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Full Length Research Paper

Exact solutions of N-Dimensional Schrödinger equation for a potential containing coulomb and quadratic terms

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We encounter Coulomb and quadratic terms in the Hamiltonian relation in many branches of physics. Here we consider the N-dimensional space Schrödinger equation in the presence of Coulomb and quadratic terms and analytically calculate the eigenfunctions and the corresponding eigenenergies.

Key words: Schrödinger equation, N-dimension, Coulomb and quadratic potentials.

INTRODUCTION

Among the different forms of physical potentials which appear in the Hamiltonian, those with spherical symmetry have received great attention within the recent years because of their wide applications (de Souza and Almeida, 2000; Roy and Roy, 2002; Alhaidari, 2002; Serra and Lipparini, 1997; Li et al., 2003; Dong et al., 1999). On the other hand, since the study of many physical systems corresponds to study an N-dimensional problem, many attempts have been made to analyze N-dimensional spaces (Dong, 2000; Arda and Sever, 2010; Milanovic and Ikovic, 1999; Cardoso and Alvarez-Nodarse, 2003; Coelho and Amaral, 2002; Chen, 2005; Kalnin et al., 2002; Oyewumi and Ogunsola, 2004). Often, the potential is assumed as a hypercentral one.

The merit of using hyperspherical coordinates for hypercentral potentials is that it simply transforms the Schrödinger equation into a single hyperradial one. In the minimal case, there exist at least two hypercentral potentials which can be analytically solved; the harmonic oscillator, which is exactly hypercentral, and the hypercoulomb potential. Although the latter is not confining, it has interesting properties and for example leads to a power-law behavior of the proton from factor (Paz, 2001; Oyewumi et al., 2008; Ikhdair and Sever, 2008; Hamzavi et al., 2010a, b, c, d; Hassanabadi et al., 2008; Hassanabadi and Rajabi, 2007; Zarrinkamar et al., 2010; Zarrinkamar et al., 2010b; Zarrinkamar et al., 2011; Hamzavi et al., 2011; Hassanabadi et al., 2010a, b, c; Hamzavi et al., 2009; Ferraris et al., 1995; Campostrini et al., 1987; Bali et al., 2000).

Here, we have studied potentials of the form:

$$V(r) = V_{h.o.}(r) + V_{hyc}(r) = ar^2 - \frac{b}{r} \qquad (1)$$

Where r denotes the hyperradius and the coefficients are both constant.

The exact-band state solutions in N-dimensions

The motion of a particle in a spherically symmetric potential in N-dimensions is written as:

$$\left\{-\frac{\hbar^2}{2\mu}\Delta_N + V(r)\right\}\psi_{n,l,m}(r,\Omega_N) = E_{n,l}\psi_{n,l,m}(r,\Omega_N) \qquad (2)$$

Where,

$$\Delta_N \equiv \nabla_N^2 = \frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\right) - \frac{\Lambda_N^2(\Omega_N)}{r^2} \qquad (3)$$

As hyperspherical harmonics are eigenfunctions of the

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operator $\Lambda_N^2(\Omega_N)$, we could write:

$$\psi_{n,l,m}(\Omega_N) = R_{n,l}(r) Y_l^m(\Omega_N) \tag{4}$$

It is well-known that $\frac{\Lambda_N^2(\Omega_N)}{r^2}$ is a generalization of the centrifugal barrier for the case of N-dimensional space and involves the angular coordinates Ω_N.

The eigenvalues of the $\Lambda_N^2(\Omega_N)$ are given by:

$$\Lambda_N^2(\Omega_N) Y_l^m(\Omega_N) = l(l+N-2) Y_l^m(\Omega_N) \tag{5}$$

Where $Y_l^m(\Omega_N)$, $R_{n,l}(r)$, $E_{n,l}$ and l represent the hyperspherical harmonics, $R_{n,l}(r)$ is the hyperradial part, the energy eigenvalues and orbital angular momentum, respectively. The hyperradial part in the N-dimensional case is:

$$R''_{n,l}(r) + \frac{N-1}{r} R'_{n,l}(r) - \frac{l(l+N-2)}{r^2} R_{n,l}(r) + \frac{2\mu^*}{\hbar^2} E_{n,l} R_{n,l}(r) - \frac{2\mu^*}{\hbar^2} a r^2 R_{n,l}(r) + \frac{2\mu^*}{\hbar^2} \frac{b}{r} R_{n,l}(r) = 0 \tag{6}$$

Before proceeding further, for the sake of simplicity, we introduce the following new parameters:

$$\alpha_n = \frac{2\mu^*}{\hbar^2} E_{n,l} \tag{7a}$$

$$\beta = \frac{2\mu^*}{\hbar^2} a \tag{7b}$$

$$\gamma = \frac{2\mu^*}{\hbar^2} b \tag{7c}$$

In order to solve Equation (6) let us consider the special limit $r \to \infty$ corresponding to:

$$\frac{d^2 R_{n,l}(r)}{d^2 r} - \beta r^2 R_{n,l}(r) = 0 \tag{8}$$

Multiplying both sides of equation (8) in $2\frac{dR_{n,l}(r)}{dr}$, assuming $2re^{-r^2} \prec\prec 4r^3 e^{-r^2}$, as really the case for large r, and finally integrating with respect to r give:

$$\lim_{r\to\infty} R_{n,l}(r) = \exp\left\{-\sqrt{\beta}\frac{r^2}{2}\right\} \tag{9}$$

Therefore we choose the term corresponding to the relative motion according to:

$$R_{n,l}(r) = N_{n,l} \exp\left\{-\sqrt{\beta}\frac{r^2}{2}\right\} f_{n,l}(r) \tag{10}$$

With $N_{n,l}$ being the normalization factor. Substitution of $R_{n,l}(r)$ from the given equations in Equation (6) gives:

$$f''_{n,l}(r) + \left(\frac{N-1}{r} - 2\sqrt{\beta} r\right) f'_{n,l}(r) + \left(\alpha_n - N\sqrt{\beta} - \frac{l(l+N-2)}{r^2} + \frac{\gamma}{r}\right) f_{n,l}(r) = 0 \tag{11}$$

Considering $h(r) = \left(\frac{N-1}{r} - 2\sqrt{\beta} r\right)$ and $g(r) = \left(\alpha_n - N\sqrt{\beta} - \frac{l(l+N-2)}{r^2} + \frac{\gamma}{r}\right)$, it could be seen that while $h(r)$ is not analytical in $r = 0$, and neither is $g(r)$, the functions $rh(r) = H(r)$ and $r^2 g(r) = G(r)$ are analytical in $r = 0$. Making use of these two new functions instead of the former ones, we have the following characteristic equation:

$$k(k-1) + kH(0) + G(0) = 0 \tag{12a}$$

$$k^2 + k(N-2) - l(l+N-2) = 0 \tag{12b}$$

$$k = \frac{-(N-2) \pm \sqrt{(N-2)^2 + 4l(l+N-2)}}{2}, \quad k_+ = l, \ k_- = -(N+l-2) \tag{12c}$$

As the wave function must be bounded in the origin, the acceptable physical limit is $k \geq 0$. Let us now consider solutions of the form:

$$f_{n,m}(r) = r^k \sum_{n=0}^{\infty} a_n r^n \tag{13}$$

Substitution of the given equation in equation (11) gives, after setting equal the corresponding powers of r on both sides,

$$a_0 \neq 0 \tag{14a}$$

$$a_1 = -\frac{\gamma a_0}{(2l+N-1)} \tag{14b}$$

$$a_n = \frac{\left[2\sqrt{\beta}(n+l+\frac{N}{2}-2) - \alpha_{n-2}\right] a_{n-2} - \gamma a_{n-1}}{\left[(n_r+l)(n_r+l+N-2) - l(l+N-2)\right]}, \quad n \geq 2 \tag{14c}$$

On the other hand, the series must be bounded for $n = n_r$. The latter leads to the equations:

$$E_{n_r-2,l} = \sqrt{\frac{2\hbar^2}{\mu^*}}a(n_r + l + \frac{N}{2} - 2) - b(\frac{a_{n_r-1}}{a_{n_r-2}}) \qquad (15)$$

It could be seen that the term a_{n+1} contains a_n and a_{n-1}, that is,

$$a_{n_r+1} = \frac{\left[2\sqrt{\beta}(n_r + l + \frac{N}{2} - 1) - \alpha_{n_r-1} + N\sqrt{\beta}\right]a_{n_r-1} - \gamma a_{n_r}}{\left[(n_r + l + 1)(n_r + l + N - 1) - l(l + N - 2)\right]} \qquad (16)$$

Naturally, for a_{n+1} to be zero, the coefficient of a_{n-1} must be zero, too. This leads to the given relation:

$$2\sqrt{\beta}(n_r + l + \frac{N}{2} - 1) + N\sqrt{\beta} = \alpha_{n_r-1} \qquad (17)$$

Which immediately gives the energy difference between the successive states n_r and n_{r+1}, after making use of Equation (7-a). To give some examples, let us consider the eigenvalue problem for $n_r = 0$

$$E_{0,l} = \sqrt{\frac{2\hbar^2}{\mu^*}}a(l + \frac{N}{2}) + b(\frac{1}{2l + N - 1}) \qquad (18)$$

Which corresponds to the wave function given as:

$$R_{2,l}(r) = N_{2,l}\exp\left\{-\sqrt{\beta}\frac{r^2}{2}\right\}r^l\left\{1 - \frac{\gamma r}{(2l + N - 1)}\right\}, \; a_0 = 1 \qquad (19)$$

Which is of course an essential ingredient in many theoretical calculations of many, or few body systems such as the two- or three-electron-quantum dots.

CONCLUSION

Exact analytical solutions of Schrödinger equation have always been of great interest in different annals of physics. Depending on the physical problem, the dimension and also the chosen potential could vary. Namely, Coulomb and quadratic terms are present in different areas of physics including the solid state physic, nuclear physics, etc. In the present problem, we have presented an analytical method for this potential and thereby have calculated the corresponding eigenfunctions as well as the eigenenergies.

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