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Full Length Research Paper

Oscillatory channel flow for non-Newtonian fluid

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This paper deals with an analytical solution of oscillatory flow of an incompressible second grade fluid in a channel. The flow in the channel is driven by suction at the permeable walls, whereas small amplitude time harmonic pressure waves are responsible for oscillations in the velocity field. The time independent axial velocity and the time dependent oscillatory axial velocity are calculated analytically. The important physical quantities like the velocity profile, amplitude of the oscillation and penetration depth of the wave are given special emphasis. The effects of second grade parameter and suction parameter on these quantities are particularly examined. A comparative study of the oscillatory flow for second grade with viscous fluid is also made.

Key words: Oscillatory channel flow, second grade fluid, shear stress, perturbation solution, Wentzel-Kramers-Brillouin (WKB) approximation.

INTRODUCTION

The history of channel flows goes back to the celebrated paper by Berman (1953) who initiated the study of two-dimensional laminar flow of incompressible viscous fluid in a rectangular channel with permeable walls. Several attempts were later made under the assumptions of small and large cross flow Reynolds number. Majdalani (1998) introduced the oscillatory viscous flow between two parallel porous plates subject to sidewall injection for the first time. Later on, some studies extended their work of oscillatory channel flows to different flow configurations (Jankowski and Majdalani, 2000, 2002; Majdalani and Roh, 2000, 2001; Majdalani, 2001). We witness that these studies of oscillatory channel flows have been undertaken in viscous fluid.

The oscillatory flows have special relevance in vibrating media with applications in oil-drilling, control of blood flow during surgical operations, manufacturing and processing of foods and paper, oil exploration and paper industry. In biology, it has applications in modeling of respiratory functions in lungs, modeling of chemical/blood dispensing in biochemistry/clinical labs, etc. Some other applications of value are: to detect the intensity of underground explosions, chemicals and material processing, isotope separation, irrigation systems, rocket propulsion, filtration mechanism, sweat cooling, cooling of electronic device, heat exchanger and many others.

It is now well established that the non-Newtonian fluids are more appropriate than viscous fluids in many practical applications. Examples of such fluids include certain oils, lubricants, mud, shampoo, ketchup, blood at low shear rate, cosmetic products, polymers and many others. Unlike the viscous fluids, all the non-Newtonian fluids (in terms of their diverse characteristics) cannot be described by a single constitutive relationship. Hence, several models of non-Newtonian fluids are proposed in the literature. Although, in general, the classification of non-Newtonian fluids is presented into three categories, namely, the differential, rate and integral types, but the differential type of fluids has been properly studied by the researchers in the field. In particular, there is a simplest subclass of differential type of fluids known as second grade which has attracted much attention in recent studies (Wenchang and Takashi, 2005; Yigit et al. 2007; Fetecau et al., 2008; Chunhong, 2009; Nazar et al., 2010; Naeem et al., 2010; Yuedong and Yanhua, 2010; Erdogan and Imrak, 2011; Norfifah et al., 2011).

The purpose of the present paper is to investigate the

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oscillatory channel flow in second grade fluid. The mathematical modeling is based on the constitutive equations of second grade fluid. This model is preferred over other models of non-Newtonian fluid, since it describes the normal stress effects. The governing equations of the unsteady second grade fluid are very nonlinear in nature. These equations are linearized under small pressure wave amplitude (of oscillating wave) and small Mach number assumptions. The analytical solutions of the ensuing problem is developed using various analytic techniques, such as separation of variable, perturbation method and Wentzel-Kramers-Brillouin (WKB) expansion method under small Mach number approximations. The transient nature of the field suggests separating the field into a time independent and time dependent parts. The solution of the time independent part is found using regular perturbation expansion. The time dependent part is further divided into an acoustic, irrotational, pressure-driven part and a vortical, rotational, vorticity-driven part. The solution of the acoustic part is straightforward and is solved in an exact fashion using separation of variable method. The solution of vortical part is obtained using perturbation method and WKB approximation. The time dependent part of oscillatory axial velocity, its amplitude and penetration depth are of particular interest in the study of oscillatory channel flows. The effects of suction and the second grade parameters on the axial velocity are analyzed and compared with those of viscous fluid. An increase in the suction parameter shows a decrease in the amplitude of the wave for the second grade fluid. The same is true for the viscous fluid; however, the rate of decrease of the amplitude with respect to suction parameter is faster in second grade fluid than in viscous fluid. The increase of second grade parameter witnesses a significant increase in the amplitude of the wave and a nominal decrease in the penetration depth. Comparing our results for $k=0$ shows a complete agreement with the results of viscous fluid. In the end, it is remarked that the present attempt will provide a base for future investigations in non-Newtonian oscillatory channel flows taking different fluid models and flow configurations.

FORMULATION OF THE PROBLEM

We consider the flow of a non-Newtonian second grade fluid between a long and narrow channel with porous walls separated by a distance $2h$. The length of the channel is L and width w. The suction is taking place from the porous plates with a uniform wall velocity v_w. The assumption that $w >> h$ enables us to treat the problem as a two-dimensional one. The geometry and the flow configuration between the channels is as shown in Figure 1. The equation of continuity and equation of motion for incompressible homogeneous second grade fluid are:

$$\frac{\partial \rho^*}{\partial t^*} + \nabla^* \cdot \left(\rho^* \mathbf{u}^*\right) = 0 \tag{1}$$

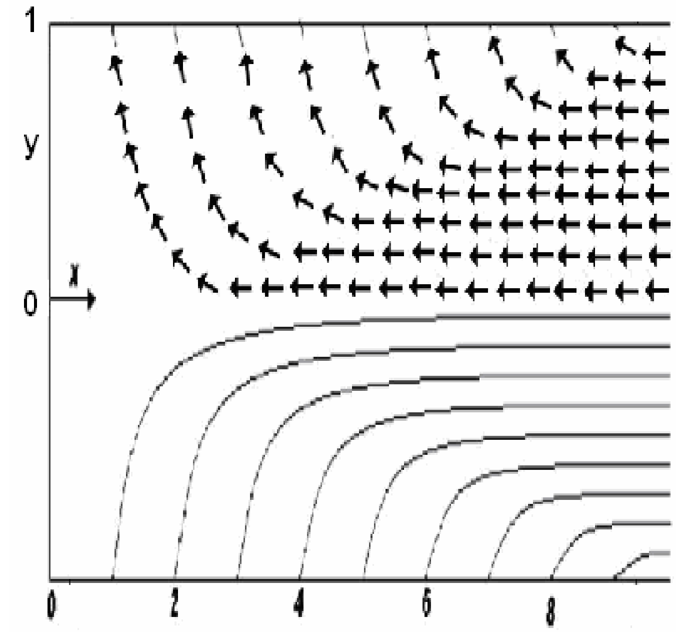

Figure 1. Geometry and flow configuration of the problem.

$$grad\left(\boldsymbol{T}^*\right) + \rho^* \boldsymbol{b} = \rho^* \frac{d\mathbf{u}^*}{dt^*} \tag{2}$$

where $\mathbf{u}^*$ is the velocity vector, ρ^* is the density, $\boldsymbol{b}$ is the body force and $\boldsymbol{T}^*$ is the Cauchy stress tensor for the second grade fluid. $\boldsymbol{T}^*$ in Equation 2 is defined by:

$$\boldsymbol{T}^* = -p^* \boldsymbol{I} + \mu \boldsymbol{A}_1^* + \alpha_1 \boldsymbol{A}_2^* + \alpha_2 \boldsymbol{A}_1^{2*} \tag{3}$$

where p^* is pressure, $\boldsymbol{I}$ is the identity tensor, μ is the dynamic viscosity, α_1 and α_2 are the material constants, $\boldsymbol{A}_1^*$ and $\boldsymbol{A}_2^*$ are first two Rivlin-Ericksen tensors. In Equation 3, $\boldsymbol{A}_1^*$ and $\boldsymbol{A}_2^*$ are given by:

$$\begin{aligned} \boldsymbol{A}_1^* &= \left(grad\,\mathbf{u}^*\right) + \left(grad\,\mathbf{u}^*\right)^{\mathrm{T}} \\ \boldsymbol{A}_2^* &= \frac{d\boldsymbol{A}_1^*}{dt} + \boldsymbol{A}_1^*\left(grad\,\mathbf{u}^*\right) + \left(grad\,\mathbf{u}^*\right)^{\mathrm{T}} \boldsymbol{A}_1^* \end{aligned} \tag{4}$$

$$\mu \ge 0, \alpha_1 \ge 0, \alpha_1 + \alpha_2 = 0$$

where $\frac{d}{dt}$ is the material time derivative, *grad* is the gradient operator and T in the superscript denotes the transpose. Substitution of Equations 3 and 4 in Equation 2 gives:

$$grad\left[\frac{1}{2}\rho^*\left|\mathbf{u}^*\right|^2 + p^* - \alpha_1\left(\mathbf{u}^*\cdot\nabla^*\mathbf{u}^* + \frac{1}{4}\left|A_1^*\right|^*\right)\right] + \rho^*\left[\mathbf{u}_{t^*}{}^* - \mathbf{u}^*\times\left(\nabla^*\times\mathbf{u}^*\right)\right] \quad (5)$$
$$= \mu\nabla^{*2}\mathbf{u}^* + \alpha_1\left[\nabla^{*2}\mathbf{u}_{t^*}{}^* + \nabla^{*2}\left(\nabla^*\times\mathbf{u}^*\right)\times\mathbf{u}^*\right] + \left(\alpha_1+\alpha_2\right)\mathrm{div}A_1^{*2} + \rho^* b,$$

where the subscript t^* denotes the partial derivative with respect to time, ∇^2 is the Laplacian operator and $(*)$ denotes the dimensional variables. Equations 1 and 5 can be made dimensionless by introducing the following dimensionless quantities:

$$x = \frac{x^*}{h},\ \ y = \frac{y^*}{h},\ \ \mathbf{u} = \frac{\mathbf{u}^*}{a_S},\ \ p = \frac{p^*}{\gamma p_S},\ \ \rho = \frac{\rho^*}{\rho_S},\ \ t = t^*\omega_S. \quad (6)$$

In Equation 6, a_s, p_s and ρ_s are the stagnation speed of the sound, the stagnation pressure and the stagnation density, respectively, while γ is the ratio of specific heats and ω_s is the frequency of the longitudinal pressure oscillation. Making use of Equation 6 in Equations 1 and 5, we arrive at the following non dimensional boundary value problem:

$$\omega\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\mathbf{u}\right) = 0 \quad (7)$$

$$\rho\left[\omega\frac{\partial\mathbf{u}}{\partial t} + \left(\mathbf{u}\cdot\nabla\right)\mathbf{u}\right] = -\nabla p + M\varepsilon\left[\frac{4}{3}\nabla\left(\nabla\cdot\mathbf{u}\right) - \nabla\times\left(\nabla\times\mathbf{u}\right)\right] \quad (8)$$
$$-k\nabla\left[\mathbf{u}\square\nabla^2\mathbf{u} + \frac{1}{4}\left|A_1\right|^2\right] + k\left[\omega\nabla^2\mathbf{u}_t + \nabla^2\left(\nabla\times\mathbf{u}\right)\times\mathbf{u}\right]$$

$$v = 0,\quad \frac{\partial u}{\partial y} = 0; \qquad \text{at}\quad y = 0, \quad (9)$$
$$v = M,\quad u = 0; \qquad \text{at}\quad y = 1,$$

where M is Mach number, ε is the reciprocal of cross flow Reynolds number (suction parameter), ω is the dimensionless wave frequency and k is the second grade parameter. These quantities are defined as:

$$M = \frac{v_w}{a_s},\ \ \varepsilon = \frac{1}{R} = \frac{\nu}{v_w h},\ \ \omega = \frac{\omega_s h}{a_s},\ k = \frac{\alpha_1}{\rho_s h^2}$$

We perturb the pressure, density and velocity in terms of small pressure wave amplitude $\bar{\varepsilon}$:

$$p\left(x,\ y,\ t\right) = p_0\left(x,\ y\right) + \bar{\varepsilon}p_1\left(x,\ y\right)e^{\left(-it\right)}, \quad (10)$$

$$\rho\left(x,\ y,\ t\right) = 1 + \bar{\varepsilon}\rho_1\left(x,\ y\right)e^{\left(-it\right)}, \quad (11)$$

$$\mathbf{u}\left(x,\ y,\ t\right) = M\mathbf{u_0}\left(x,\ y\right) + \bar{\varepsilon}\mathbf{u_1}\left(x,\ y\right)e^{\left(-it\right)}. \quad (12)$$

In what follows, we will be concerned with finding leading order time independent velocity and the time dependent perturbed velocity.

TIME INDEPENDENT SOLUTION

We first determine the time independent part of the velocity that corresponds to the leading order system in $\bar{\varepsilon}$, Equation 12. For that, we make use of Equations 10 to 12 in Equations 7 and 8 to arrive at:

$$\nabla\cdot\mathbf{u_0} = 0, \quad (13)$$

$$M^2\left(\mathbf{u_0}\cdot\nabla\right)\mathbf{u_0} = -\nabla p_0 + M^2\varepsilon\left[\frac{4}{3}\nabla\left(\nabla\cdot\mathbf{u_0}\right) - \nabla\times\left(\nabla\times\mathbf{u_0}\right)\right] \quad (14)$$
$$-kM^2\left[\nabla\left[\mathbf{u_0}\cdot\nabla^2\mathbf{u_0} + \frac{1}{4}M^2\left\{4\frac{\partial u_0}{\partial x}\frac{\partial v_0}{\partial y} - \left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}\right)^2\right\}^2\right] - \nabla^2\left\{\left(\nabla\times\mathbf{u_0}\right)\times\mathbf{u_0}\right\}\right].$$

where u_0 and v_0 are the two components of $\mathbf{u}_0$.

The stream function of the problem is defined as (Jankowski and Majdalani, 2006):

$$\Psi = -xF\left(y\right). \quad (15)$$

With this value of stream function, Equation 13 is identically satisfied and Equation 14 takes the following form:

$$M^2\left[x\left(F'^2 - FF''\right)\hat{i} + FF'\hat{j}\right] = -\frac{\partial p_0}{\partial x}\hat{i} - \frac{\partial p_0}{\partial y}\hat{j} + M^2\varepsilon\left[-xF'''\hat{i} + F''\hat{j}\right] \quad (16)$$
$$+kM^2\left[x\left\{M^2\left(4F'^2F''^2 + x^2F''^4\right) - FF^{v} - F''^2\right\}\hat{i}\right]$$
$$+kM^2\left[\left\{-2x^2F''F''' + FF''' - F'F'' + M^2\left(4F'F'' + x^2F''F'''\right)\left(4F'^2 + x^2F''^2\right)\right\}\hat{j}\right],$$

where $'$ in the superscript denotes the derivatives with respect to y. Splitting Equation 16 into component form, differentiating the i^{th} and the j^{th} components with respect to y and x, respectively and subtracting the resulting equations, we obtain:

$$F^{iv} + R\left(F'F'' - FF'''\right) + k\left(FF^{v} + F'F^{iv} - 2F''F'''\right) = 0 \quad (17)$$

The boundary conditions in F takes the form:

$$F'\left(1\right) = F\left(0\right) = F''\left(0\right) = 0\ ,\ \ F\left(1\right) = 1. \quad (18)$$

The solution of Equation 17 subject to Equation 18 is given by:

$$F\left(y\right) = \frac{1}{2}y\left(3 - y^2\right) + k\left[\frac{3}{20}y\left(y^2 - 1\right)^2\right] + O\left(k^2\right) \quad (19)$$

The effects of the second grade parameter on the time independent part of the velocity $\mathbf{u_0}$ computed from Equation 17 are shown in Figure 2. For $k = 0$, the result matches with Jankowski and Majdalani (2006) and an increase in the second grade parameter shows an increase in the velocity distribution $\mathbf{u_0}$. After having found the time independent part, we proceed to find time dependent part subsequently.

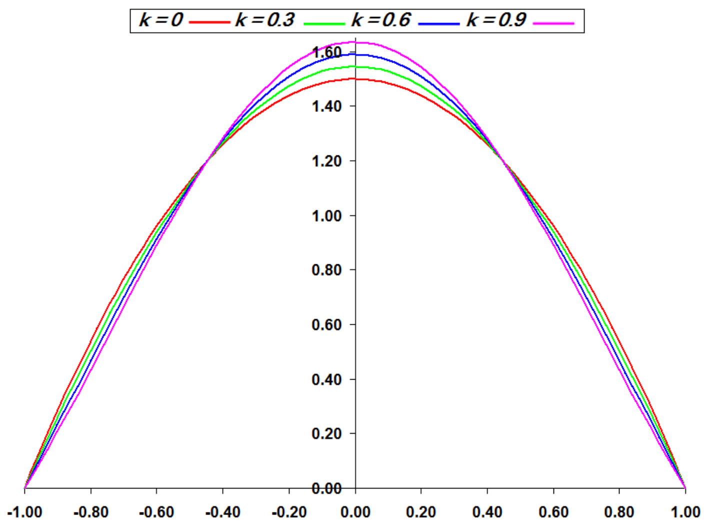

Figure 2. Effects of second grade parameter k on velocity profile $\mathbf{u}_0$.

TIME DEPENDENT PART (THE OSCILLATORY VELOCITY)

Utilizing Equations 10 to 12 in Equations 7 and 8 and taking the terms at $O(\bar{\varepsilon})$, the time dependent part of the velocity field $\mathbf{u}$, can be written as:

$$-i\omega\rho_1+\nabla\cdot\mathbf{u}_1=-M\nabla\cdot(\rho_1\mathbf{u}_0), \tag{20}$$

$$\begin{aligned}-i\omega\mathbf{u}_1&=-M\left[\nabla(\mathbf{u}_0\cdot\mathbf{u}_1)-\mathbf{u}_0\times(\nabla\times\mathbf{u}_1)-\mathbf{u}_1\times(\nabla\times\mathbf{u}_0)\right]\\&-\nabla p_1+M\varepsilon\left[\frac{4}{3}\nabla(\nabla\cdot\mathbf{u}_1)-\nabla\times(\nabla\times\mathbf{u}_1)\right]+kM\left[\nabla\{\mathbf{u}_0\cdot\nabla(\nabla\cdot\mathbf{u}_1)\}\right.\\&\left.-\nabla\{\mathbf{u}_0\cdot\nabla\times(\nabla\times\mathbf{u}_1)\}+\nabla\{\mathbf{u}_1\cdot\nabla(\nabla\cdot\mathbf{u}_0)\}-\nabla\{\mathbf{u}_1\cdot\nabla\times(\nabla\times\mathbf{u}_0)\}\right]\\&-ki\omega\left[\nabla(\nabla\cdot\mathbf{u}_1)-\nabla\times(\nabla\times\mathbf{u}_1)\right]+kM\nabla\left[\nabla\cdot\{(\nabla\times\mathbf{u}_1)\times\mathbf{u}_0+(\nabla\times\mathbf{u}_0)\times\mathbf{u}_1\}\right]\\&-kM\nabla\times\left[\nabla\times\{(\nabla\times\mathbf{u}_1)\times\mathbf{u}_0+(\nabla\times\mathbf{u}_0)\times\mathbf{u}_1\}\right]\end{aligned} \tag{21}$$

The velocity $\mathbf{u}_1$ satisfies no-slip boundary condition at the wall $\mathbf{u}_1(x,\ 1)=0$ and symmetry at the midsection of the channel $\frac{\partial\mathbf{u}_1(x,\ 0)}{\partial y}=0$. To proceed further, we decompose $\mathbf{u}_1(x,\ y)$ into an acoustic, irrotational, pressure-driven part $\hat{\mathbf{u}}$ and vortical, rotational, vorticity-driven part $\tilde{\mathbf{u}}$ as:

$$\mathbf{u}_1=\hat{\mathbf{u}}+\tilde{\mathbf{u}}, \tag{22}$$

with the properties:

$$\nabla\times\hat{\mathbf{u}}=0\ ,\ \ \nabla\cdot\tilde{\mathbf{u}}=0\ ,\ \ p_1=\hat{p},\ \ \rho_1=\hat{\rho}, \tag{23}$$

Using Equation 22 in Equations 20 and 21, separating acoustic and vortical parts yields the following set of four equations (Equations 24 to 27).

$$-i\omega\hat{\rho}+\nabla\cdot\hat{\mathbf{u}}=-M\nabla\cdot\left(\hat{\rho}\mathbf{u}_0\right) \tag{24}$$

$$\begin{aligned}-i\omega\hat{\mathbf{u}}&=-\nabla\hat{p}+\frac{4}{3}M\varepsilon\nabla\left(\nabla\cdot\hat{\mathbf{u}}\right)-M\left[\nabla\left(\hat{\mathbf{u}}\cdot\mathbf{u}_0\right)-\hat{\mathbf{u}}\times(\nabla\times\mathbf{u}_0)\right]\\&+kM\left[\nabla\left\{\mathbf{u}_0\cdot\nabla\left(\nabla\cdot\hat{\mathbf{u}}\right)\right\}+\nabla\left\{\hat{\mathbf{u}}\cdot\nabla(\nabla\cdot\mathbf{u}_0)\right\}-\nabla\left\{\hat{\mathbf{u}}\cdot\nabla\times(\nabla\times\mathbf{u}_0)\right\}\right]\\&-ki\omega\nabla\left(\nabla\cdot\hat{\mathbf{u}}\right)+kM\nabla\left[\nabla\cdot(\nabla\times\mathbf{u}_0)\times\hat{\mathbf{u}}\right]-kM\nabla\times\left[\nabla\times(\nabla\times\mathbf{u}_0)\times\hat{\mathbf{u}}\right]\end{aligned} \tag{25}$$

and

$$\nabla\cdot\tilde{\mathbf{u}}=0 \tag{26}$$

$$\begin{aligned}-i\omega\tilde{\mathbf{u}}&=-M\varepsilon\nabla\times\left(\nabla\times\tilde{\mathbf{u}}\right)-M\left[\nabla\left(\tilde{\mathbf{u}}\cdot\mathbf{u}_0\right)-\tilde{\mathbf{u}}\times(\nabla\times\mathbf{u}_0)-\mathbf{u}_0\times\left(\nabla\times\tilde{\mathbf{u}}\right)\right]\\&-kM\left[\nabla\left\{\mathbf{u}_0\cdot\nabla\times\left(\nabla\times\tilde{\mathbf{u}}\right)\right\}+\nabla\left\{\tilde{\mathbf{u}}\cdot\nabla(\nabla\cdot\mathbf{u}_0)\right\}-\nabla\left\{\tilde{\mathbf{u}}\cdot\nabla\times(\nabla\times\mathbf{u}_0)\right\}\right]+ki\omega\nabla\times\left(\nabla\times\tilde{\mathbf{u}}\right)\\&+kM\nabla\left[\nabla\cdot\left(\nabla\times\tilde{\mathbf{u}}\right)\times\mathbf{u}_0+\nabla\cdot(\nabla\times\mathbf{u}_0)\times\tilde{\mathbf{u}}\right]-kM\nabla\times\left[\nabla\times\left(\nabla\times\tilde{\mathbf{u}}\right)\times\mathbf{u}_0+\nabla\times(\nabla\times\mathbf{u}_0)\times\tilde{\mathbf{u}}\right]\end{aligned} \tag{27}$$

Solution of the acoustic part ($\hat{\mathbf{u}}$)

The axial acoustic pressure and velocity are dominant for the purpose of oscillatory channel flow under consideration (Majdalani and Roh, 2000). For perfect gas undergoing isentropic oscillations, we know that $\hat{\rho}=\hat{p}$. The Equations 24 and 25 can thus be seen as two coupled equations in pressure $\hat{p}$ and velocity $\hat{\mathbf{u}}$. Eliminating $\hat{\mathbf{u}}$ from these equations, the resulting equation in $\hat{p}$ can be readily solved upto $O(M)$. This gives axial acoustic pressure $\hat{p}$ and velocity $\hat{\mathbf{u}}$ as:

$$\hat{p}=\cos\left(\frac{\omega x}{\sqrt{1-k\omega^2}}\right)+O(M), \tag{28}$$

$$\hat{\mathbf{u}}=i\sqrt{1-k\omega^2}\,\sin\left(\frac{\omega x}{\sqrt{1-k\omega^2}}\right)+O(M). \tag{29}$$

Solution of the vortical part ($\tilde{\mathbf{u}}$)

Writing $\tilde{\mathbf{u}}$ in component form as $\tilde{\mathbf{u}}=(\tilde{u}_1,\ \tilde{u}_2)$, assuming that the ratio of the horizontal to vertical component is of $O(M)$, we can conveniently neglect $\tilde{u}_2$. Now, using $\mathbf{u}_0=\left(-xF',\ F\right)$ in Equation 27, dropping $\tilde{u}_2$ and writing $\tilde{u}_1=X(x)Y(y)$, we obtain:

$$\begin{aligned}&\varepsilon\frac{d^2Y}{dy^2}-F\frac{dY}{dy}+\left[iS_t+\left(1+x\frac{X'}{X}\right)F'\right]Y\\&=k\left[-F\frac{d^3Y}{dy^3}+\left\{iS_t+\left(x\frac{X'}{X}-1\right)F'\right\}\frac{d^2Y}{dy^2}-F''\frac{dY}{dy}+\left(1+x\frac{X'}{X}\right)F'''Y\right].\end{aligned} \tag{30}$$

where $S_t = \dfrac{\omega}{M}$ is the dimensionless Strouhal number. The x dependent function in Equation 30 is written as:

$$x\frac{X'}{X} = k_n$$

where k_n are so far unknown eigenvalues and will be determined using no slip boundary condition. Solving this equation, we get the eigenfunctions $X(x) = x^{k_n}$. The axial component of acoustic velocity can be expressed in terms of these eigenfunctions as:

$$\tilde{\mathbf{u}}(x,\ y) = u_1(x,\ y) = \sum_n c_n x^{k_n} Y_n(y). \qquad (31)$$

From no slip condition, at $y = 1$ the vortical solution is equal to the negative of the acoustic solution giving:

$$\hat{\mathbf{u}}(x,\ 1) = -\tilde{\mathbf{u}}(x,\ 1),$$

Substituting Equation 31 in the aforementioned condition, we conclude that:

$$k_n = 2n+1\ ,\quad c_n = -i\sqrt{1-k\omega^2}\,\frac{(-1)^n}{(2n+1)!}\left(\frac{\omega}{\sqrt{1-k\omega^2}}\right)^{2n+1}\ ,\quad Y_n(1) = 1$$

Using the aforementioned values of k_n and c_n, Equations 31 and 30 take the form:

$$\tilde{\mathbf{u}}(x,y) = -i\sqrt{1-k\omega^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\omega x}{\sqrt{1-k\omega^2}}\right)^{2n+1} Y_n(y). \qquad (32)$$

$$\begin{aligned}&\varepsilon\frac{d^2Y}{dy^2} - F\frac{dY}{dy} + \left[iS_t + (2n+2)F'\right]Y \\ &= k\left[-F\frac{d^3Y}{dy^3} + \left(iS_t + 2nF'\right)\frac{d^2Y}{dy^2} - F''\frac{dY}{dy} + (2n+2)F'''Y\right].\end{aligned} \qquad (33)$$

where Y satisfies the following boundary conditions.

$$Y(1) = 1\ ,\quad Y'(0) = 0. \qquad (34)$$

In order to determine the complete solution of the vortical, part we proceed to find Y appearing in Equation 32 and defined by Equations 33 and 34. The methods adopted are perturbation method and the WKB approximation.

The solution for Y

We express the solution of Equation 33 in regular perturbation expansion in k:

$$Y(y,k) \approx Y_0(y) + kY_1(y) + O(k^2) \qquad (35)$$

Using Equation 35 into Equations 33 to 34 and equating like powers of k, the leading order system is given by:

$$\varepsilon\frac{d^2Y_0}{dy^2} - F\frac{dY_0}{dy} + \left[iS_t + (2n+2)F'\right]Y_0 = 0 \qquad (36)$$

$$Y_0(1) = 1\ ,\quad Y_0'(0) = 0. \qquad (37)$$

This system is now solved using WKB approximation for small ε. For that, we write the general expression for the WKB approximation as:

$$Y_0 = e^{\frac{1}{\delta}\sum_{j=0}^{\infty}\delta^j S_j}. \qquad (38)$$

Making use of the aforementioned expression into Equation 36 and expanding up to S_1, we have:

$$\frac{1}{\varepsilon}S_0'^2 + 2S_0'S_1' + S_0'' - \frac{F}{\varepsilon}S_0' - FS_1' + iS_t + (2n+2)F' + O(\varepsilon) = 0,$$

where we assume $S_t = \frac{1}{\varepsilon}$, and $\delta = \varepsilon.$ Collecting the terms of $O(\frac{1}{\varepsilon})$ in the aforementioned equation, we write:

$$S_0'^2 - FS_0' + iS_t\varepsilon = 0. \qquad (39)$$

Equation 39 is recognized as eikonal equation and its solution is expressed as:

$$S_0'^2 - FS_0' + iS_t\varepsilon = 0. \qquad (40)$$

$$S_0 = \frac{1}{2}\int_1^y\left(F \pm \sqrt{F^2 - 4i\varepsilon S_t}\right)d\eta$$

The one term WKB solution is thus written as:

$$Y_0 = c_1 e^{\frac{1}{2\varepsilon}\int_1^y (F+\sqrt{F^2-4i\varepsilon S_t})d\eta} + c_2 e^{\frac{-1}{2\varepsilon}\int_1^y (-F+\sqrt{F^2-4i\varepsilon S_t})d\eta} \qquad (41)$$

The boundary conditions (Equation 34) and the observation that the second term is of no physical interest (being wave propagating into the surface) implies $c_1 = 1$ and $c_2 = 0$. Equation 41 thus becomes:

$$Y_0 = e^{\frac{1}{2\varepsilon}\int_1^y (F+\sqrt{F^2-4i\varepsilon S_t})d\eta}. \qquad (42)$$

The system of order $O(k)$ is written as:

$$\frac{d^2Y_1}{dy^2}-\frac{F}{\varepsilon}\frac{dY_1}{dy}+\left[iS_t+(2n+2)F'\right]\frac{Y_1}{\varepsilon}=g(y) \quad (43)$$

where

$$g(y)=-\frac{F}{\varepsilon}\frac{d^3Y_0}{dy^3}+\frac{iS_t+2nF'}{\varepsilon}\frac{d^2Y_0}{dy^2}-\frac{F''}{\varepsilon}\frac{dY_0}{dy}+(2n+2)\frac{F'''}{\varepsilon}Y_0$$

and the corresponding boundary conditions are:

$$Y_1(1)=0\,,\;\; Y_1'(0)=0. \quad (44)$$

The solution of this system is provided by the variation of parameter method. For the complementary solution, we write:

$$\varepsilon\frac{d^2Y_1}{dy^2}-F\frac{dY_1}{dy}+\left[iS_t+(2n+2)F'\right]Y_1=0. \quad (45)$$

The solution of this differential equation is readily found to be:

$$Y_{1c}=c_3e^{\frac{1}{2\varepsilon}\int_1^y(F+\sqrt{F^2-4i\varepsilon S_t})d\eta}+c_4e^{\frac{-1}{2\varepsilon}\int_1^y(-F+\sqrt{F^2-4i\varepsilon S_t})d\eta} \quad .46)$$

If we substitute Y_0 from Equation 42 in $g(y)$ and work to the terms of $O\left(\frac{1}{\varepsilon}\right)$, we get:

$$g(y)=-3(2n+2)e^{\frac{1}{2\varepsilon}\int_1^y\left(F+\sqrt{F^2-4i\varepsilon S_t}\right)d\eta} \quad (47)$$

Omitting the straight forward details, the particular solution of Equation 43 is finally written as:

$$Y_{1p}=\int_1^y\left[-\frac{6(n+1)\left(1+\frac{F^2}{8i\varepsilon S_t}\right)}{(-4i\varepsilon S_t)^{\frac{1}{2}}}\right]d\eta.e^{\frac{1}{2\varepsilon}\int_1^y\left(F+\sqrt{F^2-4i\varepsilon S_t}\right)d\eta}-\int_1^y\left[\frac{6(n+1)\left(1+\frac{F^2}{8i\varepsilon S_t}\right)}{(-4i\varepsilon S_t)^{\frac{1}{2}}}\right]d\eta.e^{\frac{-1}{2\varepsilon}\int_1^y\left(-F+\sqrt{F^2-4i\varepsilon S_t}\right)d\eta} \quad (48)$$

The general solution of the first order system is thus given by:

$$Y_1=c_1e^{\frac{1}{2\varepsilon}\int_1^y(F+\sqrt{F^2-4i\varepsilon S_t})d\eta}+c_2e^{\frac{-1}{2\varepsilon}\int_1^y(-F+\sqrt{F^2-4i\varepsilon S_t})d\eta}+\int_1^y\left[-\frac{6(n+1)\left(1+\frac{F^2}{8i\varepsilon S_t}\right)}{(-4i\varepsilon S_t)^{\frac{1}{2}}}\right]d\eta.e^{\frac{1}{2\varepsilon}\int_1^y\left(F+\sqrt{F^2-4i\varepsilon S_t}\right)d\eta}-\int_1^y\left[\frac{6(n+1)\left(1+\frac{F^2}{8i\varepsilon S_t}\right)}{(-4i\varepsilon S_t)^{\frac{1}{2}}}\right]d\eta.e^{\frac{-1}{2\varepsilon}\int_1^y\left(-F+\sqrt{F^2-4i\varepsilon S_t}\right)d\eta} \quad (49)$$

Using boundary conditions (Equation 44), and the argument as given for Y_0, we have:

$$Y_1=\int_1^y\left[-\frac{6(n+1)\left(1+\frac{F^2}{8i\varepsilon S_t}\right)}{(-4i\varepsilon S_t)^{\frac{1}{2}}}\right]d\eta.e^{\frac{1}{2\varepsilon}\int_1^y\left(F+\sqrt{F^2-4i\varepsilon S_t}\right)d\eta} \quad (50)$$

Thus, the two term perturbation solution of Equation 33 is given by:

$$Y=\left[1+k\int_1^y\left[-\frac{6(n+1)\left(1+\frac{F^2}{8i\varepsilon S_t}\right)}{(-4i\varepsilon S)^{\frac{1}{2}}}\right]d\eta\right]e^{\frac{1}{2\varepsilon}\int_1^y\left(F+\sqrt{F^2-4i\varepsilon S_t}\right)d\eta} \quad (51)$$

Substitution of Equation 51 into Equation 32, gives the following expression of vortical velocity:

$$\tilde{\mathbf{u}}(x,\;y)=-i\sqrt{1-k\omega^2}\sin\left(\frac{\omega x}{\sqrt{1-k\omega^2}}\right).\left[1+k\int_1^y\left(-\frac{6(n+1)\left(1+\frac{F^2}{8i\varepsilon S_t}\right)}{(-4i\varepsilon S_t)^{\frac{1}{2}}}\right)d\eta\right]e^{\frac{1}{2\varepsilon}\int_1^y(F+\sqrt{F^2-4i\varepsilon S_t})d\eta} \quad (52)$$

After calculating both acoustic and vortical velocities, we revert back to the time dependent oscillatory velocity. Thus, Equations 29, 52 and 22 together gives the time dependent oscillatory velocity as:

$$\mathbf{u}_1=i\left[\sin(\omega x)(1-Y_0)-k\left\{\sin(\omega x)Y_1-\left(\frac{1}{2}\omega^3x-\frac{1}{2}\omega^2\sin(\omega x)\right)(1-Y_0)\right\}\right] \quad (53)$$

where Y_0 and Y_1 are given by Equations 41 and 50, respectively. The integrals in these solutions cannot be calculated analytically and hence numerical integration is performed. Mathematically, the viscous fluid results are obtained for $k=0$, and are found to match with Jankowski and Majdalani (2006).

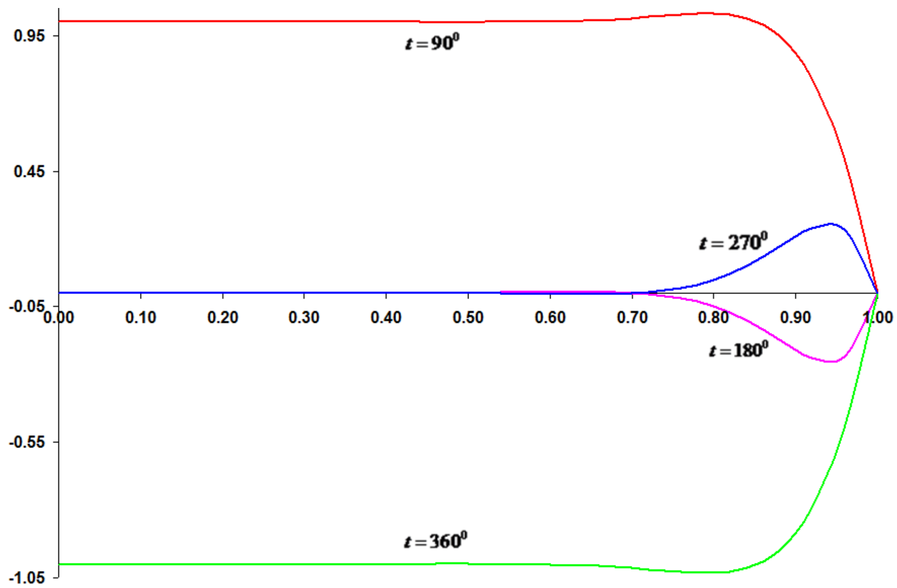

Figure 3. Plot of oscillatory axial velocity $u_1 \exp(-it)$ against y for viscous fluid $k=0$ and $\frac{x}{l}=1, m=1, S_t=20, R=10$ at dimensionless time $t = 90^o$, 180^o, 270^o and 360^o.

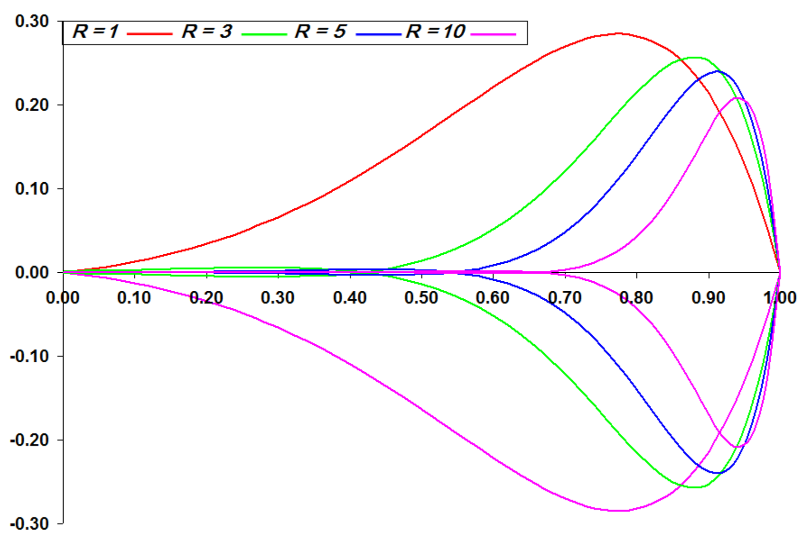

Figure 4. Effects of suction parameter R on oscillatory axial velocity $u_1 \exp(-it)$ for viscous fluid $k=0$ and $\frac{x}{l}=1, m=1, S_t=20$ at time phase $t=180^0$ and $t=360^0$

RESULTS AND DISCUSSION

Here, we would like to discuss the behavior of time dependent part, which is the crux of our discussion for oscillatory channel flow. To add credibility and to substantiate our results for the second grade fluid (Equation 53), we first plot the real part of oscillatory axial velocity $u_1 \exp(-it)$ for viscous fluid ($k=0$). Figure 3 represents the velocity distribution at four time lines ($t=90^0, 180^0, 270^0$ and 360^0) while Figure 4 shows the amplitude and penetration depth of the oscillatory wave for different values of suction parameter. The two graphs are found to

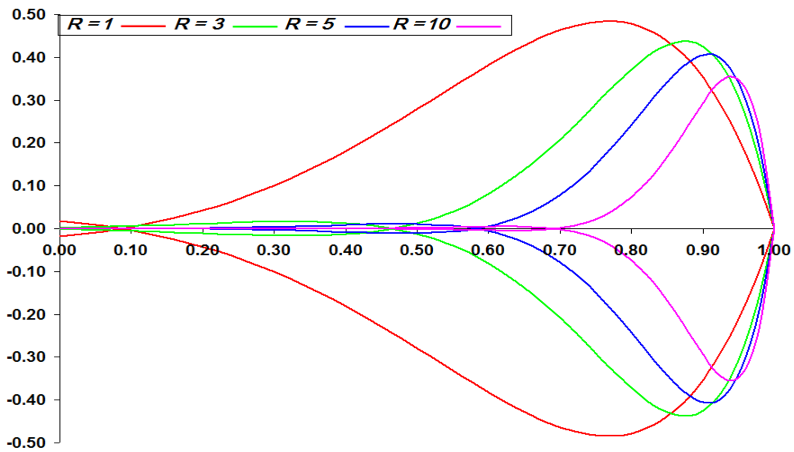

Figure 5. Effects of suction parameter R on oscillatory axial velocity $u_1 \exp(-it)$ for $\frac{x}{l}=1, m=1, S_t=20, k=0.6$ at time phase $t=180^0$ and $t=360^0$.

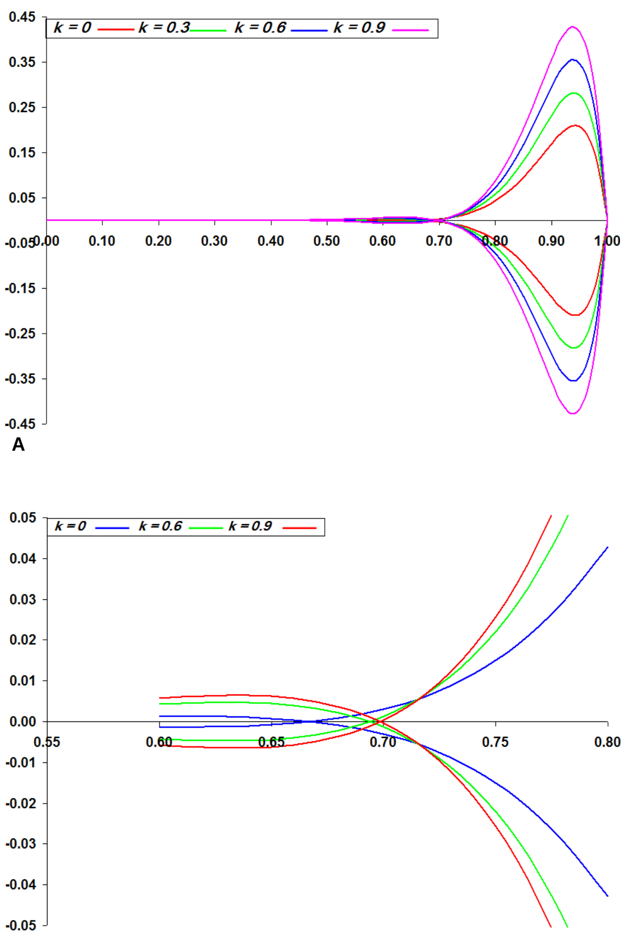

Figure 6. Effects of second grade parameter k on oscillatory axial velocity $u_1 \exp(-it)$ for $\frac{x}{l}=1, m=1, S_t=20, R=10$ at time phase $t=180^0$ and $t=360^0$.

match identically with Jankowski and Majdalani (2006).

The effects of suction parameter on the oscillatory axial velocity for the second grade fluid ($k=0.6$) are as shown in Figure 5. The graph shows the amplitude of oscillation and the penetration depth decreases as the suction parameter R increases. However, the rate of decrease of the amplitude with respect to suction parameter is faster in the second grade fluid than in the viscous fluid.

The effects of second grade fluid on the oscillatory axial velocity are as shown in Figure 6. Starting from $k=0$, the amplitude of the wave is found to increase substantially as the second grade parameter k increases (Figure 6a). The penetration depth of the wave decreases nominally with the increase in second grade parameter. This fact is separately highlighted in Figure 6b.

Conclusion

The aim of this paper is to study the analytical solution of oscillatory channel flow in second grade fluid. The velocity distribution is decomposed into time independent and time dependent parts. The solution for the time independent part is readily obtained using perturbation method. Amplitude of the time independent velocity profile is found to increase with the increase of second grade parameter. The time dependent oscillatory velocity distribution is realized by determining acoustic and vortical waves. The axial solution for the acoustic part is obtained using standard separation of variable method. The vortical part is more involved and a number of techniques, such as separation of variable, perturbation method and WKB approximations are used to reach the result. The analytical solution for oscillatory axial velocity is then built from the acoustic and vortical parts. The effects of second grade parameters on the oscillatory velocity are shown with the help of graphs. We observe significant increase in the amplitude of the wave and a small decrease in the penetration depth as the second grade parameter increases. Furthermore, the suction decreases the amplitude of the velocity distribution. These observations are as expected from the physics of second grade fluid and agree with those of viscous fluid in the limit as $k \to 0$. Finally, we may add that this is the first attempt to consider the effects of non-Newtonian second grade fluid in the theory of oscillatory channel flows and the discussion can be further extended to other types of non-Newtonian fluids with different rheological properties, different governing equations and different industrial applications.

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