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Full Length Research Paper

Isomorphism theorems for fuzzy submodules of gamma modules

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We generalize the Davvaz's paper to the fuzzy submodules of gamma modules and give some characterizations of fuzzy gamma modules.

Key words: Fuzzy ideal, fuzzy submodule, $\Gamma-$ring, gamma modules (ΓM -Module), fuzzy isomorphism theorems.

INTRODUCTION

As is well known, Zadeh (1965) introduced the notion of a fuzzy subset μ of a nonempty set X as a function from X to unit real interval $I=[0, 1]$. The notion of fuzzy groups was introduced by Rosenfeld (1971). Many scientists studied in basic properties about fuzzy groups and ideals. In these years Barnes (1966), Booth (1992), Luh (1969) and Coppage (1971) studied about gamma rings. The concept of fuzzy ideal in Γ-ring was introduced by Jun and Lee (1992). They studied some preliminary properties of fuzzy ideals of Γ-rings. In the last years Dutta and Chanda (2005) and Hong and Jun (1995) improved these works. Booth and Groenawald (1992) analised the prime modules of gamma ring. In this paper, we define fuzzy submodule of ΓM module G and give some basic characterization of it. That is if μ is fuzzy submodule of ΓM module G if and only if $\forall t\in[0,1]$, μ_t is a submodule of gamma module. We proved that if $f:G_1\to G_2$ be f invariant of G_1, G_2, ΓM module G_1, G_2 and μ be fuzzy submodule of G_1, G_2. Then $f(\mu), (f^{-1}(\mu))$ is fuzzy submodule of ΓM module $G_2, (G_1)$, respectively. Moreover we have shown that similar isomorphism theorems in algebra are also valid for fuzzy submodules of gamma modules.

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PRELIMINARIES

We will give some basic definitions and theorems.

Definition 1

Let M be a commutative additive group. M is called $\Gamma-$ring if the following conditions are satisfied. $\forall a,b,c\in M$, $\alpha,\beta,\gamma\in\Gamma$ and there is a mapping $M\times\Gamma\times M\to M$ such that;

1. $a\alpha b\in M$

2. $(a+b)\alpha c=a\alpha c+b\alpha c$
$a(\alpha+\beta)c=a\alpha c+a\beta c$
$a\alpha(b+c)=a\alpha b+a\alpha c$

3. $a\alpha(b\beta c)=(a\alpha b)\beta c$ (Barnes, 1966).

Definition 2

Let G be an additive abelian group and M be $\Gamma-$ring. $\forall g,g^{`}\in G$, $\alpha,\beta\in\Gamma$, $\forall x,y\in M$ and there is a mapping $G \times \Gamma \times M \rightarrow G$ such that;

1. $g\alpha x \in G$

2. $g\alpha(x\beta y) = (g\alpha x)\beta y$

3. $(g + g^{`})\alpha x = g\alpha x + g^{`}\alpha x$

$g\alpha(x + y) = g\alpha x + g\alpha y$

Then G is called a ΓM module (Booth and Groenewald, 1992).

Definition 3

A subgroup N of an additive abelian group ΓM module G is said to be submodule of G, if it satisfies $n\alpha x \in N$ $\forall n \in N$, $\alpha \in \Gamma$, $\forall x \in M$ (Booth and Groenewald, 1992).

Definition 4

Let μ, σ be two fuzzy subsets of G_1, G_2 respectively and the image $f(\mu)$ of μ is the fuzzy subset of G_2 defined by

$$f(\mu)(y) = \begin{cases} \sup(\mu(x) & \text{if} \quad f^{-1}(y) = x \\ 0 & otherwise \end{cases}$$

For all $y \in G_2$. Moreover, the inverse image of $f^{-1}(v)(x) = v(f(x)$ for all $x \in G_1$ (Malik and Mordeson, 1998).

Definition 5

Let Let G_1 and G_2 be ΓM modules and f be a mapping from a G_1 onto a G_2. Let. A fuzzy submodule μ of M is called f-invariant if $f(x) = f(y)$ implies that $\mu(x) = \mu(y)$ for all x, y$\in G_1$.

Now we will define a fuzzy submodule of ΓM module G by using the same argument about fuzzy submodules.

Definition 6

Let G be an additive abelian group, M be a $\Gamma-$ring and μ be a fuzzy subset of G. If $\forall g, g^{`} \in G$, $\alpha, \beta \in \Gamma$, $\forall m, m^{`} \in M$ there is a mapping $G \times \Gamma \times M \rightarrow G$ such that;

1. $\mu(0_G) = 1$
2. $\mu(g\alpha m) \geq \mu(g)$
3. $\mu(g\alpha m + g^{`}\beta m^{`}) \geq \mu(g) \wedge \mu(g^{`})$

then μ is called fuzzy submodule of G, ΓM module.

Theorem 1

Let μ and v be fuzzy submodules of ΓM module G. Then $\mu \cap v$ is also fuzzy submodule of ΓM module G.

Proof

One can immediately prove this theorem.

Example

Let μ be fuzzy subset of G and $G = \{[a \quad 0] | a \in 2□\}$, $\Gamma = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x, y \in □ \right\}$, $M = \{[c \quad 0] | c \in 2□\}$ and μ be fuzzy subset of ΓM module G such that

$$\mu(x) = \begin{cases} 1 & x = 0 = [0 \quad 0] \\ 1/2 & x \in G - \{[0 \quad 0]\} \\ 0 & x \notin G \end{cases}.$$

It is obviously seen that μ is fuzzy submodule of ΓM module G. Here with the next theorem, it is easily seen that there is direct relationship between fuzzy submodules of gamma modules and submodules of gamma modules. Then we prove essential theorems about fuzzy module structures.

Theorem 2

μ is fuzzy submodule of ΓM module G if and only if $\forall t \in [0,1]$, $\mu_t = \{x \in G | \mu(x) \geq t\}$ is a submodule of gamma module G.

Proof

One can easily show this theorem. Now we will give the relationship between images and preimage any fuzzy submodules of gamma modules. With the next two theorems, we can define correspondence theorems for fuzzy submodules.

Theorem 3

Let $f:G_1 \to G_2$ be f invariant of G_1, G_2 ΓM module and μ be fuzzy submodule of G_1. Then $f(\mu)$ is fuzzy submodule of $, G_2$ ΓM module.

Proof

Since μ is fuzzy submodule of G_1

$$f(\mu)(0_{G_2}) = \sup\{\mu(0_{G_1}):\ f(x) = f(0_{G_1}) = 0_{G_2}\}$$
$$= 1.$$

$$f(\mu)(g_2\alpha m_2) = \sup\{\mu(g_1\alpha m_1):\ f(g_1\alpha m_1) = g_2\alpha m_2\}$$
$$\geq \sup\{\mu(g_1)::\ f(g_1) = g_2\}$$
$$= f(\mu)(g_2)$$

$$f(\mu)(g_2\alpha m_2 + g_2^{`}\beta m_2^{`}) = \sup\{\mu(g_1\alpha m_1 + g_1^{`}\beta m_1^{`}):\ f(g_1\alpha m_1 + g_1^{`}\beta m_1^{`}) = g_2\alpha m_2 + g_2^{`}\beta m_2^{`}\}$$
$$\geq \sup\{\mu(g_1)\wedge\mu(g_1^{`}):\ f(g_1) = g_2, f(g_1^{`}) = g_2^{`}\}$$
$$= \sup\{\mu(g_1): f(g_1) = g_2\}\wedge\sup\{\mu(g_1^{`}):\ f(g_1^{`}) = g_2^{`}\}$$
$$= f(\mu)(g_2)\wedge f(\mu)(g_2^{`}).$$

Therefore $f(\mu)$ is fuzzy submodule of $, G_2$ ΓM module.

Theorem 4

Let $f:G_1 \to G_2$ be f invariant of G_1, G_2 ΓM module and μ be fuzzy submodule of G_2. Then $f^{-1}(\mu)$ is fuzzy submodule of $, G_1$ ΓM module.

Proof

Since μ is fuzzy submodule of G_2.

$$f^{-1}(\mu)(0_{G_1}) = \mu(f(0_{G_1}))$$
$$= \mu(0)$$
$$= 1.$$

$$f^{-1}(\mu)(g_1\alpha m_1) = \mu(f(g_1\alpha m_1))$$
$$= \mu(f(g_1)\alpha f(m_1))$$
$$\geq \mu(f(g_1))$$
$$= f^{-1}(\mu)(g_1)$$

$$f^{-1}(\mu)(g_1\alpha m_1 + g_1^{`}\beta m_1^{`}) = \mu(f(g_1\alpha m_1 + g_1^{`}\beta m_1^{`}))$$
$$= \mu(f(g_1\alpha m_1) + f(g_1^{`}\beta m_1^{`}))$$
$$= \mu(f(g_1)\alpha f(m_1) + f(g_1^{`})\beta f(m_1^{`}))$$
$$\geq \mu(f(g_1))\wedge\mu(f(g_1^{`}))$$
$$= f^{-1}(\mu)(g_1)\wedge f^{-1}(\mu)(g_1^{`}).$$

So $f^{-1}(\mu)$ is fuzzy submodule of G_1 ΓM module.

ISOMORPHISM THEOREMS

Here before the main part we have to define ΓM module homomorphism for constructing isomorphism theorems similarly in Ma et al. (2010).

Definition 1

If G and $G^{'}$ are ΓM modules, then a mapping $f:G \to G^{'}$ with for $x, y \in G$ and $\alpha \in \Gamma$ $f(x \oplus y) = f(x) \oplus f(y)$ and $f(x\alpha y) = f(x)\alpha f(y)$ is called a ΓM module homomorphism.

Firstly, we need to define equivalence class for fuzzy submodules of gamma modules with respect to fuzzy sets.

Definition 2

Let μ be a fuzzy submodule of M and $x, y \in M$. We define the following relation on M such that $x \equiv y(\mathrm{mod}\,\mu) \Leftrightarrow \mu(x-y) = \mu(0)$. We can denote this relation as $x\mu^* y$.

Lemma

The relation μ^* is an equivalence relation.

Proof

1. Since $\mu(0) = \mu(x-x)$, $x\mu^* x$ implies reflexive.
2. If $x\mu^* y$, then we have $\mu(x-y) = \mu(0) = \mu(y-x)$. Thus $y\mu^* x$. That μ^* is symmetric.
3. If $x\mu^* y$ and $y\mu^* z$ then $\mu(x-y) = \mu(y-z) = \mu(0)$. Since μ is fuzzy submodule $\mu(x-z) = \mu(x-y+y-z) \geq \mu(x-y)\wedge\mu(y-z) = \mu(0)\wedge\mu(0) = \mu(0)$

implies that $\mu(x-z)=\mu(0)$ which is $x\mu^{*}z$. Therefore μ^{*} is transitive.

Corollary 1

If $x\mu^{*}y$ then $\mu(x)=\mu(y)$.

Proof

$x\mu^{*}y$ implies $\mu(x-y)=\mu(0)$.

$$\begin{aligned}\mu(x)&=\mu(x-y+y)\\&\geq\mu(x-y)\wedge\mu(y)\\&=\mu(0)\wedge\mu(y)\\&=\mu(y).\end{aligned}$$

$$\begin{aligned}\mu(y)&=\mu(y-x+x)\\&\geq\mu(y-x)\wedge\mu(x)\\&=\mu(0)\wedge\mu(x)\\&=\mu(x).\end{aligned}$$

This completes the proof.

Let $\mu^{*}[x]$ be the equivalence class containing the element x, then G/μ will be the set of all equivalence classes, that is $G/\mu=\{\mu^{*}[x]\mid x\in G\}$. We define the operations $\oplus$ and $\square$ as the following $\mu^{*}[x]\oplus\mu^{*}[y]=\{\mu^{*}[z]\mid z\in\mu^{*}[x]+\mu^{*}[y]\}$ and $r\,\square\,\mu^{*}[x]=\mu^{*}[r\alpha x]$ for some $r\in G$ and $\alpha\in\Gamma$. With these operations, we can get quickly the following result.

Corollary 2

$(G/\mu,\oplus,\square)$ is a gamma module. The following theorem will be given without proof.

Theorem 1

Let $f:G\rightarrow G'$ be an epimorphism of gamma modules and μ, v be fuzzy submodules of G, G' respectively. Then the following statements are equivalent.

1. If f is an epimorphism, then $f(f^{-1}(v))=v$.
2. If μ is a constant on $Kerf$, then $f^{-1}(f(\mu))=\mu$.

With the help of these statements we can establish the isomorphism theorems for fuzzy submodules of gamma modules.

Theorem 2 (First isomorphism theorem)

Let $f:G\rightarrow G'$ be an epimorphism of gamma modules and μ be fuzzy submodule of G with $Kerf\subset\mu_{*}$. Then $G/\mu\cong G'/f(\mu)$.

Proof

We first prove G/μ and $G'/f(\mu)$ are gamma modules. Let $\theta:G/\mu\rightarrow G'/f(\mu)$ with $\theta(\mu^{*}[x])=f(\mu)^{*}[f(x)]$ for all $x\in G$. θ is well defined; $\mu^{*}[x]=\mu^{*}[y]$ implies $\mu(x-y)=\mu(0)$. Then we have $\mu(x)=\mu(y)$. Since $Kerf\subset\mu_{*}$ and μ is constant on $Kerf$, we get $f^{-1}(f(\mu))=\mu$. As a result of this $f^{-1}(f(\mu))(x)=f^{-1}(f(\mu))(y)$. $f(\mu)(f(x))=f(\mu)(f(y))$. This implies $f(\mu)^{*}(f(x))=f(\mu)^{*}(f(y))$. θ is a homomorphism;

$$\begin{aligned}\theta(\mu^{*}[x]\oplus\mu^{*}[y])&=\theta(\mu^{*}[z]\mid z\in\mu^{*}[x]+\mu^{*}[y])\\&=\{f(\mu)^{*}(f(z))\|z\in\mu^{*}[x]+\mu^{*}[y]\}\\&=\{f(\mu)^{*}(f(z)\|z\in\mu^{*}[x]+\mu^{*}[y]\}\\&=\{f(\mu)^{*}(f(x)+f(y))\|x\in\mu^{*}[x],y\in\mu^{*}[y]\}\\&=\{f(\mu)^{*}(f(x))+f(\mu)^{*}(f(y))\|x\in\mu^{*}[x],y\in\mu^{*}[y]\}\\&=\theta(\mu^{*}[x])\oplus\theta(\mu^{*}[y]).\end{aligned}$$

$$\begin{aligned}\theta(g\,\square\,\mu^{*}[x])&=\theta(\mu^{*}[g\alpha x])\\&=f(\mu)^{*}f(g\alpha x)\\&=f(\mu)^{*}f(g)\alpha f(x)\\&=g\,\square\,\theta(\mu^{*}[x])\end{aligned}$$

$\theta(\mu^{*}[0])=f(\mu)^{*}f(0\alpha a)=f(\mu)^{*}[0]=0$. So θ is gamma homomorphism. Also clearly θ is onto. For one to one let $f(\mu)^*(f(x)) = f(\mu)^*(f(y))$. Since $f^{-1}(f(\mu))(x) = f^{-1}(f(\mu))(y)$. Therefore $\mu(x) = \mu(y),$ and $\mu^*[x] = \mu^*[y]$. This completes the proof.

Theorem 3 (Second isomorphism theorem)

If μ and ν are fuzzy submodules of gamma modules with $\mu(0) = \nu(0)$, then $\mu_*/_{\mu \cap \nu} \cong \mu_* + \nu_*/_{\nu}$.

Proof

We certainly know that $\mu \cap \nu$ is a fuzzy submodule of μ_* and ν is a fuzzy submodule of $\mu_* + \nu_*$. Here we conclude that $\mu_*/_{\mu \cap \nu}$ and $\mu_* + \nu_*/_{\nu}$ are gamma submodules. Let us define $\theta : \mu_* \to \mu_* + \nu_*/_{\nu}$ as $\theta(x) = \nu^*([x])$ for all $x \in \mu_*$. Since $\theta(x + y) = \nu^*([x + y]) = \nu^*([x]) + \nu^*([y])$ and $\theta(g \square x) = \nu^*([g\alpha x]) = g\nu^*([x])$ then θ is homomorphism. Clearly θ is onto. Since

$$
\begin{aligned}
Ker\theta &= \{x \in \mu_* \,|\, \theta(x) = \nu^*[0]\} \\
&= \{x \in \mu_* \,|\, \nu^*[x] = \nu^*[0]\} \\
&= \{x \in \mu_* \,|\, \nu(x) = \nu(0) = \mu(0) = \mu(x)\}, \\
&= \{x \in \mu_* \,|\, x \in \nu_*\} \\
&= \mu_* \cap \nu_*,
\end{aligned}
$$

$$\mu_*/_{\mu \cap \nu} \cong \mu_* + \nu_*/_{\nu}.$$

Theorem 4 (Third isomorphism theorem)

Let μ and ν be fuzzy submodules of gamma modules with $\mu(0) = \nu(0)$ and $\nu \subset \mu$. Then

$$\frac{M/_{\nu}}{\mu_*/_{\nu}} \cong M/_{\mu}.$$

Proof

It is easily seen that $\mu_*/_{\nu}$ is a gamma submodule of $M/_{\nu}$. Let us define $\theta : M/_{\nu} \to M/_{\mu}$ with $\theta(\nu^*[x]) = \mu^*[x]$. Since $\nu^*[x] = \nu^*[y]$ implies $\mu(x\alpha y) \ge \nu(x\alpha y) = \nu(0) = \mu(0) \Rightarrow \mu^*[x] = \mu^*[y]$. θ is well defined. Moreover

$$
\begin{aligned}
\theta(\nu^*[x] \oplus \nu^*[y]) &= \theta(\{\nu^*[z] \,|\, z \in \nu^*[x] + \nu^*[y]\} \\
&= \{\mu^*[z] \,|\, z \in \nu^*[x] + \nu^*[y]\} \\
&= \mu^*[\nu^*[x]] \oplus \mu^*[\nu^*[y]] \\
&= \mu^*[x] \oplus \mu^*[y] \\
&= \theta(\nu^*[x]) \oplus \theta(\nu^*[y])
\end{aligned}
$$

$\theta(\nu^*[0]) = \mu^*[0] = 0$. Since θ is onto, θ is also epimorphism.

$$
\begin{aligned}
Ker\theta &= \{\nu^*[x] \in M/_{\nu} \,|\, \theta(\nu^*[x]) = \mu^*[0]\} \\
&= \{\nu^*[x] \in M/_{\nu} \,|\, \mu^*[x] = \mu^*[0]\} \\
&= \{\nu^*[x] \in M/_{\nu} \,|\, \mu(x) = \mu(0)\} \\
&= \{\nu^*[x] \in M/_{\nu} \,|\, x \in \mu^*\} \\
&= \mu_*/_{\nu}
\end{aligned}
$$

We conclude that $\frac{M/_{\nu}}{\mu_*/_{\nu}} \cong M/_{\mu}$. This completes the proof.

Conclusion

We study the isomorphism theorems for fuzzy submodules of gamma modules similarly in Davvaz's (2007) paper. The next work will be about some applications of fuzzy isomorphism theorems in fuzzy algebra. Also some part of second chapter in this paper was mentioned in FUZZYSS09 symposium.

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