Full Length Research Paper

Timelike tubes with Darboux frame in Minkowski 3-space

Sezai Kızıltuğ^{1*} and Yusuf Yaylı²

¹Department of Mathematics, Faculty of Science, Atatürk University, Erzurum, Turkey.

²Department of Mathematics, Faculty of Science, Ankara University, Ankara, Turkey.

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In this study, we define a timelike tube around a spacelike curve with timelike binormal by taking Darboux frame instead of Frenet frame in Minkowski 3-space E_1^3 . Subsequently, we compute the Gaussian curvature, the mean curvature, and the second Gaussian curvature of timelike tube with Darboux frame and obtained some characterizations for special curves on this timelike tube around a spacelike curve with timelike binormal.

Key words: Darboux frame, tube surface, Minkowski space-time, mean and gaussian curvatures, second gaussian curvature, asymptotic and geodesic curves, line of curvature.

INTRODUCTION

Tube surface is a special case of the canal surface. A canal surface is defined as one of the parameter family of spheres. Or, a canal surface is the envelope of a moving sphere with varying radius, defined by the trajectory $\alpha(u)$ (spine curve) of its centers and a radius function $r(u)$. If the radius function $r(u) = r$ is a constant, then the canal surface is called a tube or tubular surface. Several Geometers have studied canal surfaces and tubes surfaces, and have obtained many interesting results. Maekawa et al (1998) carried out a research on the necessary and sufficient conditions for the regularity of tube surfaces. Also, Ro and Yoon (2009) studied the tubes of Weingarten type in a Euclidean 3-space. Abdel-Aziz and Khalifa (2011) studied the Weingarten timelike tube surfaces around a spacelike curve in Minkowski 3-Space E_1^3 . Recently, Dogan and Yayli (2012) investigated tubes with Darboux frame in Euclidean 3-space. Surface theory had been a popular topic for many researchers in many aspects. Furthermore, using the curves, surfaces, and canal surfaces are the most popular in computer aided geometric design such as designing models of internal and external organs, preparing of terrain-infrastructures, constructing of blending surfaces, reconstructing of shape, robotic path planning.

In this study, we investigate the timelike tube surface

around a spacelike curve with timelike binormal by taking the Darboux frame instead of Frenet frame in Minkowski 3-Space E_1^3 and the characterizations of some special curves on this timelike tube are given.

PRELIMINARIES

The Minkowski 3-space E_1^3 is the Euclidean 3-space $\mathbb{R}^3$ provided with the indefinite inner product given as:

$$\langle , \rangle = -d_{y_1}^2 + d_{y_2}^2 + d_{y_3}^2$$

Where,

(y_1, y_2, y_3) is natural coordinates of E_1^3 .

Since $\langle , \rangle$ is indefinite inner product, recall that a vector $\beta \in E_1^3$ can have one of the three causal characters it can be spacelike if $\langle \beta, \beta \rangle > 0$ and or $\beta = 0$, timelike if $\langle \beta, \beta \rangle < 0$ and null (lightlike) if $\langle \beta, \beta \rangle = 0$ and $\beta \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(u)$ in E_1^3 can locally called be as spacelike, if its velocity vector $\alpha'(u)$ is spacelike. Recall that the norm of a vector is given by $\|\beta\| = \sqrt{|\langle \beta, \beta \rangle|}$ and that the spacelike $\alpha(u)$ is said to be of unit speed if $\langle \alpha'(u), \alpha'(u) \rangle = +1$. Moreover, the velocity of curve $\alpha(u)$ is the function $\vartheta(v) = \|\alpha'(u)\|$.

*Corresponding author. E-mail: skiziltug24@hotmail.com.

Denote by $\{T(u), N(u), B(u)\}$ the moving Frenet frame along the curve $\alpha(u)$ in the Minkowski 3-space E_1^3 . Then Frenet formula of $\alpha(u)$ in the space E_1^3 is defined by (Neill, 1983)

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

Where,

$$\langle T, T \rangle = \langle N, N \rangle = 1, \langle B, B \rangle = -1, \langle T, N \rangle = \langle T, B \rangle = \langle N, B \rangle = 0$$

and κ and τ are curvature and torsion of the spacelike curve $\alpha(u)$, respectively. Then $\alpha(s)$ is a spacelike curve with timelike principal normal N and spacelike binormal B . The vector product of the vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is defined by

$$x \times y = \det \begin{bmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_2y_1 - x_1y_2).$$

We denote a timelike surface M in E_1^3 by:

$$M(u, v) = (m_1(u, v), m_2(u, v), m_3(u, v))$$

Let U be the standard unit of normal vector field on a surface M defined by:

$$U = \frac{M_u \times M_v}{\|M_u \times M_v\|}$$

Where $M_u = \frac{\partial M(u, v)}{\partial u}$ and $M_v = \frac{\partial M(u, v)}{\partial v}$. Then the first fundamental form I and the second fundamental form II of a surface M are defined by (Gray, 1999), respectively

$$I = Edu^2 + 2Fdudv + Gdv^2$$

$$II = edu^2 + 2fdudv + gdv^2$$

Where,

$$E = \langle M_u, M_u \rangle, F = \langle M_u, M_v \rangle, G = \langle M_v, M_v \rangle$$

$$e = \langle M_{uu}, U \rangle, f = \langle M_{uv}, U \rangle, g = \langle M_{vv}, U \rangle.$$

On the other hand, the Gaussian curvature K and the mean curvature H are:

$$K = \frac{eg - f^2}{EG - F^2}$$

$$H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)},$$

respectively.

If the second fundamental form is non-degenerate; $eg - f^2 \neq 0$. In this case, one define formally the second Gaussian curvature K_{II} a similar one to Brioschi's formula for the Gaussian curvature obtained on M replacing the components of the first fundamental form E, F, G by those of the second fundamental form e, f, g as (Khalifa, 2011).

$$K_{II} = \frac{1}{(eg - f^2)^2} \begin{vmatrix} -\frac{1}{2}e_{vv} + f_{uv} - \frac{1}{2}g_{uu} & \frac{1}{2}e_u & f_u - \frac{1}{2}e_v \\ f_v - \frac{1}{2}g_u & e & f \\ \frac{1}{2}g_v & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_v & g_u \\ \frac{1}{2}e_v & e & f \\ \frac{1}{2}g_u & f & g \end{vmatrix}.$$

TIMELIKE TUBES WITH DARBOUX FRAME IN E_1^3

In this section, we define a timelike tube surface around spacelike curve with timelike binormal by taking Darboux frame instead of Frenet frame and compute the coefficients of first and second fundamental form, the Gaussian curvature K , the mean curvature H , and the second Gaussian curvature K_{II} for this timelike tube, respectively.

Let $\alpha = \alpha(u): (a, b) \rightarrow E_1^3$ be a spacelike unit speed curve with a timelike binormal B , where u is the arc length parameter of α . Consider M as a timelike tube surface parametrized by (Khalifa, 2011)

$$M(u, v) = \alpha(u) + r(N(u)\cosh v + B(u)\sinh v).$$

Let M be timelike surface and $\alpha = \alpha(u): (a, b) \rightarrow E_1^3$ be unit speed spacelike curve on the timelike surface M . Then, Darboux frame $\{T, Y = Z \times T, Z\}$ is well-defined along the spacelike curve α where T is the tangent of α and Z is the unit normal of M . The derivative formulae of the Darboux frame of $\alpha(u)$ is given by:

$$\begin{bmatrix} T' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} 0 & k_g & -k_n \\ k_g & 0 & t_r \\ k_n & t_r & 0 \end{bmatrix} \begin{bmatrix} T \\ Y \\ Z \end{bmatrix} \quad (1)$$

$$\langle T, T \rangle = \langle Z, Z \rangle = 1, \langle Y, Y \rangle = -1$$

$$\langle T, Y \rangle = \langle T, Z \rangle = \langle Y, Z \rangle = 0.$$

In this formulae k_g, k_n and t_r are called the geodesic curvature, the normal curvature and the geodesic torsion, respectively (Uğurlu and Kocayigit, 1996). The relations between geodesic curvature, the normal curvature, the geodesic torsion and κ, τ are given as follows:

$$k_g = \kappa \cosh v, k_n = \kappa \sinh v, t_r = \tau + \frac{dv}{du}.$$

Besides, in the differential geometry of surfaces, for a curve $\alpha(u)$ lying on a surface M the following are

well-known:

- i) $\alpha(s)$ is a geodesic curve if and only if $k_g = 0$,
- ii) $\alpha(s)$ is an asymptotic curve if and only if $k_n = 0$,
- iii) $\alpha(s)$ is a principal line if and only if $t_r = 0$.

Let the center spacelike curve $\alpha(u)$ be on the timelike surface M . Since the characteristic circles of canal surface lie in the plane which is perpendicular to the tangent center of spacelike curve $\alpha(u)$, we can write timelike tube surface with Darboux frame as:

$$M(u, v) = \alpha(u) + r(Y(u)\cosh v + U(u)\sinh v) \quad (2)$$

where U is the unit normal of the surface M along the curve $\alpha(u)$.

Let M be a timelike tube surface with Darboux frame in E_1^3 given in Equation 2. So, from the derivative formulas of Darboux frame, partial differentiation of M with respect to u and v are as follows:

$$M_u = (1 + rk_g \cosh v + rk_n \sinh v)T + rt_r \sinh v Y + rt_r \cosh v U,$$

$$M_v = r \sinh v Y + r \cosh v U.$$

Therefore, we find the components of the first fundamental form of M to be:

$$E = (1 + rk_g \cosh v + rk_n \sinh v)^2 + r^2 t_r^2, F = -r^2 t_r, G = r^2. \quad (3)$$

On the other hand, the unit surface normal vector field Z is obtained by:

$$Z = \frac{M_u \times M_v}{\|M_u \times M_v\|} = \cosh v Y + \sinh v U. \quad (4)$$

The second order partial differentials of M are found as:

$$\begin{aligned} M_{uu} &= [r(k'_g + k_n t_r) \cosh v + r(k'_n + k_g t_r) \sinh v]T \\ &\quad + [r(k_g^2 + t_r^2) \cosh v + k_g + r(k_n k_g + t'_r) \sinh v]Y \\ &\quad + [r(t'_r - k_g k_n) \cosh v - k_n + r(t_r^2 - k_n^2) \sinh v]U, \end{aligned}$$

$$M_{uv} = (rk_g \sinh v + rk_n \cosh v)T + rt_r \cosh v Y + rt_r \sinh v U,$$

$$M_{vv} = r \cosh v Y + r \sinh v U.$$

From Equation 4 and the last equations we find the second fundamental for coefficients as follows:

$$e = (k_g \cosh v + k_n \sinh v)(-1 + rk_g \cosh v + rk_n \sinh v) + rt_r^2, f = rt_r, g = r. \quad (5)$$

Thus, the Gaussian curvature K , the mean curvature H , and the second Gaussian curvature K_{II} are given by:

$$\begin{aligned} K &= \frac{k_g \cosh v + k_n \sinh v}{r(-1 + rk_g \cosh v + rk_n \sinh v)}, \\ H &= \frac{2r(-k_g \cosh v - k_n \sinh v) + 1}{2r(-1 + rk_g \cosh v + rk_n \sinh v)}, \end{aligned} \quad (6)$$

$$K_{II} = \frac{(3 + k_g \cosh 2v + k_n \sinh 2v - 12rk_n \sinh^3 v - 12rk_g \cosh^3 v - 8r^2 k_n^2 \sinh^4 v - 8r^2 k_g^2 \cosh^4 v) \operatorname{sech}^2 v}{8r(-1 + rk_g \sinh v + rk_n \cosh v)^2},$$

respectively.

Where k_g , k_n , and t_r are the geodesic curvature, the normal curvature, and the geodesic torsion of $\alpha(u)$, respectively.

SOME CHARACTERIZATIONS FOR SPECIAL CURVES ON THIS TIMELIKE TUBE SURFACE

In this section, we investigate the relation between parameter curves and special curves such as geodesic curves, asymptotic curves, and lines of curvature on this timelike tube surface $M(u, v)$.

Theorem 1

For the timelike tube surface $M(u, v)$:

- i) v - parameter curves are also geodesic.
- ii) u - parameter curves are also geodesic if and only if k_g , k_n and t_r of $\alpha(u)$ satisfy the equation system:

$$\begin{aligned} k_g \sinh v + k_n \cosh v + rk_g k_n \cosh 2v - \frac{1}{2}r(k_n^2 - k_g^2) \sinh 2v - rt'_r &= 0 \\ -(k'_g + k_n t_r) \cosh v - (k'_n + k_g t_r) \sinh v &= 0. \end{aligned} \quad (7)$$

Proof

For u - and v - parameter curves, we get

$$Z \times M_{vv} = r \sinh v \cosh v T - r \sinh v \cosh v T = 0$$

$$\begin{aligned} Z \times M_{uu} &= \left[k_g \sinh v + k_n \cosh v + rk_g k_n \cosh 2v - \frac{1}{2}r(k_n^2 - k_g^2) \sinh 2v - rt'_r \right] T \\ &\quad + r[k'_g \sinh v \cosh v - k'_n (\sinh v)^2 - k_g t_r (\sinh v)^2 + k_n t_r \sinh v \cosh v] Y \\ &\quad - r[k'_g (\cosh v)^2 + k'_n \sinh v \cosh v + k_g t_r \sinh v \cosh v + k_n t_r (\cosh v)^2] U, \end{aligned}$$

- i) Since $Z \times M_{vv} = 0$, v -parameter curves are also geodesics.

ii) Because T , Y and U are linearly independent, $Z \times M_{uu} = 0$ if and only if

$$k_g \sinh v + k_n \cosh v + r k_g k_n \cosh 2v - \frac{1}{2} r (k_n^2 - k_g^2) \sinh 2v - r t_r' = 0,$$

$$k_g' \sinh v \cosh v - k_n' (\sinh v)^2 - k_g t_r' (\sinh v)^2 + k_n t_r' \sinh v \cosh v = 0,$$

$$k_g' (\cosh v)^2 + k_n' \sinh v \cosh v + k_g t_r' \sinh v \cosh v + k_n t_r' (\cosh v)^2 = 0$$

By the last two equations, we have

$$-(k_g' + k_n t_r') \cosh v - (k_n' + k_g t_r') \sinh v = 0.$$

Then k_g , k_n and t_r hold the equation system

$$k_g \sinh v + k_n \cosh v + r k_g k_n \cosh 2v - \frac{1}{2} r (k_n^2 - k_g^2) \sinh 2v - r t_r' = 0$$

$$-(k_g' + k_n t_r') \cosh v - (k_n' + k_g t_r') \sinh v = 0.$$

This completes proof.

Corollary 1

Let $\alpha(u)$ be a geodesic on timelike tube surface M . If u -parameter curves are also geodesic on $M(u, v)$, then the curvatures k and τ of $\alpha(u)$ satisfy the equation:

$$r\tau^2 - k \sinh v + rk^2 (\sinh v)^2 = c$$

Where, c is a constant.

Proof

Since the center curve $\alpha(u)$ is an geodesic curve, $k_g = 0$. If we replace $k_g = 0$ in Equation 7, we obtain:

$$k \cosh v (1 - r k \sinh v) - r \tau' = 0,$$

$$-k \tau \cosh v - k' \sinh v = 0.$$

In the first equation above, if we leave $k \cosh v$ alone and substitute this in the second equation we will get:

$$r \tau \tau' - k' \sinh v + r k k' (\sinh v)^2 = 0.$$

If we integrate the last equation, it follows that

$$r \tau^2 - k \sinh v + r k^2 (\sinh v)^2 = c.$$

Corollary 2

Let $\alpha(u)$ be a asymptotic on timelike tube surface M . If

u -parameter curves are also asymptotic on $M(u, v)$, then the curvatures k and τ of $\alpha(u)$ satisfy the equation

$$r \tau^2 - k \cosh v + r k^2 (\cosh v)^2 = c$$

Where, c is a constant.

Proof

Since the center curve $\alpha(u)$ is an asymptotic curve, $k_n = 0$. If we replace $k_n = 0$ in Equation 7, we obtained:

$$k \sinh v (1 - r k \cosh v) - r \tau' = 0,$$

$$-k \tau \sinh v - k' \cosh v = 0.$$

In the first equation above, if we leave $k \sinh v$ alone and substitute this in the second equation we will get:

$$r \tau \tau' - k' \cosh v + r k k' (\cosh v)^2 = 0.$$

If we integrate the last equation, it follows that

$$r \tau^2 - k \cosh v + r k^2 (\cosh v)^2 = c.$$

Theorem 2

For the timelike tube surface $M(u, v)$,

i) v -parameter curves cannot also be asymptotic curves on $M(u, v)$,

ii) u -parameter curves are also asymptotic curve on $M(u, v)$ if and only if $M(u, v)$ is generated by a moving sphere with the radius function.

$$r = \frac{k_g \cosh v + k_n \sinh v}{(k_g \cosh v + k_n \sinh v)^2 + t_r^2} = c \quad (8)$$

such that r is a constant.

Proof

i) As $\langle Z \times M_{vv} \rangle = -r(\cosh v)^2 + r(\sinh v)^2 = r \neq 0$ v -parameter curves cannot also be asymptotic curves on $M(u, v)$.

ii) u -parameter curves are also asymptotic curve on $M(u, v)$ if and only if

$$\langle Z \times M_{uu} \rangle = (k_g \cosh v + k_n \sinh v)(r k_g \cosh v + r k_n \sinh v - 1) + r t_r^2 = 0.$$

From this, we get the radius function:

$$r = \frac{k_g \cosh v + k_n \sinh v}{(k_g \cosh v + k_n \sinh v)^2 + t_r^2} = c$$

This completes proof.

Corollary 3

Let u -parameter curves are also asymptotic curves on $M(u, v)$.

i) If the center curve $\alpha(u)$ is a geodesic curve on $M(u, v)$, then

$$r = \frac{k \sinh v}{k^2 (\sinh v)^2 + \tau^2} = c.$$

ii) If the center curve $\alpha(u)$ is a asymptotic curve on $M(u, v)$, then:

$$r = \frac{k \cosh v}{k^2 (\cosh v)^2 + \tau^2} = c.$$

iii) If the center curve $\alpha(u)$ is a line of curvature on $M(u, v)$, then:

$$r = \frac{1}{k_g \cosh v + k_n \sinh v} = c.$$

Proof

Because u - parameter curves are also asymptotic curves on $M(u, v)$, from Equation 8:

$$r = \frac{k_g \cosh v + k_n \sinh v}{(k_g \cosh v + k_n \sinh v)^2 + t_r^2} = c$$

i) Since $\alpha(u)$ is a geodesic curve on $M(u, v)$, $k_g = 0$ So, $k_n = k$, and $t_r = \tau$. If we replace these in Equation 8, we

$$r = \frac{k \sinh v}{k^2 (\sinh v)^2 + \tau^2} = c.$$

obtained:

ii) Since $\alpha(u)$ is a asymptotic curve on $M(u, v)$, $k_n = 0$ So, $k_g = k$ and $t_r = \tau$. If we replace these in Equation 8),

$$r = \frac{k \cosh v}{k^2 (\cosh v)^2 + \tau^2} = c.$$

we obtained:

iii) Since $\alpha(u)$ is a line of curvature on $M(u, v)$, $t_r = 0$. If we replace these in Equation 8, we will get:

$$r = \frac{1}{k_g \cosh v + k_n \sinh v} = c.$$

Theorem 3

The parameter curves of $M(u, v)$ are also lines of curvature if and only if the center curve $\alpha(u)$ is a line of curvature on $M(u, v)$.

Proof

From Equations 3 and 5 we have:

$$F = -r^2 t_r, f = r t_r.$$

According to theorem of line of curvature, the parameter curves on a surface are also lines of curvature if and only if $F = f = 0$. From $F = f = 0$, it concludes that $t_r = 0$, that is, $\alpha(u)$ is a line of curvature on $M(u, v)$.

Theorem 4

For the timelike tube surface $M(u, v)$,

i) If the center curve $\alpha(u)$ is a geodesic curve on $M(u, v)$, then the Gaussian curvature K , the mean curvature H , and the second Gaussian curvature K_{II} of surface $M(u, v)$ are as follows:

$$K = \frac{k \sinh v}{r(-1 + rk \sinh v)}$$

$$H = \frac{-2rk \sinh v + 1}{r(-1 + rk \sinh v)}$$

$$K_{II} = \frac{(3 + \sinh 2v - 12rk(\sinh v)^3 + 8r^2 k^2 (\sinh v)^4)(\cosh v)^2}{8r(-1 + rk \sinh v)^2}$$

ii) If the center curve $\alpha(u)$ is a asymptotic curve on $M(u, v)$, then the Gaussian curvature K , the mean curvature H and the second Gaussian curvature K_{II} of surface $M(u, v)$ are as follows:

$$K = \frac{k \cosh v}{r(-1 + rk \cosh v)}$$

$$H = \frac{-2rk \cosh v + 1}{r(-1 + rk \cosh v)}$$

$$K_{II} = \frac{(3 + \cosh 2v - 12rk(\cosh v)^3 + 8r^2 k^2 (\cosh v)^4)(\sinh v)^2}{8r(-1 + rk \cosh v)^2}$$

Proof

i) Because $\alpha(u)$ is a geodesic curve on $M(u, v)$, $k_g = 0$. So, $k_n = k$ and $t_r = \tau$. If we replace these in Equation 6, we obtained:

$$K = \frac{k \sinh v}{r(-1 + rk \sinh v)}$$

$$H = \frac{-2rk \sinh v + 1}{r(-1 + rk \sinh v)}$$

$$K_{II} = \frac{(3 + \sinh 2v - 12rk(\sinh v)^3 + 8r^2k^2(\sinh v)^4)(\operatorname{csch} v)^2}{8r(-1 + rk \sinh v)^2}$$

ii) Because $\alpha(u)$ is a asymptotic curve on $M(u, v)$, $k_n = 0$. So, $k_g = k$ and $t_r = \tau$. If we replace these in Equation 6, we obtain:

$$K = \frac{k \cosh v}{r(-1 + rk \cosh v)}$$

$$H = \frac{-2rk \cosh v + 1}{r(-1 + rk \cosh v)}$$

$$K_{II} = \frac{(3 + \cosh 2v - 12rk(\cosh v)^3 + 8r^2k^2(\cosh v)^4)(\operatorname{sech} v)^2}{8r(-1 + rk \cosh v)^2}$$

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