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Full Length Research Paper

Relativistic scattering state solutions of the Makarov potential

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In this article, we exactly investigate scattering states solutions of the Klein-Gordon equation with the Makarov potential. The normalized polar angle wave functions are obtained by using the Nikiforov-Uvarov method and the normalized radial wave functions of scattering states are expressed in terms of confluent hypergeometric functions. We have also discussed analytical properties of the scattering amplitude.

Key words: Scattering state, Klein-Gordon equation, Makarov potential.

INTRODUCTION

There has been continuous interest in studying of the solutions scattering states within the framework of non-relativistic and relativistic quantum mechanic for central and non-central potentials (Dombey et al., 2000; Kennedy, 2002; Rojas and Villalba, 2005). Dong and Lozada-Gassou (2004) studied the scattering of the two-dimensional Dirac particle by the Coulomb potential. Alhaidari (2005) obtained L^2-series of scattering and bound state solutions of the three-dimensional Schrödinger equation for a large class of non-central potentials that includes, as special cases, The Aharonov-Bohm, Hartmann and magnetic monopole potentials. Chen et al. (2006) studied relativistic scattering with the Coulomb plus a new ring-shaped potential. Wei et al. (2008) investigated approximately analytical scattering state solutions of the l -wave Schrödinger equation for the Eckart potential by a proper approximation to the centrifugal term. Wei et al. (2008) also studied approximately analytical scattering state solutions of the l -wave Schrödinger equation for the Manning-Rosen potential.

Cannata et al. (2008) investigated scattering properties of a square PT -symmetric well potential. Wie et al. (2009) presented approximately analytical bound and scattering state solutions of the arbitrary l -wave Klein-Gordon equation for the mixed Manning-Rosen potentials by an improved new approximation to the centrifugal term. Chen et al. (2009) studied exact solutions of the scattering states of the Klein-Gordon equation with Hartmann potential under the conditions that vector potential is equal to the scalar potential and that the vector potential is equal to the minus scalar potential. Chen et al. (2010) also presented exact solutions of scattering states of the Schrödinger with the Makarov potential using the partial-wave method. Panella et al. (2010) studied scattering states of the one-dimensional Dirac equation in the framework of a position-dependent mass under the action of the Woods-Saxon external potential and derived exact expressions for the reflection and transmission coefficients. Arda et al. (2010) obtained the scattering solutions of the one-dimensional Schrödinger equation for the Woods-Saxon potential within the position-dependent mass formalism. The Makarov potential (Makarov et al., 1967) is a kind of non-central physical potential defined as

$$V(r,\theta)=-\frac{\alpha}{r}+\frac{\beta}{r^2\sin^2\theta}+\frac{\gamma\cos\theta}{r^2\sin^2\theta}, \tag{1}$$

where α, β and γ are positive constants (Chen et al., 2010).

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SCATTERING STATES SOLUTIONS OF KLEIN-GORDON EQUATION WITH MAKAROV POTENTIAL

The Klein-Gordon equation with scalar potential $S(r,\theta)$ and vector potential $V(r,\theta)$ is (in units where $\hbar = c = 1$)

$$[\mathbf{P}^2 + (M + S(\mathbf{r}))^2]\psi(\mathbf{r}) = [E - V(\mathbf{r})]^2\psi(\mathbf{r}), \quad (2)$$

where $\mathbf{P} = -i\nabla$ is the three-dimensional momentum operator, E is the relativistic energy and M denotes the mass. When Makarov scalar potential is equal to the vector potential one, Equation (2) becomes

$$\left[-\nabla^2 + 2(M+E)(-\frac{\alpha}{r} + \frac{\beta}{r^2\sin^2\theta} + \frac{\gamma\cos\theta}{r^2\sin^2\theta})\right]\psi(r,\theta,\phi) = (E^2 - M^2)\psi(r,\theta,\phi). \quad (3)$$

Let

$$\psi(r,\theta,\varphi) = \frac{R(r)}{r}H(\theta)\Phi(\phi), \quad (4)$$

and separating variables in Equation (3), we obtain

$$\frac{d^2u(r)}{dr^2} + \left\{\left(E^2 - M^2\right) + \frac{2(M+E)\alpha}{r} - \frac{\lambda'}{r^2}\right\}u(r) = 0, \quad (5a)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH(\theta)}{d\theta}\right) + \left[\lambda' - \frac{m^2}{\sin^2\theta} - \frac{2(M+E)(\beta+\gamma\cos\theta)}{\sin^2\theta}\right]H(\theta) = 0, \quad (5b)$$

$$\frac{d^2\Phi(\phi)}{d\phi^2} + m^2\Phi(\phi) = 0, \quad (5c)$$

where λ' and m^2 are separation constants. It is well known that the solution of Equation (5c) is

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi} \qquad m = 0, \pm 1, \pm 2, ... \quad (6)$$

Solution of polar angle part

We are now going to drive Eigen values and Eigen functions of polar angle part of the Klein-Gordon equation with Makarov potential that is solution of Equation (5b), by using the Nikiforov-Uvarov method.

Nikiforov-Uvarov method

The Nikiforov-Uvarov method (Nikiforov and Uvarov, 1988) can be used to solve second order differential equations with an appropriate coordinate transformation $x = x(r)$,

$$\psi_n''(x) + \frac{\tilde{\tau}(x)}{\sigma(x)}\psi_n'(x) + \frac{\tilde{\sigma}(x)}{\sigma^2(x)}\psi_n(x) = 0, \quad (7)$$

where $\sigma(x)$ and $\tilde{\sigma}(x)$ are polynomials, at most of second-degree, and $\tilde{\tau}(x)$ is a first-degree polynomial. To find particular solution Equation (7) by separating of variables, one deals with the transformation as follows

$$\psi_n(x) = \phi(x)y_n(x), \quad (8)$$

and it reduces to an equation of hypergeometric type

$$\sigma(x)y_n''(x) + \tau(x)y_n'(x) + \lambda y_n(x) = 0. \quad (9)$$

$\phi(x)$ is defined as logarithmic derivative

$$\frac{\phi'(x)}{\phi(x)} = \frac{\pi(x)}{\sigma(x)}. \quad (10)$$

$y_n(x)$ is the hypergeometric-type function whose polynomials solutions are given by the Rodrigues relation

$$y_n(s) = \frac{B_n}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right], \quad (11)$$

where B_n is the normalization constant and the weight function $\rho(x)$ must be satisfy the condition

$$\frac{d}{ds}w(x)=\frac{\tau(x)}{\sigma(x)}w(x), \qquad w(x)=\sigma(x)\rho(x). \tag{12}$$

The function $\pi(x)$ and the parameter λ required for this method are defined as follows

$$\pi(x)=\frac{\sigma'-\tilde{\tau}}{2}\pm\sqrt{\left(\frac{\sigma'-\tilde{\tau}}{2}\right)^2-\tilde{\sigma}+K\sigma}, \tag{13a}$$

$$\lambda=K+\pi'(x). \tag{13b}$$

In order to find the value of K , the expression under the square root must be square of polynomial. Thus, a new Eigen value equation is

$$\lambda=\lambda_n=-n\tau'(x)-\frac{n(n-1)}{2}\sigma''(x), \tag{14}$$

Where

$$\tau(x)=\tilde{\tau}(x)+2\pi(x), \tag{15}$$

and the derivative of $\tau(x)$ is negative (Nikiforov and Uvarov, 1988). Therefore, by using transformation $x=\cos\theta$, Equation (5b) becomes

$$\frac{d^2H(x)}{dx^2}+\frac{-2x}{(1-x^2)}\frac{dH(x)}{dx}+\frac{1}{(1-x^2)^2}\left[\lambda'(1-x^2)-\gamma'x-m'^2\right]H(x)=0, \tag{16}$$

where $\gamma'=2(M+E)\gamma$ and $m'^2=m^2+2(M+E)\beta$. Let

$$\lambda'=l'(l'+1), \tag{17}$$

and by comparing Equation (16) with Equation (7), we obtain

$$\begin{aligned}&\tilde{\tau}(x)=-2x,\\&\sigma(x)=1-x^2,\\&\tilde{\sigma}(x)=-l'(l'+1)x^2-\gamma'x+l'(l'+1)-m'^2.\end{aligned} \tag{18}$$

Using Equation (13a), $\pi(x)$ is found as

$$\pi(x)=\begin{cases}\pm\sqrt{\dfrac{m'^2+u}{2}}x+\sqrt{\dfrac{m'^2-u}{2}} & for \quad K=l'(l'+1)-\dfrac{m'^2}{2}-\dfrac{1}{2}u,\\ \pm\sqrt{\dfrac{m'^2-u}{2}}x+\sqrt{\dfrac{m'^2+u}{2}} & for \quad K=l'(l'+1)-\dfrac{m'^2}{2}+\dfrac{1}{2}u,\end{cases} \tag{19}$$

Where $u=\sqrt{m'^4-\gamma'^2}$. For polynomial $\tau(x)$ from Equation (15), we choose

$\pi(x)=-\sqrt{\dfrac{m'^2+u}{2}}x+\sqrt{\dfrac{m'^2-u}{2}}$ for

$K=l'(l'+1)-\dfrac{m'^2}{2}-\dfrac{1}{2}u$,

we obtain

$$\tau(x)=-2\left(1+\sqrt{\frac{m'^2+u}{2}}\right)x+2\sqrt{\frac{m'^2-u}{2}}. \tag{20}$$

From Equations (13b) and (14), one obtains

$$\lambda=K+\pi'(x)=l'(l'+1)-\frac{m'^2}{2}-\frac{1}{2}u-\sqrt{\frac{m'^2+u}{2}}, \tag{21a}$$

$$\lambda_n=\lambda=-\tilde{n}\tau'(x)-\frac{\tilde{n}(\tilde{n}-1)}{2}\sigma''(x)=2\tilde{n}\left(1+\sqrt{\frac{m'^2+u}{2}}\right)+\tilde{n}(\tilde{n}-1), \tag{21b}$$

where $\tilde{n}$ is non-negative integer. By equating Equations (21), we obtain

$$l'=\tilde{n}+\sqrt{\frac{m'^2+u}{2}}=\tilde{n}+\sqrt{\frac{m'^2+\sqrt{m'^4-\gamma'^2}}{2}}. \tag{22}$$

For wave functions of polar angle part, using Equations (10 to 12, 18 to 20), one obtains

$$\phi(x)=(1-x)^{\frac{B+C}{2}}(1+x)^{\frac{B-C}{2}},$$
$$\rho(x)=(1-x^2)^B\left(\frac{1+x}{1-x}\right)^{-C},$$
$$y_{\tilde{n}}(x)=B_{\tilde{n}}(1-x)^{-(B+C)}(1+x)^{-(B+C)}\frac{d^{\tilde{n}}}{dx^{\tilde{n}}}\left[(1+x)^{\tilde{n}+B-C}(1-x)^{\tilde{n}+B+C}\right], \quad (23)$$

Where;

$$B=\sqrt{\frac{m'^2+u}{2}},$$
$$C=\sqrt{\frac{m'^2-u}{2}}. \quad (24)$$

$y_{\tilde{n}}(x)$ is expressed in terms of Jacobi polynomials, given $\approx P_{\tilde{n}}^{(B+C,B-C)}(x)$. Substituting $\phi(x)$ and $y_{\tilde{n}}(x)$ into Equation (8), we obtain

$$H_{\tilde{n}}(x)=N_{\tilde{n}}(1-x)^{\frac{B+C}{2}}(1+x)^{\frac{B-C}{2}}P_{\tilde{n}}^{(B+C,B-C)}(x), \quad (25)$$

Where $N_{\tilde{n}}$ is normalization constant determined by

$$\int_{-1}^{+1}\left[H_{\tilde{n}}(x)\right]^2dx=1, \quad (26)$$

and using the relation of Jacobi polynomials, the $N_{\tilde{n}}$ becomes

$$N_{\tilde{n}}=\sqrt{\frac{(2\tilde{n}+2\beta+1)\Gamma(\tilde{n}+1)\Gamma(\tilde{n}+2\beta+1)}{2\Gamma(\tilde{n}+\beta+\gamma+1)\Gamma(\tilde{n}+\beta-\gamma+1)}}. \quad (27)$$

So, we can rewrite $H_{\tilde{n}}(\theta)$ as follows, $(x=\cos\theta)$

$$H_{\tilde{n}}(\theta)=\sqrt{\frac{(2\tilde{n}+2\beta+1)\Gamma(\tilde{n}+1)\Gamma(\tilde{n}+2\beta+1)}{2\Gamma(\tilde{n}+\beta+\gamma+1)\Gamma(\tilde{n}+\beta-\gamma+1)}}\times(1-\cos\theta)^{\frac{B+C}{2}}(1+\cos\theta)^{\frac{B-C}{2}}P_{\tilde{n}}^{(B+C,B-C)}(\cos\theta). \quad (28)$$

Solution of the radial part

We now study Equation (5a). Substituting Equation (17) into Equation (5a) allows us to obtain

$$\frac{d^2R(r)}{dr^2}+\left\{(E^2-M^2)+\frac{2(M+E)\alpha}{r}-\frac{l'(l'+1)}{r^2}\right\}R(r)=0. \quad (29)$$

For scattering states, $|E|\succ M$. Let

$$k=\sqrt{E^2-M^2}, \quad (30a)$$
$$s=(M+E)\alpha, \quad (30b)$$

Equation (29) can be rewritten as

$$\frac{d^2R(r)}{dr^2}+\left\{k^2+\frac{2s}{r}-\frac{l'(l'+1)}{r^2}\right\}R(r)=0. \quad (31)$$

By using transformation $R(r)=(kr)^{l'+1}e^{ikr}f(r)$ and introducing new variable $z=-2ikr$, Equation (31) reduces to

$$z\frac{d^2f}{dz^2}+(2l'+2-z)\frac{df}{dz}-(l'+1-\frac{is}{k})f=0. \quad (32)$$

The confluent hyper-geometric solutions of the aforestated differential equation can be given by (Landau and Lifshitz, 1997)

$$f(r)=F(l'+1-\frac{is}{k},2l'+2,-2ikr). \quad (33)$$

Therefore, the normalized analytical radial wave functions of scattering states are obtained as

$$R_{kl'}(r)=N(kr)^{l'+1}e^{ikr}F(l'+1-\frac{is}{k},2l'+2,-2ikr), \quad (34)$$

Where N is the normalization constant to be determined.

For calculation of normalized constant and phase shifts, we study the asymptotic form of the aforestated equation at large r. Thus, when $r \to \infty$, one can obtain

$$F(l'+1-\frac{is}{k}, 2l'+2, -2ikr) \xrightarrow[r\to\infty]{} \frac{\Gamma(2l'+2)}{\Gamma(l'+1-\frac{is}{k})} e^{-2ikr} (2kr)^{-(l'+1+\frac{is}{k})} e^{i\pi(l'+1+\frac{is}{k})/2} + \frac{\Gamma(2l'+2)}{\Gamma(l'+1+\frac{is}{k})} (2kr)^{-(l'+1-\frac{is}{k})} e^{-i\pi(l'+1-\frac{is}{k})/2}. \tag{35}$$

Substituting Equation (35) into (34) leads to

$$R_{kl'}(r) \xrightarrow[r\to\infty]{} \frac{N\,\Gamma(2l'+2)e^{-\pi s/2k}}{\left|\Gamma(l'+1-\frac{is}{k})\right| 2^{l'+1}} 2\sin(kr + \delta_{l'} - \pi l'/2 + s\,(\ln 2kr)/k\,), \tag{36}$$

where $\delta_{l'}$ is a real number (Chen et al., 2009; Chen et al., 2010). According to Landau and Lifshitz (1997)

$$R_{kl}(r) \xrightarrow[r\to\infty]{} 2\sin(kr + \delta_l - \pi l/2 + Z\,(\ln 2kr)/k\,), \tag{37}$$

is the radial wave function of scattering states for Coulomb potential in the non relativistic case. The Makarov potential is the Coulomb potential surrounded by a ring-shaped inverse square potential, where ring-shaped part is a short-distance potential and it has no influence on the asymptotic expression of the wave function at large r. Therefore, the asymptotic expression of the Makarov potential in the relativistic case is equal to that of the Coulomb potential in the non-relativistic case when $r \to \infty$ (Chen et al., 2010) that is;

$$R_{kl'}(r) \xrightarrow[r\to\infty]{} 2\sin(kr + \delta_l' - \pi l/2 + s\,(\ln 2kr)/k\,). \tag{38}$$

Here δ_l' is the phase shifts. By comparing Equations (38) and (36), the normalization constant is obtained as

$$N = \frac{2^{l'+1}\left|\Gamma(l'+1-\frac{is}{k})\right| e^{\pi s/2k}}{\Gamma(2l'+2)}, \tag{39}$$

and the phase shifts δ_l' are obtained as

$$\delta_l' = \delta_{l'} + \pi(l-l')/2 = \arg\Gamma(l'+1-\frac{is}{k}) + \pi(l-l')/2. \tag{40}$$

SOME DISCUSSION ON SCATTERING AMPLITUDE

According to the general theory of the partial wave method (Landau and Lifshitz, 1997), the scattering amplitude is

$$f\,(\theta) = \sum_{l=0}^{\infty}(2l+1)\left[\frac{e^{2i\delta_l}-1}{2ik}\right] P_l(\cos\theta),$$

where l is the angular quantum number. For analytical properties of scattering amplitude, we have to discuss the properties of $\Gamma(l'+1-\frac{is}{k})$. From the definition of Gamma function

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z\,(z+1)} = \frac{\Gamma(z+3)}{z\,(z+1)(z+2)} = ..., \tag{42}$$

we know that $z = 0, -1, -2, -3, ...$ are the first order poles of the $\Gamma(z)$. Thus, the first order poles of $\Gamma(l'+1-\frac{is}{k})$ are at

$$l'+1-\frac{is}{k}=0,-1,-2,-3,...=-n_r \qquad (n_r=0,1,2,3,...), \tag{43}$$

Or

$$\frac{s}{k}=-i\,(l'+1+n_r), \tag{44}$$

at these poles, the corresponding energy levels are given by (Equation (22))

$$\alpha\sqrt{\frac{M+E}{M-E}}=n_r+l'+1=n_r+\tilde{n}+\sqrt{\frac{m'^2+\sqrt{m'^4-\gamma'^2}}{2}}+1, \tag{45}$$

The expression in equation (45) is the bound states solution of the Makarov potential for spin-0 particles. It is well known that the energy levels of the scattering states reduce to those of the bound states at the poles of the scattering amplitude (Wei et al., 2009).

CONCLUSIONS

In this article, we have studied the exact solutions of the scattering states of the Klein-Gordon equation with the Makarov potential. We presented the normalized radial wave functions of scattering states by the partial-wave method and normalized polar angle part by the Nikiforov-Uvarov method. We also found that energy eigenvalues of bound states at the poles of the scattering amplitude.

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