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Full Length Research Paper

Study on unified implicit methods for solving variational inequalities

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In this paper, we suggest and analyze some new implicit methods for solving the variational inequalities. These new methods include the extra gradient method of Korpelevich (1976) and the modified extra gradient method of Noor (2004, 2010) as special cases. We also consider the convergence of this new implicit iterative method under some conditions. The ideas and techniques of this paper may stimulate further research in this dynamic area of pure and applied sciences.

Key words: Variational inequality, implicit iterative method, convergence.

INTRODUCTION

Variational inequalities, the origin of which can be traced back to Stampacchia (1964) are being used to study a wide class of diverse unrelated problems arising in various branches of pure and applied sciences in a unified framework. It is well known that the variational inequalities are equivalent to the fixed point problem. This alternative equivalent formulation has played an important and fundamental role in the existence, numerical methods and other aspects of the variational inequalities. There are several iterative methods for solving the variational inequalities such as projection method and its variant form including the extra gradient method of Korpelevich (1976) and the modified extra gradient method of Noor (2004). These two extra gradient type methods are two very distinct and different generalizations of the projection iterative methods.

Motivated and inspired by the ongoing research in this direction, we use the equivalent fixed point formulation to suggest and analyze some new implicit iterative methods for solving the variational inequalities. We have shown that these new implicit methods include the extra gradient method of Korpelevich (1976) and modified extra gradient method of Noor (2004) as special cases. Under some conditions, we consider the convergence criteria of this method. We hope that the ideas and techniques of this paper stimulate further research in this area of pure and applied sciences.

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PRELIMINARIES

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. Let K be a non empty, closed and convex set in H.

For a given nonlinear operator $T: \mathrm{H} \to H$, we consider the problem of finding $u \in K$ such that

$$\langle Tu, v-u \rangle \geq 0, \ \forall v \in K \tag{1}$$

this is called the variational inequality, introduced and studied by Stampacchia (1964).

For the applications, formulations, numerical methods and other aspects of the equilibrium variational inequalities (Giannessi and Maugeri, 1995, Giannessi et al., 2001; Glowinski et al., 1981; Noor, 1998, 2004, 2009, 2010, 2011; Noor et al., 1993, 2011, 2011b, 2011c; Noor et al., 2012, 2012a; 2012b).

We now recall some well known results and concepts.

Lemma 1

Let K be a non empty, closed and convex set in H. For a given $z \in H$, $u \in K$ satisfies the inequality

$$\langle u-z, v-u \rangle \geq 0, \quad \forall\, v \in K,$$

If and only if,

$$u = P_K z,$$

where P_K is the projection of H onto the closed and convex set K.

It is well known that the projection operator P_K is non expansive, that is,

$$\| Tu - Tv \| \leq \| u - v \|, \quad \forall\, u, v \in H.$$

This property plays a very important role in the studies of the variational inequalities and related optimization.

Using Lemma 1, one can easily show that the variational inequality (1) is equivalent to finding $u \in K$ such that:

$$u = P_K [u - \rho T u], \tag{2}$$

where $\rho > 0$ is a constant.

Definition 1

An operator $T : H \rightarrow H$ is said to be strongly monotone, if and only if, there exists a constant $\alpha > 0$ such that:

$$\langle Tu - Tv, u - v \rangle \geq 0, \quad \forall u, v \in H.$$

And Lipschitz continuous, if there exist a constant $\beta > 0$ such that:

$$\| Tu - Tv \| \leq \beta \| u - v \|, \quad \forall\, u, v \in H.$$

MAIN RESULTS

In this section, we use the fixed point formulation (2) to suggest a new unified implicit method for solving the variational inequalities 1 and this is the main motivation of this paper. Using the equivalent fixed point formulation, one can suggest the following iterative method for solving the variational inequality (1).

Algorithm 1

For a given $u_0 \in K$, find the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = P_K [u_n - \rho T u_n], \quad n = 0, 1, 2, \ldots$$

Algorithm 1 is known as the projection iterative method. For the convergence analysis of Algorithm 1 (Noor, 2004) For a given $\lambda \in [0,1]$, we can rewrite (2) as:

$$u = P_K [(1-\lambda) u + u - \rho T u].$$

This fixed point formulation is used to suggest the following iterative method for solving variational inequality (1) as:

Algorithm 2

For a given $u_0 \in K$, find the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = P_K [\lambda (u_n - u_{n+1}) + u_{n+1} - \rho T u_{n+1}], \quad n = 0, 1, 2, \ldots$$

Note that Algorithm 2 is an implicit type iterative method. Clearly, Algorithm 2 includes the implicit method of Noor (2004) and the classical implicit method as special cases. For $\lambda = 0$, Algorithm 2 becomes the implicit projection iterative method for solving the variational inequality according to Noor (2010) and for $\lambda = 1$, we obtain the classical implicit method.

In order to implement this method, we use the predictor-corrector technique. We use Algorithm 1 as the predictor and Algorithm 2 as the corrector. Consequently, we obtain the following two-step iterative method for solving the variational inequality (1).

Algorithm 3

For a given $u_0 \in K$, find the approximate solution u_{n+1} by the iterative schemes:

$$y_n = P_K [u_n - \rho T u_n] \tag{3}$$

$$u_{n+1} = P_K [\lambda (u_n - y_n) + y_n - \rho T y_n], \quad n = 0, 1, 2, \ldots \tag{4}$$

Algorithm 3 is a new two-step implicit iterative method for solving the variational inequality (1). For $\lambda = 0,$ Algorithm 3 reduces to the following iterative method for solving variational inequality (1).

Algorithm 4

For a given $u_0 \in K,$ find the approximate solution u_{n+1} by the iterative schemes:

$$y_n = P_K[u_n - \rho T u_n]$$
$$u_{n+1} = P_K[y_n - \rho T y_n], \qquad n = 0,1,2,...$$

which is known as the modified double projection method and is according to Noor (2004).

For $\lambda = 1,$ Algorithm 3 reduces to the following method for solving variational inequality (1).

Algorithm 5

For a given $u_0 \in K,$ find the approximate solution u_{n+1} by the iterative schemes:

$$y_n = P_K[u_n - \rho T u_n]$$
$$u_{n+1} = P_K[u_n - \rho T y_n], \qquad n = 0,1,2,...$$

which is known as the extra gradient method and is according to Korpelevich (1976)

This clearly shows that Algorithm 3 is a unified implicit method and includes the previously known implicit and predictor-corrector methods as special cases. We now consider the convergence criteria of Algorithm 3 and this is the main motivation of our next result.

Theorem 1

Let the operator T be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0.$ If there exists a constant $\rho > 0$ such that:

$$|\rho - \frac{\alpha}{\beta^2}| < \sqrt{\alpha^2 - \beta^2 \lambda(2-\lambda)}, \;\; \alpha > \beta\sqrt{\lambda(2-\lambda)}, \;\; \lambda < 1. \qquad (5)$$

then, the approximate solution u_{n+1} obtained from Algorithm 3 converges strongly to the exact solution $u \in K$ satisfying the variational inequality (1).

Proof: Let $u \in K$ be a solution of (1) and $y_n,\ u_{n+1}$ be the approximate solution obtained from Algorithm 3. Then, from (2) and (3), we have,

$$\begin{aligned} \| y_n - u \| &= \| P_K[u_n - \rho T u_n] - P_K[u - \rho T u] \| \\ &\le \| u_n - u - \rho(T u_n - T u) \|. \end{aligned} \qquad (6)$$

From the strongly monotonicity and Lipschitz continuity of the operator T , we obtain:

$$\begin{aligned} \| u_n - u - \rho(Tu_n - Tu) \|^2 &= \langle u_n - u - \rho(Tu_n - Tu), u_n - u - \rho(Tu_n - Tu) \rangle \\ &= \| u_n - u \|^2 - 2\langle Tu_n - Tu, u_n - u \rangle + \rho^2 \| Tu_n - Tu \|^2 \\ &\le (1 - 2\alpha\rho + \beta^2\rho^2) \| u_n - u \|^2 . \end{aligned} \qquad (7).$$

From (6) and (7), we obtain:

$$\begin{aligned} \| y_n - u \| &\le \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\, \| u_n - u \| \\ &= \theta \| u_n - u \|, \end{aligned} \qquad (8)$$

Where

$$\theta = \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}. \qquad (9)$$

From (2), (4), (8) and (9), we have:

$$\begin{aligned} \| u_{n+1} - u \| &= \| P_K[\lambda(u_n - y_n) + y_n - \rho T y_n] - P_K[u - \rho T u] \| \\ &\le \lambda \| u_n - y_n \| + \| y_n - u - \rho(T y_n - T u) \| \\ &\le \lambda \| u_n - y_n \| + \theta \| y_n - u \| \\ &\le \lambda \| u_n - u \| + \lambda \| y_n - u \| + \theta \| y_n - u \| \\ &= [\lambda + \theta(\lambda + \theta)] \| u_n - u \| \\ &= \theta_1 \| u_n - u \|, \end{aligned} \qquad (10)$$

where,

$$\theta_1 = [\lambda + \theta(\lambda + \theta)]. \qquad (11).$$

From (5), it follows that $\theta_1 < 1.$ Thus, the fixed point problem (2) has a unique solution and consequently the iterative solution u_{n+1} obtained from Algorithm 3 converges to u , the exact solution of (2).

CONCLUSION

In this paper, we have used the equivalence between the variational inequality and the fixed point problem to suggest and analyze some new implicit iterative methods for solving the variational inequality. We have also shown that the new implicit method includes the extra gradient method of Korpelevich (1976) and the modified extra gradient method of Noor (2004, 2010, 2011) as special cases. We have also discussed the convergence criteria of the proposed new iterative methods under some suitable conditions. We have also shown that this technique can be used to suggest several iterative methods for solving various classes of equilibrium and variational inequalities problems. Results proved in this paper may inspire further research in this area.

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