positive solutions of the equation.

Key words: Difference equation, stability, periodicity, invariant interval, global stability.

INTRODUCTION

The study of properties of rational difference equations has been an area of intense interest in recent years. Related to this subject, are researches done by Devault (2001), Dourbaki (2008), Gibbons (2002), Jia (2010), Kulenovic (2003), Li (2005a, b), Saleh (2006), Sebdani (2006) and Tang (2010a, b). Our aim in this paper is to investigate the dynamical behavior of the following nonlinear rational difference equation:

$$x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-k}}{A+Bx_n}, n = 0,1,... \quad (1)$$

Where initial conditions $x_{-k},..., x_0 \in (0,\infty)$ and the parameters $\alpha, \beta, \gamma, A, B \in (0,\infty)$ and $k \geq 1$.

Definition 1

Let Equation 1 be some interval of real numbers and let:

$$f : I \times I \to I \quad (2)$$

be a continuously differentiable function. Then for every set of initial conditions:

$y_{-k},..., y_0 \in I$, the difference equation:

$$y_{n+1} = f(y_n, y_{n-k}) \ , \ n \in N_0 \quad (3)$$

has a unique solution: $\{y_n\}_{n=-k}^{\infty}$.

A point $\bar{y}$ is called an equilibrium point of Equation 3 if:

$$\bar{y} = f(\bar{y}, \bar{y}) \quad (4)$$

That is,

$$y_n = \bar{y} \ , \ \text{for} \ n \geq 0, \quad (5)$$

is a solution of Equation 3, or equivalently $\bar{y}$ is a fixed point of f.

Definition 2

Let $\bar{y}$ be an equilibrium point of Equation 3 and assume that I is some interval of real numbers.

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(i) The equilibrium $\bar{y}$ is called "locally stable" (or stable) if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, if $y_{-k},...,y_0 \in I$ and $|y_{-k} - \bar{y}| + ... + |y_0 - \bar{y}| < \delta$, we have $|y_n - \bar{y}| < \varepsilon$ for all $n \geq -k$.

(ii) The equilibrium $\bar{y}$ of Equation 3 is called "locally asymptotically" stable (asymptotically stable) if it is locally stable and if there exists $\gamma > 0$ such that, if I $y_{-k},...,y_0 \in I$ and $|y_{-k} - \bar{y}| + ... + |y_0 - \bar{y}| < \gamma$, $\lim_{n\to\infty} y_n = \bar{y}$.

(iii) The equilibrium $\bar{y}$ of Equation 3 is called a "global attractor" if, for every $y_{-k},...,y_0 \in I$, we have $\lim_{n\to\infty} y_n = \bar{y}$.

(iv) The equilibrium $\bar{y}$ of Equation 3 is called "global asymptotically stable" if it is locally stable and is a global attractor.

(v) The equilibrium $\bar{y}$ of Equation 3 is called "unstable" if it is not stable.

(v) The equilibrium $\bar{y}$ of Equation 3 is called a "source", or a "repeller", if there exists $r > 0$ such that, for all $y_{-k},...,y_0 \in I$ and $|y_{-k} - \bar{y}| + ... + |y_0 - \bar{y}| < r$, there exists $N \geq 1$ such that $|y_N - \bar{y}| \geq r$.

An interval $J \subseteq I$ is called an "invariant interval" for Equation 3 if:

$$y_{-k},...,y_0 \in J \Rightarrow y_n \in J \quad \forall n > 0 \tag{6}$$

That is, every solution of Equation 3 with the initial conditions in J remains in J.

Let:

$$P = \frac{\partial f}{\partial u}(\bar{y},\bar{y}) \quad \text{and} \quad Q = \frac{\partial f}{\partial v}(\bar{y},\bar{y})$$

Where, $f(u,v)$ is the function in Equation 3 and $\bar{y}$ is an equilibrium of Equation 3. Then the equation:

$$y_{n+1} = Py_n + Qy_{n-k} \ , \ n = 0,1,... \tag{7}$$

is called the "linearized equation" associated with Equation 3 about the equilibrium point $\bar{y}$.
Its characteristic equation is:

$$\lambda^{k+1} - P\lambda^k - Q = 0 \tag{8}$$

Theorem 1

(i) If all the roots of Equation 8 lie in the open disk $|\lambda| < 1$, then the equilibrium $\bar{y}$ of Equation 3 is asymptotically stable.
(ii) If at least one root of Equation 8 has absolute value greater than one, then the equilibrium $\bar{y}$ of Equation 3 is unstable (Kocic, 1993) (linearized stability).

Theorem 2

Assume that $P, Q \in R$ and $k \in 1, 2,...$. Then (Kocic, 1993):

$$|P| + |Q| < 1 \tag{9}$$

is a sufficient condition for the asymptotic stability of the difference equation:

$$y_{n+1} = Py_n + Qy_{n-k} \ , \ n = 0,1,... \tag{10}$$

Lemma 1

Consider the difference Equation 3 (Li, 2005).

Let: $I = [a,b]$ be an interval of real numbers and assume that $f : [a,b] \times [a,b] \to [a,b]$ is a continuous function satisfying the following properties:

(i) $f(x,y)$ is non decreasing in each of its arguments.
(ii) The equation:

$$f(x,x) \tag{11}$$

has a unique positive solution in the interval $[a,b]$. Then Equation 3 has a unique equilibrium point $\bar{y} \in [a,b]$ and every solution of Equation 3 converges to $\bar{y}$.

Lemma 2

Consider the difference Equation 3 (DeVault et al., 2001). Let $I = [a,b]$ be an interval of real numbers and assume that $f:[a,b]\times[a,b]\to[a,b]$ is a continuous function satisfying the following properties.

(a) $f(x,y)$ is a non increasing function in x and a non decreasing function in y.
(b) If $(m,M)\in[a,b]\times[a,b]$ is a solution of the following system:

$$m=f(M,m) \quad , \quad M=f(m,M) \tag{12}$$

then $m=M$.

Then Equation 3 has a unique equilibrium point $\bar{y}\in[a,b]$ and every solution of Equation 3 converges to $\bar{y}$.

LOCAL STABILITY AND PERIOD-TWO SOLUTIONS

Here, we will investigate the local stability and the periodic character of Equation 1. By the change of variables $\frac{\beta}{B}y_n=x_n$ reduces Equation 1 to the following difference equation:

$$y_{n+1}=\frac{p+y_n+qy_{n-k}}{r+y_n} \quad , \quad n=0,1,\ldots \tag{13}$$

Where, $p=\frac{\alpha B}{\beta^2}$, $q=\frac{\gamma}{\beta}$ and $r=\frac{A}{\beta}$ with $p,q,r\in(0,\infty)$, $y_{-k},\ldots,y_0\in(0,\infty)$.

Equation 13 has a unique positive equilibrium $\bar{y}$:

$$\bar{y}=\frac{(q+1-r)+\sqrt{(q+1-r)^2+4p}}{2} \tag{14}$$

The linearized equation associated with Equation 13 about the unique positive equilibrium $\bar{y}$ is:

$$z_{n+1}-\frac{1-\bar{y}}{r+\bar{y}}z_n-\frac{q}{r+\bar{y}}z_{n-k}=0 \tag{15}$$

and its characteristic equation is:

$$\lambda^{k+1}-\frac{1-\bar{y}}{r+\bar{y}}\lambda^k-\frac{q}{r+\bar{y}}=0 \tag{16}$$

Theorem 3

(a) Assume that k is odd. Then Equation 13 has a prime period-two solution:

$$\ldots,\varphi,\psi,\varphi,\psi,\ldots \tag{17}$$

if and only if:

$$r+1=q \tag{18}$$

Furthermore when Equation 18 holds (17) is a prime period-two solution of Equation 13 if and only if:

$$\psi>1 \quad \text{and} \quad \varphi=\frac{p+\psi}{\psi-1} \tag{19}$$

(b) Assume that k is even. Then Equation 13 has no period-two solution.

Proof

Assume for the sake of contradiction that there exist two distinct non negative real numbers φ and ψ such that $\ldots,\varphi,\psi,\varphi,\psi,\ldots$ is a prime period-two solution of Equation 13.

(a) Let k be odd. Then $x_{n+1}=x_{n-k}$ and φ, ψ satisfy the following system:

$$\varphi=\frac{p+\psi+q\varphi}{r+\psi}, \quad \psi=\frac{p+\varphi+q\psi}{r+\varphi} \tag{20}$$

Subtracting both sides of the aforementioned two equations, we obtain:

$$(\varphi-\psi)(r+1-q)=0 \tag{21}$$

Since $\varphi\neq\psi$, $r+1=q$. The reverse part is clear by simple computation. So it is omitted.

Now, when Equation 18 holds, $\ldots,\varphi,\psi,\varphi,\psi,\ldots$ is a prime period-two solution of Equation 13 if and only if $\varphi+\psi=\varphi\psi-p$ with $\varphi,\psi\in(0,\infty)$ and $\varphi\neq\psi$. That is equivalent to Equation 19.

(b) Let k be even. Then $x_n=x_{n-k}$ and φ, ψ satisfy the following system:

$$\varphi = \frac{p+\psi+q\psi}{r+\psi}, \quad \psi = \frac{p+\varphi+q\varphi}{r+\varphi} \tag{22}$$

Subtracting both sides of the aforementioned two equations, we obtain:

$$(\varphi-\psi)(r+1+q)=0 \tag{23}$$

Since $\varphi \neq \psi$, $(r+1+q)=0$; this contradicts the hypothesis that $r, q > 0$.

BOUNDEDNESS AND INVARIANT INTERVALS

Here, we will investigate the boundedness and invariant intervals of Equation 13.

Let $\{y_n\}_{n=-k}^{\infty}$ be a nonnegative solution of Equation 13. Then we have the following identities: For $n \in N_0$,

$$y_{n+1}-1=\frac{[y_{n-k}-(r-p)/q]q}{r+y_n}$$

$$y_{n+1}-\frac{r-p}{q}=\frac{q\left[y_{n-k}-\frac{r-p}{q}\right]+(r+y_n)\left[1-\frac{r-p}{q}\right]}{r+y_n}$$

$$y_{n+1}-\frac{p}{r-q}=\frac{q\left[y_{n-k}-\frac{p}{r-q}\right]+y_n\left[1-\frac{p}{r-q}\right]}{r+y_n}(r-q)$$

$$y_{n+1}-y_{n-k}=\frac{(r-q)\left[\frac{p}{r-q}-y_{n-k}\right]+y_n(1-y_{n-k})}{r+y_n} \tag{24}$$

If $r=p+q$, then the unique positive equilibrium $\bar{y}=1$ and in Equation (24:

$$y_{n+1}-1=\frac{(y_{n-k}-1)q}{r+y_n}$$

$$y_{n+1}-y_{n-k}=\frac{(p+y_n)(1-y_{n-k})}{r+y_n} \tag{25}$$

If $q=r$, then the unique positive equilibrium $\bar{y}=\frac{1+\sqrt{1+4p}}{2}$ and in Equation 24,

$$y_{n+1}-y_{n-k}=\frac{p+(1-y_{n-k})y_n}{r+y_n} \tag{26}$$

Let,

$$f(x,y)=\frac{p+x+qy}{r+x} \tag{27}$$

Then,

$$\frac{\partial f}{\partial x}=\frac{(r-p)-qy}{(r+x)^2} \text{ and } \frac{\partial f}{\partial y}=\frac{q}{r+x} \tag{28}$$

Lemma 3

Let $f(x,y)$ be defined in Equation 27. Then the following statements are true:

(i) Assume that $r > p$. Then $f(x,y)$ is increasing in each of its arguments for $y \le \frac{r-p}{q}$ and it is increasing in y and decreasing in x for $y > \frac{r-p}{q}$.

(ii) Assume that $r \le p$. Then $f(x,y)$ is decreasing in x and increasing in y for $x \ge 0$.

CASE 1

$r > q$

Lemma 4

Assume that $r > p+q$ and $\{y_n\}_{n=-k}^{\infty}$ is a non negative solution of Equation 13. Then the following statements are true:

(i) If for some $N \ge 0$, $y_{N-k} < \frac{r-p}{q}$, then $y_{N+1} < 1$.

(ii) If for some $N \ge 0$, $y_{N-k} = \frac{r-p}{q}$, then $y_{N+1} = 1$.

(iii) If for some $N \ge 0$, $y_{N-k} > \frac{r-p}{q}$, then $y_{N+1} > 1$.

(iv) If for some $N \ge 0$, $y_{N-k} \ge \frac{p}{r-q}$, then $y_{N+1} \ge \frac{p}{r-q}$

(v) If for some $N \geq 0$, $y_{N-k} \leq \frac{p}{r-q}$, then $y_{N+1} > y_{N-k}$.

(vi) If for some $N \geq 0$, $y_{N-k} \geq 1$, then $y_{N+1} < y_{N-k}$.

(vii) Equation 13 possesses an invariant interval $\left[\frac{p}{r-q}, \frac{r-p}{q}\right]$ and $\bar{y} \in (p/(r-q), (r-p)/q)$. The interval $\left[\frac{p}{r-q}, 1\right]$ is also an invariant interval of Equation 13 and $\bar{y} \in (p/(r-q), 1)$.

Proof

When $p/(r-q) < 1 < (r-p)/q$ holds, (i) to (vi) can be easily seen from the identities in Equation 24.

(vii) From Lemma 3. (i) the function $f(x, y)$ is increasing in each of its arguments for $y \leq \frac{r-p}{q}$. For

$$y_{-k}, \ldots, y_0 \in \left[\frac{p}{r-q}, \frac{r-p}{q}\right],$$

$$y_1 = f(y_0, y_{-k}) \geq f\left(\frac{p}{r-q}, \frac{p}{r-q}\right) = \frac{p(r+1)}{r^2 - qr + p} > \frac{p}{r-q}$$

$$y_1 = f(y_0, y_{-k}) \leq f\left(\frac{r-p}{q}, \frac{r-p}{q}\right) = 1 < \frac{r-p}{q} \quad (29)$$

implies that $y_1 \in (p/(r-q), 1) \subseteq \left[\frac{p}{r-q}, \frac{r-p}{q}\right]$.

By the induction, $y_n \in (p/(r-q), 1) \subseteq \left[\frac{p}{r-q}, \frac{r-p}{q}\right]$ for every $n \in N$. On the other hand, $r > p+q$ implies that:

$$\bar{y} = \frac{(q+1-r) + \sqrt{(q+1-r)^2 + 4p}}{2} < \frac{(q+1-r) + \sqrt{(q+1-r)^2 + 4(r-q)}}{2} = 1 \quad (30)$$

Furthermore, $\bar{y}$ is the positive root of the quadratic equation:

$$y^2 + (r-q-1)y - p = 0 \quad (31)$$

Since,

$$\left(\frac{p}{r-q}\right)^2 + (r-q-1)\frac{p}{r-q} - p = \frac{p(p+q-r)}{(r-q)^2} < 0$$

then we have that $\bar{y} > p/(r-q)$. That is, $\bar{y} \in [p/(r-q), 1] \subset [p/(r-q), (r-p)/q]$.

Lemma 5

Assume that $r = p+q$ and $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of Equation 13. Then the following statements are true:

(i) If for some $N \geq 0$, $y_{N-k} > 1$, then $y_{N+1} > 1$.

(ii) If for some $N \geq 0$, $y_{N-k} = 1$, then $y_{N+1} = 1$.

(iii) If for some $N \geq 0$, $y_{N-k} < 1$, then $y_{N-k} < 1$.

(iv) If for some $N \geq 0$, $y_{N-k} > 1$, then $y_{N+1} < y_{N-k}$.

(v) If for some $N \geq 0$, $y_{N-k} < 1$, then $y_{N+1} > y_{N-k}$.

Lemma 6

Assume that $q < r < p+q$ and $\{y_n\}_{n=-k}^{\infty}$ is a nonnegative solution of Equation 13. Then the following statements are true:

(i). If for some $N \geq 0$, $y_{N-k} \leq \frac{p}{r-q}$, then $y_{N+1} \leq \frac{p}{r-q}$.

(ii). If for some $N \geq 0$, $y_{N-k} > \frac{p}{r-q}$, then $y_{N+1} < y_{N-k}$.

(iii). If for some $N \geq 0$, $y_{N-k} \leq 1$, then $y_{N+1} > y_{N-k}$.

(iv). Equation 13 possesses an invariant interval $\left[1, \frac{p}{r-q}\right]$ and $\bar{y} \in (1, p/(r-q))$.

Furthermore, if $r \leq p$ then $y_n \geq 1$ for all $n > N$. If $r > p$, then the following statements are also true:

(a) If for some $N \geq 0$, $y_{N-k} > \frac{r-p}{q}$, then $y_{N+1} > 1$.

(b) If for some $N \geq 0$, $y_{N-k} = \frac{r-p}{q}$, then $y_{N+1} = 1$.

(c) If for some $N \geq 0$, $y_{N-k} < \frac{r-p}{q}$, then $y_{N+1} < 1$.

Proof

When $(r-p)/q < 1 < p/(r-q)$ holds, (i) to (iii) can easily be seen from identities in Equation 24.

(iv) From Lemma 2 (i) the function $f(x, y)$ is decreasing in x and increasing in y in $1, p/(r-q)$.

For $y_{-k}, \ldots, y_0 \in \left[1, \frac{p}{r-q}\right]$,

$$y_1 = f(y_0, y_{-k}) \leq f(1, \frac{p}{r-q}) \leq f(\frac{r-p}{q}, \frac{p}{r-q})$$

$$= \frac{\frac{pr}{r-q} + \frac{r-p}{q}}{r + \frac{r-p}{q}} < \frac{p}{r-q}$$

$$y_1 = f(y_0, y_{-k}) \geq f(\frac{p}{r-q}, 1)$$

$$\geq f(\frac{p}{r-q}, \frac{r-p}{q}) = 1 > \frac{r-p}{q} \qquad (32)$$

Which implies that $y_1 \in 1, p/(r-q)$. By the induction, $y_n \in 1, p/(r-q)$ for every $n \in N$.

On the other hand, $r < p + q$ implies that:

$$\bar{y} = \frac{(q+1-r) + \sqrt{(q+1-r)^2 + 4p}}{2}$$

$$> \frac{(q+1-r) + \sqrt{(q+1-r)^2 + 4(r-q)}}{2} = 1 \qquad (33)$$

Then like the proof of the Lemma 4 (vii), it can be proved that $\bar{y} \in 1, p/(r-q)$.

CASE 2

$r \leq q$

Lemma 7

Assume that $r \leq q$. Then the interval $1, \infty$ is an invariant interval of Equation 13 and $\bar{y} > 1$. If $r > p$, then the following statements are also true:

(i) If for some $N \geq 0$, $y_{N-k} > \frac{r-p}{q}$, then $y_{N+1} > 1$.

(ii) If for some $N \geq 0$, $y_{N-k} = \frac{r-p}{q}$, then $y_{N+1} = 1$.

(iii) If for some $N \geq 0$, $y_{N-k} < \frac{r-p}{q}$, then $y_{N+1} < 1$.

Proof

When $(r-p)/q < 1$ holds, (i) to (iii) can be easily seen from the identities in Equation 24. Further, $r \leq q$ implies that:

$$\bar{y} = \frac{(q+1-r) + \sqrt{(q+1-r)^2 + 4p}}{2}$$

$$\geq \frac{1 + \sqrt{1+4p}}{2} > 1 \qquad (34)$$

GLOBAL ASYMPTOTIC STABILITY FOR THE CASE

$r \geq q$

Here, we will discuss the global attractivity of the unique positive equilibrium of Equation 13.

Theorem 4

Assume that $r \geq q$. Then the positive equilibrium $\bar{y}$ of Equation 13 is a global attractor. The proof is finished by considering the following four cases (Theorem 6, 8, 10 and 12).

Theorem 5

Assume that $r > p + q$ holds and $y_n {}_{n=-k}^{\infty}$ is a nonnegative solution of Equation 13. If $y_0 \in [p/(r-q), (r-p)/q]$, then $y_n \in [p/(r-q), (r-p)/q]$ for $n \in N$. Furthermore, every nonnegative solution of Equation 13 lies eventually in the interval $[p/(r-q), (r-p)/q]$.

Proof

Assume that $p/(r-q) < 1 < (r-p)/q$ holds.

If $y_0 \in [p/(r-q), (r-p)/q]$, from Lemma 4 (i) and (iv), we have $p/(r-q) < y_n < 1 < (r-p)/q$ for $n \geq 1$.

Theorem 6

Assume that $r > p+q$ holds. Then the unique positive equilibrium $\bar{y}$ of Equation 13 is a global attractor of all nonnegative solutions of Equation 13.

Proof

From Theorem 5 and Lemma 4 (vii) imply that every non negative solution of Equation 13 eventually enters the interval $[p/(r-q), (r-p)/q]$. Furthermore, from Lemma 3 (i) the function $f(x,y)$ is increasing in each of its arguments in $[p/(r-q), (r-p)/q]$ and the equation:

$$y = \frac{p+y+qy}{r+y} \quad (35)$$

has a unique positive solution on the interval $[p/(r-q), (r-p)/q]$. From Lemma 1, Equation 13 is a global attractor of all non negative solutions of Equation 13.

Theorem 7

Assume that $r = p+q$ and $y_n \,{}_{n=-k}^{\infty}$ is a nontrivial nonnegative solution of Equation 13. Then the sequence $y_n \,{}_{n=0}^{\infty}$ is monotonic and $\lim_{n\to\infty} y_n = 1$.

Proof

When $r = p+q$ holds, we know that $\bar{y} = 1$. From Lemma 5, if $y_0 > 1$, then $y_n \,{}_{n=0}^{\infty}$ is decreasing and bounded below 1. If $y_0 = 1$, then $y_n = 1$ for $n \in N$. If $y_0 < 1$, then $y_n \,{}_{n=0}^{\infty}$ is increasing and bounded above by 1. For each cases the sequence $y_n \,{}_{n=0}^{\infty}$ converges to 1.

Theorem 8

Assume that $r = p+q$. Then the unique positive equilibrium $\bar{y}$ of Equation 13 is a global attractor of all non negative solutions of Equation 13.

Theorem 9

Assume that $q < r < p+q$ holds and $y_n \,{}_{n=-k}^{\infty}$ is a nonnegative solution of Equation 13. If $y_0 \in [1, p/(r-q)]$, then $y_n \in [1, p/(r-q)]$ for $n \in N$. Furthermore, every non negative solution of Equation 13 lies eventually in the interval $[1, p/(r-q)]$.

Proof

Assume that $(r-p)/q < 1 < p/(r-q)$ holds. If $y_0 \in [1, p/(r-q)]$, from Lemma 6 (i) and (iv), we have $1 \leq y_n \leq p/(r-q)$ $y_n \in [1, p/(r-q)]$ for $n \geq 1$.

Theorem 10

Assume that $q < r < p+q$ holds. Then the unique positive equilibrium $\bar{y}$ of Equation 13 is a global attractor of all nonnegative solutions of Equation 13.

Proof

From Theorem 9 imply that every non negative solution of Equation 13 eventually enters the interval $[1, p/(r-q)]$. Furthermore, from Lemma 3 (i) the function $f(x,y)$ is non increasing in x and non decreasing in y in $[1, p/(r-q)]$. Let $m, M \in [1, p/(r-q)]$ be a solution of the system.

$$m = \frac{p+M+qm}{r+M}, \quad M = \frac{p+m+qM}{r+m} \quad (36)$$

Then we find that $(m-M)(r+1-q) = 0$. Since $r+1 \neq q$, we get that $m = M$. From Lemma 2, Equation 13 is a global attractor of all nonnegative solutions of Equation 13.

Theorem 11

Assume that $r = q$ holds and $y_n \,{}_{n=-k}^{\infty}$ is a nonnegative solution of Equation 13. Then every nonnegative solution of Equation 13 lies eventually in the interval $1,\infty$.

Proof

When $r \le p$

$$y_{n+1} \ge \frac{r + y_n + q y_{n-k}}{r + y_n} \ge 1 \quad, n \in N_0 \tag{37}$$

So, it is true for this case.

Assume that $r > p$. So, $(r-p)/q < 1$ holds.

If $y_0 \ge (r-p)/q$, then from Lemma 7 (i) and (ii), we have $y_1 \ge 1$. If $y_0 < (r-p)/q$, then from Lemma 7 (iii), we have $y_1 < 1$. If $y_1 \ge (r-p)/q$, the proof is aforementioned.

Assume for the sake of contradiction $y_n < (r-p)/q$ for all $n \in N$. From identities in Equation 24, we get $y_n < y_{n+1} < (r-p)/q$ for $n \ge 1$, from which it follows the sequence y_n is increasing and there is a finite $\lim_{n\to\infty} y_n \le (r-p)/q$; this contradicts the fact that $\bar{y} = \left[1 + \sqrt{1+4p} \;\; / 2 \right] > 1$.

Theorem 12

Assume that $r = q$ holds. Then the unique positive equilibrium $\bar{y}$ of Equation 13 is a global attractor of all non negative solutions of Equation 13.

Proof

From Lemma 3 (ii), the function $f(x, y)$ is non increasing in x and non decreasing in y in $1,\infty$.

Let $m, M \in 1,\infty$ be a solution of the system:

$$m = \frac{p + M + qm}{r + M}, \quad M = \frac{p + m + qM}{r + m} \tag{38}$$

Then we find that $(m-M)(r+1-q) = 0$. Since $r + 1 \ne q$, we get $m = M$. From Lemma 2, Equation 13 is a global attractor of all nonnegative solutions of Equation 13.

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