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Homotopy analysis method with modified Reimann-Liouville derivative for space fractional diffusion equation

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In this paper, we applied the homotopy analysis method (HAM) to construct the analytical solutions of the space fractional diffusion equations. The derivatives are defined in the Jumarie's fractional derivative sense. The explicit solutions of the equations have been presented in the closed form by using initial conditions. Two typical examples have been discussed. The results reveal that the method is very effective and simple. On the basis of computational work and subsequent numerical results, it is worth noting that the advantage of the homotopy analysis methodology is that it displays a fast convergence of the solution.

Key words: Analytical solution, fractional diffusion equation, Reimann-Liouville fractional derivative, homotopy analysis method.

INTRODUCTION

In recent years, analysis of fractional differential equations by different methods and techniques, which are obtained from the classical differential equations in mathematical physics, engineering, vibration and oscillation by replacing the second order time derivative by a fractional derivative of order α satisfying $0 < \alpha \leq 1$, have been a field of growing interest as evident from literature survey such as, Adomian decomposition method (Momani, 2005a; Momani and Ibrahim, 2007; Momani, 2005b), variational iteration method and modified decomposition method (Das, 2008), variational iteration method (Momani et al., 2007), generalized differential transform method (Odibat et al., 2008). Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes.

Recently, a new modified Riemann-Liouville left derivative is proposed by Jumarie (1993, 2006). Comparing with the classical caputo derivative, the definition of the fractional derivative is not required to satisfy higher integer-order derivative than α. Secondly, α th derivative of a constant is zero. For these merits, Jumarie modified derivative we successfully applied in the probability calculus (2009) and fractional Laplace problem (Jumarie, 2009 a, b).

The solution of a fractional differential equation is much involved. In general, there exists no method that yields an exact solution for a fractional differential equation. Only approximate solutions can be derived using the linearization or perturbation methods. The homtopy anlysis method is relatively a new approach providing an analytical approximation to linear and nonlinear problems, and is particularly valuable as tool for scientists, engineers, and applied mathematicians, because it provide immediate and visible symbolic terms of analytic solutions, as well as a numerical approximate solution to both linear and nonlinear differential equations without linearization or discretization.

In this paper, we will consider space fractional diffusion equation by homotopy analysis method. The derivatives

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are understood in the modified Riemann-Liouville sense. By the present method, numerical results can be obtained with using a few iterations. The homotopy analysis method (Liao, 2003a; b) contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series for large value of t. Unlike, other numerical methods are given low degree of accuracy for large values of t. Therefore, the homotopy analysis method (HAM) handles linear and inhomogeneous problems without any assumption and restriction (Liao, 2009).

Firstly, we consider a one-dimensional fractional diffusion equation considered in (Meerschaert et al., 2006):

$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + q(x,t), \quad (1)$$

on a finite domain $x_L < x < x_R$ with $1 < \alpha \le 2$. We assume that the diffusion coefficient (or diffusivity) $d(x) > 0$. We also assume an initial condition $u(x,t=0) = s(x)$ for $x_L < x < x_R$ and Dirichlet boundary conditions of the form $u(x_L,t) = 0$ and $u(x_R,t) = b_R(t)$. Equation 1 uses a Riemann fractional derivative of order α.

Secondly, we consider a two-dimensional fractional diffusion equation considered in Tadjeran et al. (2006):

$$\frac{\partial u(x,y,t)}{\partial t} = d(x,y)\frac{\partial^{\alpha} u(x,y,t)}{\partial x^{\alpha}} + e(x,t)\frac{\partial^{\beta} u(x,y,t)}{\partial y^{\beta}} + q(x,y,t), \quad (2)$$

on finite rectengular domain $x_L < x < x_H$ and $y_L < y < y_H$, with fractional orders $1 < \alpha \le 2$ and $1 < \beta \le 2$, where the diffusion coefficients $d(x) > 0$ and $e(x,y) > 0$. The 'forcing' function $q(x,y,t)$ can be used to represent sources and sinks. We will assume that the fractional diffusion equation has a unique and sufficiently smooth solution under the following initial and boundary conditions. Assume the initial condition $u(x,y,t=0) = f(x,y)$ for $x_L < x < x_H$, $y_L < y < y_H$ and Dirichlet boundary condition $u(x,y,t) = B(x,y,t)$ on the boundary (perimeter) of the rectengular region $x_L \le x \le x_H$, $y_L \le y \le y_H$, with the additional restiriction that $B(x_L,y,t) = B(x,y_L,t) = 0$. In physics applications, this means that the left/lower boundary is set far away enough from evolving that no significant concentrations reach that boundary. The classical dispersion equation in two-dimensions is given by $\alpha = \beta = 2$. The values of $1 < \alpha < 2$, or $1 < \beta < 2$ model a super diffusive process in that coordinate. Equation 2 also uses Riemann fractional derivatives of order α and β.

Modified Riemann-Liouville derivative

Assume $f : R \to R, x \to f(x)$ denote a continuous (but not necessarily differentiable) function and let the partition $h > 0$ in the interval [0, 1]. Through the fractional Riemann Liouville integral

$${}_0I_x^{\alpha} f(x) = \frac{1}{\Gamma\alpha}\int_0^x (x-\xi)^{\alpha-1} f(\xi)d\xi \ , \alpha > 0, \quad (3)$$

The modified Riemann-Liouville derivative is defined as

$${}_0D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-\xi)^{n-\alpha}(f(\xi) - f(0))\, d\xi, \quad (4)$$

where $x \in [0,1]$, $n-1 \le \alpha < n$ and $n \ge 1$.

Jumarie's derivative is defined through the fractional difference

$$\Delta^{\alpha} = (FW-1)^{\alpha} f(x) = \sum_0^{\infty}(-1)^k \binom{\alpha}{k} f[x + (\alpha - k)h], \quad (5)$$

where $FWf(x) = f(x+h)$. Then the fractional derivative is defined as the following limit,

$$f^{(\alpha)} = \lim_{h\to 0}\frac{\Delta^{\alpha} f(x)}{h^{\alpha}}. \quad (6)$$

The proposed modified Riemann –Liouville derivative as shown in Equation 4 is strictly equivalent to Equation 6. Meanwhile, we would introduce some properties of the fractional modified Riemann –Liouville derivative in Equations 7 and 8.

(a) Fractional Leibniz product law

$${}_0D_x^{\alpha}(uv) = u^{(\alpha)}v + uv^{(\alpha)}, \quad (7)$$

(b) Fractional Leibniz formulation

$${}_0I_x^{\alpha}\,{}_0D_x^{\alpha} f(x) = f(x) - f(0), 0 < \alpha \le 1, \quad (8)$$

Therefore, the integration by part can be used during the fractional calculus

$${}_aI_b^{\alpha}u^{(\alpha)}v = (uv)/_a^b - {}_aI_b^{\alpha}uv^{(\alpha)}. \quad (9)$$

(c) Integration with respect to $(d\xi)^{\alpha}$

Assume $f(x)$ denote a continuous $R \to R$ function, we use the following quality for the integral with respect to $(d\xi)^{\alpha}$

$$ {}_0I_x^\alpha f(x) = \frac{1}{\Gamma\alpha}\int_0^x (x-\xi)^{\alpha-1} f(\xi)\, d\xi, 0<\alpha\le 1, \\ = \frac{1}{\Gamma(1+\alpha)}\int_0^x f(\xi)(d\xi)^\alpha, \quad (10) $$

HOMOTOPY ANALYSIS METHOD (HAM)

We consider the following differential equation:

$$ FD[u(x,t)] = 0, \quad (11) $$

where *FD* is a nonlinear operator for this problem, *x* and *t* denote an independent variable, $u(x,t)$ is an unknown function. In the frame of homotopy analysis method (HAM), we can construct the following zeroth-order deformation:

$$ (1-q)L(U(x,t;q)-u_0(x,t)) = q\hbar H(x,t)FD(U(x,t;q)), \quad (12) $$

where $q\in[0,1]$ is the embedding parameter, $\hbar\neq 0$ is an auxiliary parameter, $H(x,t)\neq 0$ is an auxiliary function, *L* is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $U(x,t;q)$ is an unknown function of the independent variables *x, t* and *q*.
Obviously, when $q=0$ and $q=1$, it holds respectively.

$$ U(x,t;0) = u_0(x,t), \quad U(x,t;1) = u(x,t), \quad (13) $$

Using the parameter *q*, we expand $U(x,t;q)$ in Taylor series as follows:

$$ U(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\, q^m, \quad (14) $$

Where

$$ u_m = \frac{1}{m!}\frac{\partial^m U(t;q)}{\partial^m q}\bigg|_{q=0}. $$

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function $H(x,t)$ are selected such that the series (12) is convergent at $q=1$, then due to Equation 12 we have

$$ u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) \quad (15) $$

Let us define the vector

$$ \vec{u}_n(x,t) = \{u_0(x,t), u_1(x,t), \ldots, u_n(x,t)\} $$

Differentiating Equations 10 *m* times with respect to the embedding parameter *q*, then setting $q=0$ and finally dividing them by $m!$, we have the so-called *m*th-order deformation equation

$$ L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t) R_m(\vec{u}_{m-1}), \quad (16) $$

Where

$$ R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} FD(U(t;q))}{\partial^{m-1} q}\bigg|_{q=0}, $$

And

$$ \chi_m = \begin{cases} 0 & m\le 1, \\ 1 & m>1. \end{cases} $$

Finally, for the purpose of computation, we will approximate the HAM solution of Equation 9 by the following truncated series:

$$ \phi_m(t) = \sum_{k=0}^{m-1} u_k(t). $$

NUMERICAL APPLICATIONS

In this section, we apply the proposed algorithm of homotopy analysis method (HAM) using Jumarie's approach for fractional order diffusion equation:

Example 1: We consider a one-dimensional fractional diffusion equation for Equation 1, as taken (Meerschaert et al., 2006; Ray et al., 2008):

$$ \frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + q(x,t), \quad (17) $$

on a finite domain $0<x<1$, with the diffusion coefficient

$$ d(x) = \Gamma(2.2)x^{2.8}/6 = 0.183634\ x^{2.8}, \quad (18) $$

the source/sink function

$$ q(x,t) = -(1+x)e^{-t}x^3, \quad for\ 0<x<1, \quad (19) $$

with the initial conditions $u(x,0)=x^3$ and the boundary conditions

$$ u(0,t)=0, \quad u(1,t)=e^{-t}, \ for\ t>0. \quad (20) $$

According to Equation 12, the zeroth-order deformation can be given by

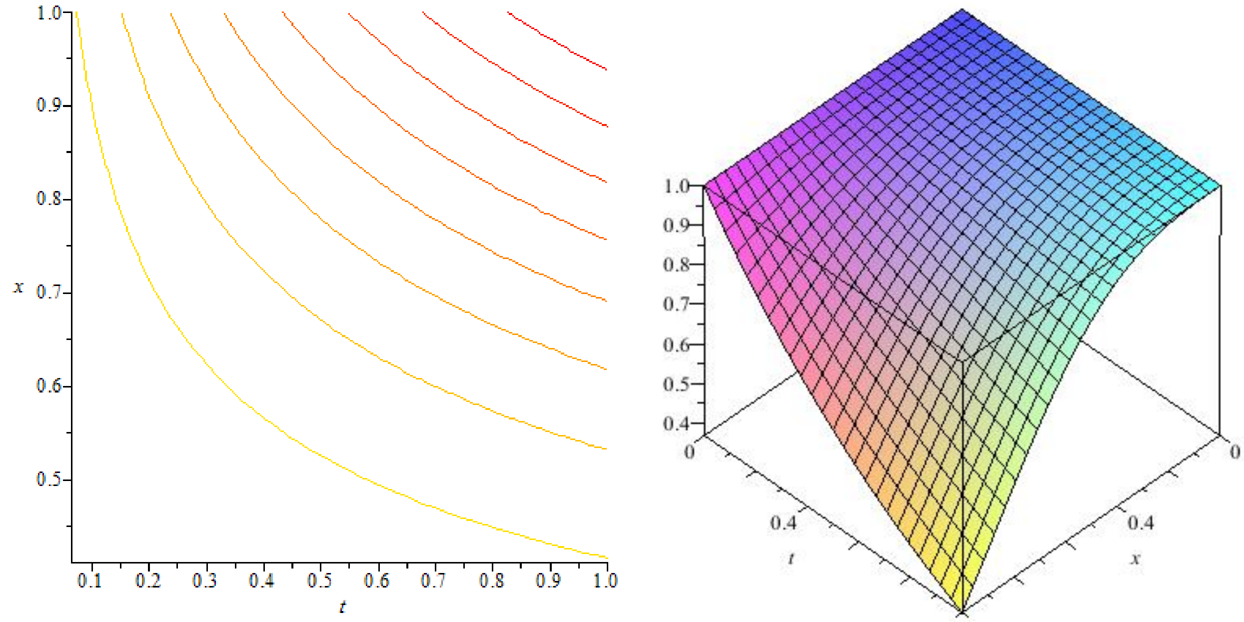

Figure 1. The surface shows the solution $u(x,t)$ for Equation 17.

$$(1-q)L(U(x,t;q)-u_0(x,t)) = q\hbar H(x,t)\left(\frac{\partial u(x,t)}{\partial t} - d(x)\frac{\partial^{1.8}u(x,t)}{\partial x^{1.8}} - q(x,t)\right), \quad (21)$$

We choose the auxiliary linear operator $L(U(x,t;q)) = D_\alpha^t U(x,t;q)$, with the property $L(C)=0$, where C is an integral constant. We also choose the auxiliary function to be $H(x,t)=1$. Hence, the *m*th-order deformation can be given by:

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t) R_m(\vec{u}_{m-1}),$$

Where

$$R_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}(x,t)}{\partial t} - d(x)\frac{\partial^{1.8}u_{m-1}(x,t)}{\partial x^{1.8}} - q(x,t) \quad (22)$$

Now the solution of the *m*th-order deformation Equation 14 for $m \geq 1$ become

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[R_m(\vec{u}_{m-1})]. \quad (23)$$

Consequently, the first few terms of the HAM series solution for $\hbar = -1$ are as follows:

$$u_0 = e^{-t}x^3 + e^{-t}x^4 - x^4,$$

$$u_1(x,t) = (-e^{-t}+1)x^4 + \frac{4(-e^{-t}+1-t)x^5}{2.2},$$

$$u_2(x,t) = \frac{4(e^{-t}-1+t)x^5}{2.2} + \frac{80(e^{-t}-\frac{t}{2!}-1+t)x^5}{3.2\times 2.2^2},$$

$$\vdots$$

It obvious that the noise terms appear between the components u_0 and u_1, and these are all canceled. The closed form solution is $u(x,t) = e^{-t}x^3$.

The surface (Figure 1) shows the solution $u(x,t)$ for equation (17).

Example 2: Now, we consider a two-dimensional fractional diffusion equation for Equation 2, considered in (Tadjeran et al., 2006; Ray et al., 2008):

$$\frac{\partial u(x,y,t)}{\partial t} = d(x,y)\frac{\partial^{1.8}u(x,y,t)}{\partial x^{1.8}} + e(x,t)\frac{\partial^{1.6}u(x,y,t)}{\partial y^{1.6}} + q(x,y,t), \quad (24)$$

on a finite rectengular domain $0<x<1$, $0<y<1$, for $0 \leq t \leq T_{end}$ with the diffusion coefficients

$$d(x,y) = \Gamma(2.2)x^{2.8}y/6, \quad (25)$$

$$e(x,y) = 2xy^{2.6}/\Gamma(4.6), \quad (26)$$

and the forcing function

$$q(x,y,t) = -(1+2xy)e^{-t}x^3y^{3.6}, \quad (27)$$

with the initial condition

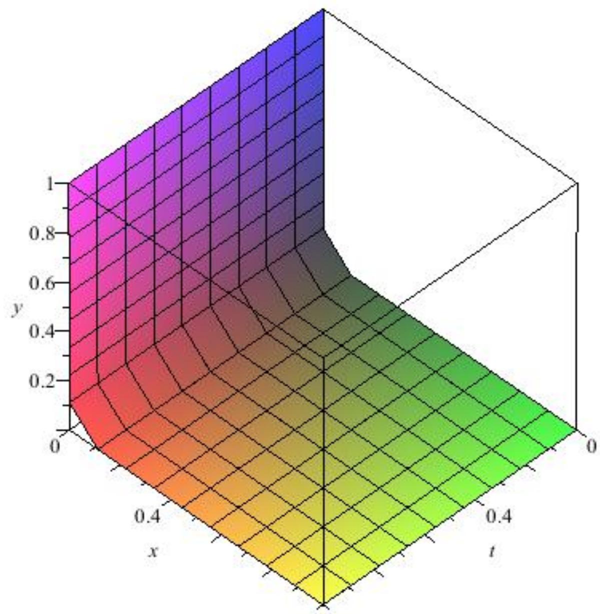

Figure 2. The surface shows the solution $u(x,t)$ for Equation 24.

$$u(x,y,0) = x^3 y^{3.6}, \tag{28}$$

and Dirichlet boundary conditions on the rectangle in the form

$$u(x,0,t) = u(0,y,t) = 0, \quad u(x,1,t) = e^{-t} x^3, \tag{29}$$

and

$$u(1,y,t) = e^{-t} y^{3.6}, \quad \text{for all } t \ge 0. \tag{30}$$

According to Equation 12, the zeroth-order deformation can be given by

$$(1-q)L(U(x,t;q) - u_0(x,t)) = q\hbar H(x,t). \left(\frac{\partial u(x,y,t)}{\partial t} - d(x,y)\frac{\partial^{1.8} u(x,y,t)}{\partial x^{1.8}} - e(x,t)\frac{\partial^{1.6} u(x,y,t)}{\partial y^{1.6}} - q(x,y,t) \right), \tag{31}$$

We choose the auxiliary linear operator $\mathrm{L}(\mathrm{U}(x,t;q)) = \mathrm{D}_\alpha^t \mathrm{U}(x,t;q)$, with the property $L(C) = 0$, where C is an integral constant. We also choose the auxiliary function to be $\mathrm{H}(x,t) = 1$. Hence, the *m*th-order deformation can be given by:

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t) R_m(\vec{u}_{m-1}),$$

Where

$$R_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}(x,y,t)}{\partial t} - d(x,y)\frac{\partial^{1.8} u_{m-1}(x,y,t)}{\partial x^{1.8}} - e(x,t)\frac{\partial^{1.6} u_{m-1}(x,y,t)}{\partial y^{1.6}} - q(x,y,t) \tag{32}$$

Now the solution of the *m*th-order deformation in Equation 14 for $m \ge 1$ become

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[R_m(\vec{u}_{m-1})]. \tag{33}$$

Consequently, the first few terms of the HAM series solution for $\hbar = -1$ are as follows

$$u_0 = x^3 y^{3.6} e^{-t} + 2x^4 y^{4.6} e^{-t} - 2x^4 y^{4.6},$$

$$u_1(x,t) = x^4 y^{4.6}(-e^{-t} + 1) + \left(\frac{8}{2.2} + \frac{2 \times 4.6}{3}\right) x^5 y^{5.6}(-e^{-t} + 1 - t),$$

$$u_2(x,t) = \frac{1106}{165} x^5 y^{5.6}(e^{-t} - 1 + t) + \frac{9101827}{272250} x^6 y^{6.6}(e^{-t} - 1 + t - \frac{t^2}{2!}),$$

$$\vdots$$

It obvious that the "noise" terms appear between the components u_0 and u_1, and these are all canceled. The closed form solution is $u(x,y,t) = x^3 y^{3.6} e^{-t}$.

The surface (Figure 2) shows the solution $u(x,t)$ for equation (24).

Conclusion

In this paper, the application of homotopy analysis method (HAM) was extended to obtain explicit and numerical solutions of linear and inhomogeneous space fractional diffusion equations with initial and boundary conditions. The obtained results and computational work demonstrate the reliability of the algorithm, reconform the convergence of the suggested algorithm and its wider applicability to fractional differential equations. The advantage of HAM is the auxiliary parameter which provides a convenient way of controlling the convergence region of series solutions. It is clear that the solutions agree with the exact solutions. Further, the proposed technique is fully capable of coping with the nonlinearity of such physical problems. İt may be concluded that this suggested technique is a nice addition to the existing techniques for solving nonlinear problems of diverse fields.

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