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Full Length Research Paper

Optimal homotopy asymptotic method solution to convection heat transfer flow

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In this paper, we apply optimal homotopy asymptotic method (OHAM) to compute an approximation to solve the problem of forced convection over a horizontal flat plate. We investigate in this work, that OHAM, being independent of free parameter, provides greater accuracy and yields a good agreement. Moreover, its convergence domain can be easily adjusted and controlled, which shows that OHAM is more effective and easy than the other approximate method.

Key words: Convection heat transfer, optimal homotopy asymptotic method (OHAM), nonlinear equation, convergence constant h.

INTRODUCTION:

Most scientific problems, such as heat transfer, are inherently nonlinear. Most scientists believe that only a limited number of these problems can be solved; otherwise, most of them cannot be solved analytically by using traditional ways. Therefore, these nonlinear equations should be solved using other methods. Some of them are solved using numerical techniques. In the numerical method, stability and convergence should be considered so as to avoid divergence or inappropriate results. In the analytical perturbation method, we need to exert the small parameter in the equation (Nayfeh, 1985). Therefore, finding the small parameter and exerting it into the equation are difficulties of this method. Since there are some limitations with the common perturbation method, and also because the common perturbation method based on upon the existence of a small parameter, developing the method for different applications is very difficult. Therefore, many different powerful mathematical methods, such as artificial parameter method, have been recently introduced to do away with the small parameters. A thorough review on these methods is given by He (2006). There also exist some analytical approaches, such as Adomian decomposition method, the KBM method, Lindstedt-Poincare method, elliptic perturbation method, the homotopy perturbation method by He (2006, 2005) and the variational iteration method by He (2005, 1998). One of the semi-exact methods is the homotopy perturbation method (He, 1999, 2000, 2006, 2005; Ganji and Sadighi, 2006; Rafei and Ganji, 2006; Siddiqui et al., 2006a, b; Beléndez and Hernández, 2007). The applications of these methods in different fields of nonlinear equations, integro-differential equations, fluid mechanics and heat transfer have been studied in Cai et al. (2006), Cveticanin (2006), El-Shahed (2005), Abbasbandy (2006), Ganji and Rafei (2006) and Ganji (2006).

In this paper, we apply optimal homotopy asymptotic method (OHAM), which is a development on homotopy perturbation method (Marinca and Herisanu, 2010), to the problem of forced convection over a horizontal flat plate for finding the approximate solution. The OHAM is valid not only for small parameters, but it has validity and great potential also for solving nonlinear problems in science and engineering. The method has been used by many authors including some of the above in a wide variety of scientific and engineering applications to solve different types of governing differential equations: Linear and nonlinear, homogeneous and non-homogeneous, and

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coupled and decoupled as well.

BASIC IDEA OF OHAM

The basic idea of OHAM was studied by several authors (Marinca and Nicolae, 2008; Marinca et al., 2008; Herisanu et al., 2008; Marinca and Nicolae, 2009; Islam et al., 2010; Ali et al., 2010; Idrees et al., 2010a; Idrees et al., 2010b; Idrees et al., 2010c; Ali et al., 2010; Shah et al., 2010). We applied the OHAM to the following differential equation:

$$L(u(x)) + g(x) + N(u(x)) = 0, \quad B\left(u, \frac{du}{dx}\right) = 0. \tag{1}$$

Here L is a linear operator, x denotes independent variable, $u(x)$ is an unknown function, $g(x)$ is a known function, N is a nonlinear operator and B is a boundary operator. According to OHAM a deformation equation is constructed:

$$\begin{aligned} &(1-p)[L(\phi(x,p)+g(x)] = H(p)[L(\phi(x,p)+g(x)+N(\phi(x,p))], \\ &B\left(\phi(x,p), \frac{\partial \phi(x,p)}{\partial x}\right) = 0, \end{aligned} \tag{2}$$

where $p \in [0,1]$ is an embedding parameter, $H(p)$ is a nonzero auxiliary function for $p \neq 0$ and $H(0)=0, \phi(x,p)$ is an unknown function. Obviously, for $p=0$ and $p=1$ the unknown function holds $\phi(x,0)=u_0(x)$ and $\phi(x,1)=u(x)$, respectively.

Thus, as p varies from 0 to 1, the solution $\phi(x,p)$ varies from $u_0(x)$ to the solution $u(x)$, where $u_0(x)$ is obtained from Equation (2) for $p=0$:

$$L(u_0(x)) + g(x) = 0, \quad B\left(u_0, \frac{du_0}{dx}\right) = 0. \tag{3}$$

We choose auxiliary function $H(p)$ in the form

$$H(p, C_i) = pC_1 + p^2 C_2 + \ldots, \tag{4}$$

where $C_1, C_{2,}\ldots$ are constants to be determined latter.

For solution, expanding $\phi(x,p,C_i)$ in Taylor's series about p, we obtain:

$$\phi(x,p,C_i) = u_0(x) + \sum_{k=1}^{\infty} u_k(x,C_i)\, p^k, \; i = 1,2,3, \tag{5}$$

Now substituting Equation (5) into Equation (2), and equating the coefficient of like powers of p, we obtain the following linear equations.

Zeroth order problem is given by Equation (3), and the first and second order problems are given by the Equations (6) and (7), respectively:

$$L(u_1(x)) = C_1 N_0(u_0(x)), \quad B\left(u_1, \frac{du_1}{dx}\right) = 0, \tag{6}$$

$$\begin{aligned} &L(u_2(x)) - L(u_1(x)) = C_2 N_0(u_0(x)) + \\ &C_1\left[L(u_1(x)) + N_1(u_0(x), u_1(x))\right], \quad B\left(u_2, \frac{du_2}{dx}\right) = 0. \end{aligned} \tag{7}$$

The general governing equations for $u_k(x)$ are given by:

$$\begin{aligned} &L(u_k(x)) - L(u_{k-1}(x)) = C_k N_0(u_0(x)) + \\ &\sum_{i=1}^{k-1} C_i\left[L(u_{k-i}(x)) + N_{k-i}(u_0(x), u_1(x), \ldots, u_{k-1}(x))\right], \\ &k = 2,3,\ldots, \qquad B\left(u_k, \frac{du_k}{dx}\right) = 0 \end{aligned} \tag{8}$$

where $N_m(u_0(x), u_1(x), \ldots, u_{k-1}(x))$ is the coefficient of p^m in the expansion of $N(\phi(x,p,C_i))$ about the embedding parameter p.

$$N(\phi(x,p,C_i)) = N_0(u_0(x)) + \sum_{m=1}^{\infty} N_m(u_0, u_1, \ldots, u_m)\, p^m. \tag{9}$$

It has been observed that the convergence of the series (5) depends upon the auxiliary constants $C_1, C_{2,}\ldots$. If it is convergent at $p=1$, one has

$$\tilde{u}(x, C_1, C_2, \ldots, C_m) = u_0(x) + \sum_{i=1}^{m} u_i(x, C_1, C_2, \ldots, C_m). \tag{10}$$

Substituting Equation (10) into Equation (1) (general problem), it results in the following residual:

$$R(x, C_1, C_2, \ldots, C_m) = L(\tilde{u}(x, C_1, C_2, \ldots, C_m)) + g(x) + N(\tilde{u}(x, C_1, C_2, \ldots, C_m)). \tag{11}$$

If $R = 0$, then $\tilde{u}$ will be the exact solution. Generally it does not happen, especially in nonlinear problems.

For the determinations auxiliary constants of C_i for *i*=1, 2, ….*m*, we choose *a* and *b* in a manner which leads to the optimum values of C_i, s for the convergent solution of the desired problem. There are many methods like Galerkin's method, Ritz method, collocation method to find the optimal values of C_i for i=1, 2, ….m. We apply the method of least squares as under:

$$J\left(C_1, C_2, ..., C_m\right) = \int_a^b R^2\left(x, C_1, C_2, ..., C_m\right) dx, \tag{12}$$

where, R is the residual, $R = L(\tilde{u}) + g(x) + N(\tilde{u})$ and

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = ... = \frac{\partial J}{\partial C_m} = 0, \tag{13}$$

where a and b are properly chosen numbers to locate the desired $C_i (i = 1, 2, ..., m)$. With these constants known, the approximate solution (of order m) is well-determined.

GOVERNING EQUATIONS

Boundary layer flow over a flat plate is governed by the continuity and the Navier-stokes equations. Under the boundary layer assumptions and a constant property assumption, the continuity and Navier-stokes equations become (Kays and Crawford, 1993):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{14}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + v\frac{\partial^2 u}{\partial y^2} + g\beta\left(T - T_\infty\right). \tag{15}$$

Under a boundary layer assumption, the energy transport equation is also simplified.

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 v}{\partial y^2} \tag{16}$$

From Equations (15) and (16), the solutions of the energy and momentum equations are coupled. However, the bouncy force may be neglected if there is a pressure gradient perpendicular to the gravitational force. Thus, in the case of the forced convection over a horizontal flat plate, the solution to the momentum equation is decoupled from the energy equation. However, the solution of the energy equation is still linked to the momentum solution. The following dimensionless variables are introduced in the transformation given by

$$\eta = \frac{y}{\sqrt{x}}\mathrm{Re}_x^{0.5}, \tag{17}$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}. \tag{18}$$

where θ is non-dimensional form of the temperature and the Reynolds number is defined as:

$$\mathrm{Re} = \frac{u_\infty x}{v}. \tag{19}$$

Using equations (14) through (18), the governing equations can be reduced to two equations where f is a function of the similarity variables η :

$$f''' + \frac{1}{2}f f'' = 0, \qquad \varepsilon\theta'' + \frac{1}{2}f\theta' = 0, \tag{20}$$

where $\varepsilon = \frac{1}{pr}$ and f is related to the velocity u is given by

$$f' = \frac{u}{u_\infty}. \tag{21}$$

The reference velocity is the free stream velocity of forced convection. The boundary conditions are obtained by considering the similarity of variables. By using the forced convection technique presented in Kays and Crawford (1993) and Esmaeilpour and Ganji (2007), we have

$$f(0) = 0, \quad f'(0) = 0, \quad \theta(0) = 1, \quad f'(\infty) = 1, \quad \theta(\infty) = 0 \tag{22}$$

OHAM SOLUTION FOR FLOW OVER A FLAT PLATE

Here, we apply the OHAM to nonlinear system (20) consisting of ordinary differential equations. According to the OHAM, we can construct a homotopy of system (20) as follows:

$$\begin{aligned}&(1-p)\left(f''' + pf''' + p^2 f''' + p^3 f'''\right)\\ &= \left(pc_1 + p^2 c_2 + p^3 c_3\right)\begin{pmatrix} f''' + pf''' + p^2 f''' + p^3 f''' \\ +\frac{1}{2}\left(f + pf + p^2 f + p^3 f\right)\left(f'' + pf'' + p^2 f'' + p^3 f''\right)\end{pmatrix}\end{aligned} \tag{23}$$

and

$$1-p\ \ \varepsilon\ \ \theta''+p\theta''+p^2\theta''+p^3\theta''$$
$$=\ pc_1+p^2c_2+p^3c_3\ \begin{pmatrix}\varepsilon\ \ \theta''+p\theta''+p^2\theta''+p^3\theta'' \\ +\dfrac{1}{2}\ \ f+pf+p^2f+p^3f\quad \theta'+p\theta'+p^2\theta'+p^3\theta'\end{pmatrix} \tag{24}$$

We consider f and θ as the following:

$$\left.\begin{aligned} f &= f_0+pf_1+p^2f_2+p^3f_3+\ldots\ldots, \\ \theta &= \theta_0+p\theta_1+p^2\theta_2+p^3\theta_3+\ldots\ldots. \end{aligned}\right\} \tag{25}$$

Considering $f_0'''=0,\ \ \theta_0''=0$ and substituting f and θ from Equation (25) into Equation (24) and after simplification and rearranging depends on powers of p-terms, we get:

$$p^0 = f_0'''\quad ,\ \varepsilon\theta_0'' \tag{26}$$

$$f_0\ \ 0\ =0,\quad f_0'\ \ 0\ =0,\quad f_0'\ \ \infty\ =1,\quad \theta_0\ \ 0\ =0,\quad \theta_0\ \ \infty\ =0, \tag{27}$$

$$p^1\ =-\frac{1}{2}c_1f_0f_0''-f_0'''-c_1f_0'''-f_1'''\ \ ;-\frac{1}{2}c_1f_0\theta_0'-\varepsilon\,\theta_0''-\varepsilon\,c_1\theta_0''+\varepsilon\,\theta_1'' \tag{28}$$

$$f_1\ \ 0\ =0,\quad f_1'\ 0\ =0,\quad f_1'\ \ \infty\ =1,\quad \theta_1\ \ 0\ =0,\quad \theta_1\ \ \infty\ =0, \tag{29}$$

$$p^2\ =\ \left.\begin{matrix}-\dfrac{1}{2}c_2f_0f_0''-\dfrac{1}{2}c_1f_1f_0''-\dfrac{1}{2}c_1f_0f_1''-c_2f_0'''-f_1'''-c_1f_1'''+f_2'''\ ; \\ -\dfrac{1}{2}c_2f_0\theta_0'-\dfrac{1}{2}c_1f_1\theta_0'-\dfrac{1}{2}c_1f_0\theta_1'-\varepsilon c_2\theta_0''-\varepsilon\theta_1''-\varepsilon c_1\theta_1''+\varepsilon\theta_2''\end{matrix}\right\} \tag{30}$$

$$f_2\ \ 0\ =0,\quad f_2'\ \ 0\ =0,\quad f_2'\ \ \infty\ =0,\quad \theta_2\ \ 0\ =0,\quad \theta_2\ \ \infty\ =0, \tag{31}$$

$$p^3\ =\ \left.\begin{matrix}-\dfrac{1}{2}c_3f_0f_0''-\dfrac{1}{2}c_2f_1f_0''-\dfrac{1}{2}c_1f_2f_0''-\dfrac{1}{2}c_2f_0f_1''\quad -\dfrac{1}{2}c_1f_1f_1'' \\ -\dfrac{1}{2}c_1f_0f_2''-c_3f_0'''-c_2f_1'''-f_2'''-c_1f_2'''+f_3'''\ ; \\ -\dfrac{1}{2}c_3f_0\theta_0'-\dfrac{1}{2}c_2f_1\theta_0'-\dfrac{1}{2}c_1f_2\theta_0'-\dfrac{1}{2}c_2f_0\theta_1'-\dfrac{1}{2}c_1f_1\theta_1' \\ -\dfrac{1}{2}c_1f_0\theta_2'-\varepsilon c_3\theta_0''-\varepsilon c_2\theta_1''-\varepsilon\theta_2''-\varepsilon c_1\theta_2''+\varepsilon\theta_3''\end{matrix}\right\} \tag{32}$$

$$f_3\ \ 0\ =0,\quad f_3'\ 0\ =0,\quad f_3'\ \ \infty\ =0,\quad \theta_3\ \ 0\ =0,\quad \theta_3\ \ \infty\ =0, \tag{33}$$

By solving Equations (26) to (32) with boundary conditions (27) to (33), we get:

$$f_0=\frac{\eta^2}{10}\ , \tag{34}$$

$$f_1=\frac{-625\eta^2c_1+2\eta^5c_1}{12000}, \tag{35}$$

$$f_2=\frac{1}{20160000}\begin{pmatrix}-1050000\eta^2c_1+3360\eta^5c_1-643750\eta^2c_1^{\,2}-140\eta^5c_1^{\,2} \\ +11\eta^8c_1^{\,2}-1050000\eta^2c_2+3360\eta^5c_2\end{pmatrix} \tag{36}$$

$$f_3=\frac{1}{1064480000}\begin{pmatrix}-554400000\eta^2c_1+1774080\eta^5c_1-679800000\eta^2c_1^{\,2} \\ -147840\eta^5c_1^{\,2}+11616\eta^8c_1^{\,2}-14688375\eta^2c_1^{\,2} \\ -725670\eta^5c_1^{\,3}+2541\eta^8c_1^{\,3}+20\eta^{11}c_1^{\,3}-55400000\eta^2c_2 \\ +1774080\eta^5c_1-679800000\eta^2c_1c_2-147840\eta^5c_1c_2 \\ +11616\eta^8c_1c_2-554400000\eta^2c_3+1774080\eta^5c_3\end{pmatrix} \tag{37}$$

$$f=f_0+f_1+f_2+f_3 \tag{38}$$

Put Equations (34) to (37) in Equation (38), we get:

$$f=\frac{\eta^2}{10}+\frac{-625\eta^2c_1+2\eta^5c_1}{12000}+\frac{1}{20160000}\begin{pmatrix}-1050000\eta^2c_1+3360\eta^5c_1-643750\eta^2c_1^{\,2} \\ -140\eta^5c_1^{\,2}+11\eta^8c_1^{\,2}-1050000\eta^2c_2+3360\eta^5c_2\end{pmatrix}$$
$$+\frac{1}{1064480000}\begin{pmatrix}-554400000\eta^2c_1+1774080\eta^5c_1-679800000\eta^2c_1^{\,2} \\ -147840\eta^5c_1^{\,2}+11616\eta^8c_1^{\,2}-14688375\eta^2c_1^{\,2}-725670\eta^5c_1^{\,3} \\ +2541\eta^8c_1^{\,3}+20\eta^{11}c_1^{\,3}-55400000\eta^2c_2+1774080\eta^5c_1 \\ -679800000\eta^2c_1c_2-147840\eta^5c_1c_2+11616\eta^8c_1c_2 \\ -554400000\eta^2c_3+1774080\eta^5c_3\end{pmatrix} \tag{39}$$

$$\theta_0 = \frac{5-\eta}{5} \quad (40)$$

$$\theta_1 = \frac{125\eta c_1 - \eta^4 c_1}{1200\varepsilon} \quad (41)$$

$$\theta_2 = \frac{1}{10080000\varepsilon^2}\begin{pmatrix} 1050000\varepsilon\eta c_1 - 8400\varepsilon\eta^4 c_1 + 78125\eta c_1{}^2 \\ +565625\varepsilon\eta c_1{}^2 + 4375\eta^4 c_1{}^2 - 4025\varepsilon\eta^4 c_1{}^2 \\ -40\varepsilon\eta^7 c_1{}^2 - 4\varepsilon\eta^7 c_1{}^2 + 1050000\varepsilon\eta c_2 \\ -8400\varepsilon\eta^4 c_2 \end{pmatrix} \quad (42)$$

$$\theta_3 = \frac{1}{36288000000\varepsilon^3}\begin{pmatrix} 3780000000\varepsilon^2\eta c_1 - 30240000\varepsilon^2\eta^4 c_1 + 562500000\varepsilon\eta c_1^2 \\ +4072500000\varepsilon^2\eta c_1^2 + 31500000\varepsilon\eta^4 c_1^2 - 28980000\varepsilon^2\eta^4 c_1^2 \\ -288000\varepsilon\eta^7 c_1^2 - 28800\varepsilon^2\eta^7 c_1^2 - 224609375\eta c_1^3 \\ +363281250\varepsilon\eta c_1^3 + 862812500\varepsilon^2\eta c_1^3 + 1171875\eta^4 c_1^3 \\ +16031250\varepsilon\eta^4 c_1^3 - 4833750\varepsilon^2\eta c_1^3 + 75000\eta^7 c_1^3 - 130500\varepsilon\eta^7 c_1^3 \\ -13800\varepsilon^2\eta^7 c_1^3 - 560\eta^{10} c_1^3 - 168\varepsilon\eta^{10} c_1^3 - 22\varepsilon^2\eta^{10} c_1^3 \\ +3780000000\varepsilon^2\eta c_2 - 30240000\varepsilon^2\eta^4 c_2 + 562500000\varepsilon\eta c_1 c_2 \\ +4072500000\varepsilon^2\eta c_1 c_2 + 31500000\varepsilon\eta^4 c_1 c_2 - 28980000\varepsilon^2\eta^4 c_1 c_2 \\ -288000\varepsilon\eta^7 c_1 c_2 - 28800\varepsilon^2\eta^7 c_1 c_2 + 3780000000\varepsilon^2\eta c_3 \\ -30240000\varepsilon^2\eta^4 c_3 \end{pmatrix} \quad (43)$$

$$\theta = \theta_0 + \theta_1 + \theta_2 + \theta_3 \quad (44)$$

Put Equations (40) to (43) in Equation (44), we get:

$$\begin{aligned} \theta = {} & \frac{5-\eta}{5} + \frac{-125\eta c_1 - \eta^4 c_1}{1200\varepsilon} \\ & + \frac{1}{10080000\varepsilon^2}\begin{pmatrix} 1050000\varepsilon\eta c_1 - 8400\varepsilon\eta^4 c_1 + 78125\eta c_1{}^2 \\ +565625\varepsilon\eta c_1{}^2 + 4375\eta^4 c_1{}^2 - 4025\varepsilon\eta^4 c_1{}^2 \\ -40\varepsilon\eta^7 c_1{}^2 - 4\varepsilon\eta^7 c_1{}^2 + 1050000\varepsilon\eta c_2 - 8400\varepsilon\eta^4 c_2 \end{pmatrix} \\ & + \frac{1}{36288000000\varepsilon^3}\begin{pmatrix} 3780000000\varepsilon^2\eta c_1 - 30240000\varepsilon^2\eta^4 c_1 + 562500000\varepsilon\eta c_1^2 \\ +4072500000\varepsilon^2\eta c_1^2 + 31500000\varepsilon\eta^4 c_1^2 - 28980000\varepsilon^2\eta^4 c_1^2 \\ -288000\varepsilon\eta^7 c_1^2 - 28800\varepsilon^2\eta^7 c_1^2 - 224609375\eta c_1^3 + 75000\eta^7 c_1^3 \\ -130500\varepsilon\eta^7 c_1^3 - 13800\varepsilon^2\eta^7 c_1^3 - 560\eta^{10} c_1^3 - 168\varepsilon\eta^{10} c_1^3 \\ -22\varepsilon^2\eta^{10} c_1^3 + 3780000000\varepsilon^2\eta c_2 - 30240000\varepsilon^2\eta^4 c_2 \\ +562500000\varepsilon\eta c_1 c_2 + 4072500000\varepsilon^2\eta c_1 c_2 + 31500000\varepsilon\eta^4 c_1 c_2 \\ -28980000\varepsilon^2\eta^4 c_1 c_2 - 288000\varepsilon\eta^7 c_1 c_2 - 28800\varepsilon^2\eta^7 c_1 c_2 \\ +3780000000\varepsilon^2\eta c_3 - 30240000\varepsilon^2\eta^4 c_3 \end{pmatrix} \end{aligned} \quad (45)$$

Using, the method of least square to obtain the unknown convergent constants in f:

$$c_1 = -0.9874136915279734;\ c_2 = 0.20338113796491977;\ c_3 = 0.046365760104753885$$

Substituting these values in Equation (39), we get:

$$\begin{aligned} f = {} & \frac{\eta^2}{10} + \frac{1}{12000}\ 617.13355720498 34\eta^2 - 1.9748273830559469\eta^5 \\ & + \frac{1}{20160000}\begin{pmatrix} 195587.07363907702\eta^2 - 2770.8473917222263\eta^5 \\ +10.72484378038 59\eta^8 \end{pmatrix} \\ & + \frac{1}{1064480000}\begin{pmatrix} 2.409315026252579918\eta^2 - 724519.0736390\eta^5 \\ +6546.437473466595\eta^8 - 19.254286524093 94\eta^{11} \end{pmatrix} \end{aligned} \quad (46)$$

Table 1. The results of OHAM, HPM (Esmaeilpour and Ganji, 2007) and NM (Bejan, 1995) methods for $f \ \ \eta$.

η	$f \ \ \eta$ **HPM**	$f \ \ \eta$ **NM**	$f \ \ \eta$ **OHAM**
0	0	0	0
0.2	0.0069699	0.0066412	0.0065356
0.4	0.0278758	0.0266762	0.0261391
0.6	0.0626959	0.0597215	0.0587927
0.8	0.1113738	0.1061082	0.10445
1.0	0.1738016	0.1655717	0.163024
1.2	0.2498038	0.2379487	0.23437
1.4	0.3391216	0.3229815	0.318277
1.6	0.4414008	0.4203207	0.414454
1.8	0.5561797	0.5295180	0.522526
2.0	0.6828833	0.6500243	0.642019
2.2	0.8208206	0.7811933	0.772368
2.4	0.9691873	0.9222901	0.912911
2.6	1.1270772	1.0725059	1.0629
2.8	1.2935005	1.2309773	1.22149
3.0	1.4674133	1.3968082	1.38781
3.2	1.6477584	1.5690949	1.56093
3.4	1.8335195	1.7469501	1.73989
3.6	2.0237911	1.9295251	1.92378
3.8	2.2178650	2.1160298	2.11172
4.0	2.4153361	2.3057464	2.30291
4.2	2.6162294	2.4980396	2.49668
4.4	2.8211494	2.6923609	2.69246
4.6	3.0314545	2.8882480	2.88982
4.8	3.2494582	3.0853206	3.08845
5.0	3.4786579	3.2832736	3.28806

Again by using, the method of Least Square to obtain the unknown constants in: θ

$c_1 = -0.9874136915279734,\ c_2 = 0.20338113796491977,\ c_3 = 0.046365760104753885$

Substituting these values in Equation (45), we get:

$$\begin{aligned}\theta = \frac{5-\eta}{5} + \frac{1}{1200} \ \ & -123.42671144099668\eta + 0.9874136915279734\eta^4 \\ & + \frac{1}{10080000}\begin{pmatrix} -195587.073630907702\eta + 6927.118479305565\eta^4 \\ -42.8993751215436\eta^7 \end{pmatrix} \\ & + \frac{1}{36288000000}\begin{pmatrix} -1.6427147906304502\eta + 1.2349763800020106\eta^4 \\ -178539.20382181625\eta^7 + 722.0357446535227\eta^{10} \end{pmatrix}\end{aligned} \tag{47}$$

(Tables 1 and 2 and Figures 1 to 4.)

Conclusion

In this study, we discussed heat transfer problem with a small parameter, but the new technique was used to solve the coupled nonlinear equations. The procedure has explicit, effective and distinct advantages over the existing approximation methods. Moreover, the approximate solution obtained here is valid not only for weakly nonlinear equations, but also for highly nonlinear ones.

The convergence and low error is remarkable, clearly for $\varepsilon \approx 1$, which represents that the OHAM has a very high accuracy.

Finally, it has been observed that OHAM reveals very good agreement with numerical results.

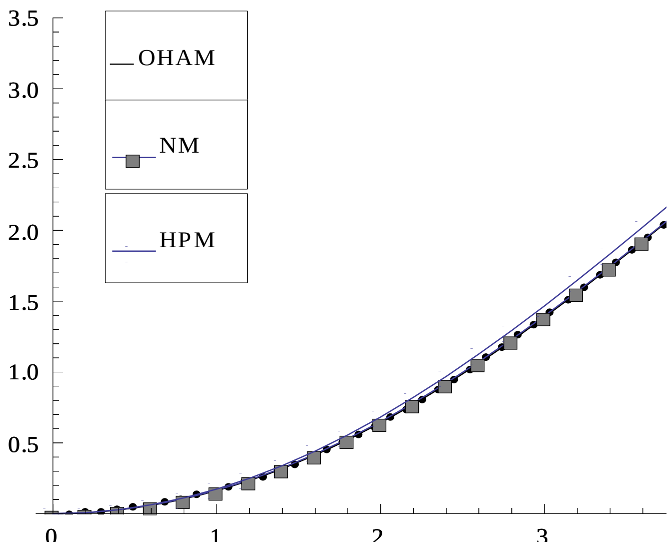

Figure 1. The comparison of the answers resulted by OHAM, HPM and NM for $f(\eta)$.

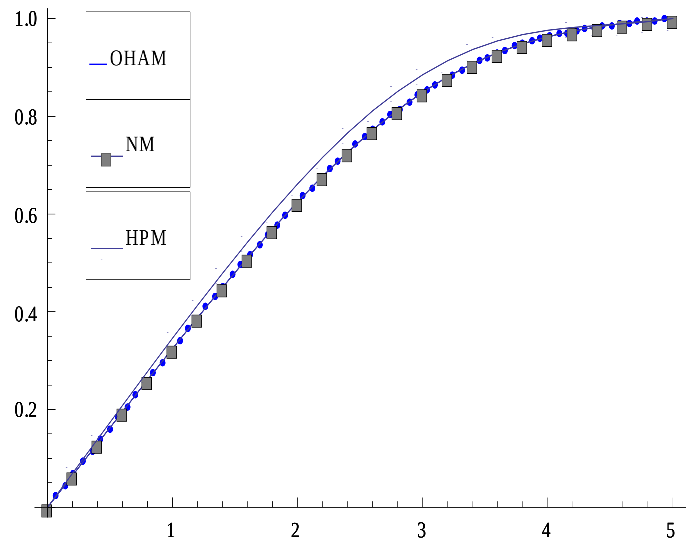

Figure 2. The comparison of the answers resulted by OHAM, HPM and NM for $f'(\eta)$.

Table 2. The results of OHAM and HPM (Esmaeilpour and Ganji, 2007), NM (Bejan, 1995) methods for $f'(\eta)$.

η	$f'(\eta)$ **HPM**	$f'(\eta)$ **NM**	$f'(\eta)$ **OHAM**
0	0	0	0
0.2	0.0696975	0.0664077	0.0653542
0.4	0.1393444	0.1327641	0.130667
0.6	2.088105	0.1989372	0.195832
0.8	0.2778800	0.2647094	0.260673
1.0	0.3462538	0.3297800	0.324945
1.2	0.4135539	0.3937761	0.388339
1.4	0.4793309	0.4562617	0.450488
1.6	0.5430747	0.5167567	0.510975
1.8	0.6042289	0.5747581	0.569345
2.0	0.6622097	0.6297657	0.62512
2.2	0.7164291	0.6813103	0.677819
2.4	0.7663226	0.7289819	0.726977
2.6	0.8113803	0.7724550	0.772174
2.8	0.8511819	0.8115096	0.813057
3.0	0.8854328	0.8460444	0.84937
3.2	0.9140010	0.8760814	0.880976
3.4	0.9369507	0.9017612	0.907886
3.6	0.9545718	0.9233296	0.930268
3.8	0.9673977	0.9411181	0.948451
4.0	0.9762106	0.9555182	0.96292
4.2	0.9820237	0.9669570	0.974274
4.4	0.9860369	0.9758708	0.983168
4.6	0.9895542	0.9826835	0.990212
4.8	0.9938540	0.9877895	0.995818
5.0	0.9999999	0.9915419	1

Table 3. The results of OHAM and HPM (Ganji, 2007), NM (Bejan, 1995) methods for $\theta(\eta)$.

η	$\theta(\eta)$ **HPM**	$\theta(\eta)$ **NM**	$\theta(\eta)$ **OHAM**
0	1	1	1
0.2	0.9303024	0.9335922	0.934646
0.4	0.8606555	0.8672358	0.869333
0.6	0.7911894	0.8010627	0.804168
0.8	0.7221199	0.7352908	0.739327
1.0	0.6537461	0.6702199	0.675055
1.2	0.5864460	0.6062238	0.611661
1.4	0.5206690	0.5437381	0.549512
1.6	0.4569252	0.4832432	0.489025
1.8	0.3957710	0.4252418	0.430655
2.0	0.3377909	0.3702342	0.37488
2.2	0.2835708	0.3186896	0.322181
2.4	0.2336773	0.2710180	0.273023
2.6	0.1886196	0.2275449	0.227826
2.8	0.1488180	0.1884903	0.186943
3.0	0.1145671	0.1439554	0.15063
3.2	0.0959989	0.1239183	0.119024
3.4	0.0630492	0.0882386	0.0921138
3.6	0.0554281	0.0666702	0.0697324
3.8	0.0326022	0.0588819	0.0515487
4.0	0.0237893	0.0314817	0.0370799
4.2	0.0179762	0.0330429	0.0257259
4.4	0.0139630	0.0241292	0.0168318
4.6	0.0104457	0.0173165	0.0097884
4.8	0.0061459	0.0122105	0.00418158
5.0	3.359E-10	0.0084581	-1.17853E-16

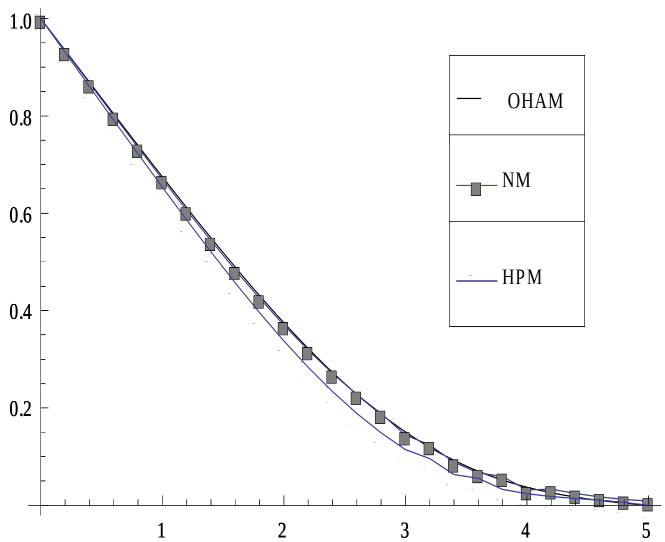

Figure 3. Comparison of the answers resulted by OHAM, HPM and NM for $\theta\ \eta\ \ at\,\mathrm{Pr}=1.$

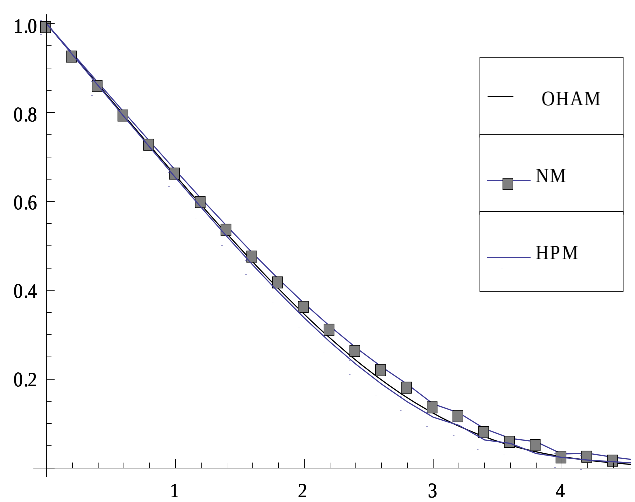

Figure 4. Comparison of the answers resulted by OHAM, HPM and NM for $\theta\ \eta\ \ at\,\mathrm{Pr}=1.2$.

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