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Full Length Research Paper

Comparison of variance component estimation methods for horizontal control networks

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In this study, some well-known variance component estimation methods, called conventional, Helmert, MINQUE, AUE, and Förstner, developed to determine the stochastic model, for the adjustment of geodetic networks, have been compared. In doing so, concrete deciding criteria, using statistical tests, have been defined and the determination of superior model has been studied. For the validation of the models, numerical experiments using data, from a part of the Istanbul Metropolitan Triangulation Network (Asian side), have been performed.

Key words: Variance, stochastic model, variance estimation, weights, Helmert method.

INTRODUCTION

To compare the stochastic models used in the adjustment of horizontal control networks, which require the highest accuracy, firstly, concrete comparing criteria must be defined by using statistical tests.

If the hypothesis given subsequently is valid, reliable results can be received by using the comparison criteria which will be explained later.

"All gross and systematic errors have been eliminated before adjustment". "There is no functional model error".

STOCHASTIC MODELS USED IN HORIZONTAL CONTROL NETWORKS

Some well-known stochastic models are used in this study.

Conventional stochastic model (Model 1)

The standard deviation of a measured direction, evaluated from a group, can be obtained either from the Ferrero Equation

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$$s_0 = \pm\sqrt{\frac{\frac{\sum w^2}{6}}{n}} \qquad (1)$$

Where

s_0 : a priori standard deviation of a measured direction
w: misclosure of the ith triangel
n: number of triangles in network
or from station adjustment

$$s_d = \pm\sqrt{\frac{\sum v^2}{(n-1)(s-1)}} \qquad (2)$$

Where:

s_d : standard deviation of a direction in a series observation
v: residual errors of direction observations
n: number of series observation of directions
s: number of directions in a station

$$s_0 = \pm\frac{s_d}{\sqrt{n}} \qquad (3)$$

Where:

s_0 : a priori standard deviation of a direction in mean series observation.

All corrections such as elipsoidal and projectional corrections etc. were made for lengths in İstanbul network. So, corrected lenghts were used in this article. The centring errors are irrelevant for the length of lines in the Istanbul network.

The a priori variances of the distance measurements can be obtained using either Equation (4), given by the instrument manufacturer.

$$s_s = \pm a + b \times S \quad (4)$$

Where:
S: measured distance
or Equations (5) and (6)

$$s_0 = \pm\sqrt{\frac{[gg]}{2n}} \quad (5)$$

$$s_s = \pm\frac{s_0}{\sqrt{2}} \quad (6)$$

Where:
g: double-difference measurement for the distance
s_0: standard deviation for a distance measurement
s_s: standard deviation for double measurement.

After the computation of the variances of distances and directions, assuming that the a priori variance determined for a group of directions, is the a priori variance (s_0^2) for a unit weight measurement of the weight of the other direction groups and the weights of distances in the network, can be obtained from the Equations given.

Weights of the direction groups with unit weight $P_i = 1$ weights of the other direction groups

$$P_j = \frac{s_0^2}{s_j^2} \quad (7)$$

Weights of the distances $P_s = \frac{s_0^2}{s_s^2} \quad (\frac{m\,gon^2}{mm^2})$

Estimation of variance components using Helmert method (Model 2)

The variance components are estimated iteratively with the Helmert method. The observations are grouped to determine the variance components in this method. The Helmert Equation is given subsequently (Garafarend and Schaffrin,1979; Sahin et al., 1992).

$$H\bar{s}_k^{-2} = c \quad (8)$$

Where

$$c_i = v_i^T \Pi(N^{-1}N_j N^{-1}N_i) \quad (i \neq j) \quad (9)$$

$$h_{ij} = Tr(N^{-1}N_j N^{-1}N_i) \quad (i \neq j) \quad (10)$$

$$h_{ii} = k_i - 2Tr(N^{-1}N_i) + Tr(N^{-1}N_i N^{-1}N_i) \quad (11)$$

$$s_k = S_k^2 \quad (12)$$

$$\begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1m} \\ h_{21} & h_{22} & \cdots & h_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ h_{m1} & h_{m2} & \cdots & h_{mm} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} \quad (13)$$

$$N_i = A_i^T P_i A_i \quad (14)$$

A: design matrix of full column rank relating the observations to the unknown parameters.
P: the assigned weight matrix
N: normal matrices
c: function of P
S_i: estimated values of variance factors.

The Helmert Equation (8) is solved and the variance ($\bar{s}_k^{-2}$) of each group is obtained. Then the weights (P_k) of the observation groups are computed using the following Equation.

$$P_{k+1} = \frac{P_k}{\bar{s}_k^{-2}} \quad (15)$$

If $\bar{s}_k^{-2}$ is not equal to 1 for all k=1, 2,..., m, iteration is continued. When $\bar{s}_k^{-2}$ = 1 for all observation groups, the iteration is finished (Sahin et al., 1992; Yavuz, 2000).

Estimation of variance components using MINQUE (Model 3)

This method is described in Equations 16 and 19. The observations are to be grouped to determine the variance components. The MINQUE Equation for the variance components is given subsequently (Chen et al., 1990; Rao, 1971).

$$\sigma = (\sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2)^T = [S(\sigma^{(0)})]^{-1} q(\sigma^{(0)}) \quad (16)$$

where

$\sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2$: variance-covariance components which will be estimated

The (i, j)th element of matrix $S(\sigma^{(0)})$ is

$$S_{ij} = Tr[R(\sigma^{(0)}) T_i R(\sigma^{(0)}) T_j] \quad (17)$$

$$q_i = l^T R(\sigma^{(0)}) T_i R(\sigma^{(0)}) l \quad (18)$$

$$R(\sigma^{(0)}) = (C_l^{(0)})^{-1}[I - A[A^T (C_l^{(0)})^{-1} A]^{-1} A^T (C_l^{(0)})^{-1}] \quad (19)$$

Where:
l : the vector of observations
I : identity matrix
$T_1, T_2, \ldots, T_m$: corresponding matrices for variance-covariance components
Tr : Trace

After some steps, The MINQUE Equation (16) is solved and the variance of each group is obtained. Then the weights of the observation groups are computed using the following equation.

$$P_{i+1} = \frac{P_i}{\sigma} \quad (20)$$

If σ is not equal to 1 for all i=1, 2,..., m the iteration is continued. When $\sigma = 1$ for all observation groups, the iteration is completed (Sahin et al., 1992; Yavuz, 2000).

AUE method (Almost unbiased estimation) (Model 4)

This method is described by Horn et.al. (1975) and Lucas (1985).

The AUE Equation used to determine the variance components is given subsequently.

$$\sigma_i^2 = \frac{l^T W H_i W l}{Tr(W H_i)} \quad (21)$$

Where:

$$W = P - PAN^{-1}A^T P \quad (22)$$

$$H_i = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & P_i^{-1} & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \quad (23)$$

l: the vector of observations
A: design matrix of full column rank relating the observations to the unknown parameters
P: the assigned weight matrix
N: normal matrices

In this method, the variance components are estimated by grouping the observations. After some steps, the AUE Equation (2, 12) is solved and the variance of each group is obtained. The a priori weights can be selected as $P_1^{(0)} = P_2^{(0)} = \ldots\ldots = P_i^{(0)} = 1$ for all groups (Sahin et al., 1992; Yavuz, 2000). After each iteration, the weights of the observation groups are computed as follows:

$$P_{i+1} = \frac{P_i}{\sigma} \quad (24)$$

Where:
P_i : the weight in ith iteration
If σ is not equal to 1 for all i=1, 2,..., m the iteration is continued. When σ=1 for all observation groups, the iteration stops (Sahin et al., 1992; Yavuz, 2000).

Förstner method (Model 5)

The equation given by Persson (1981) for the estimation of variance components is given subsequently.

$$\sigma_i^2 = \frac{v_i^T P_i v_i}{z_i} \quad (25)$$

Where:
i: number of observation group

$$N_i = A_i^T P_i A_i \quad (26)$$

$$v_i = A_i x - l_i \quad (27)$$

$$z_i = k_i - Tr(P_i A_i N^{-1} A_i^T) \quad (28)$$

$$i = 1, 2, \ldots\ldots, m$$

Where:
k_i : number of observations in ith group

N_i : normal matrices in ith group

v_i : least squares residuals in ith group

x : vector containing the unknown parameters

P_i : the weight matrix in ith group

The observations are to be grouped to determine the variance components. After some steps the Förstner Equation (25) is solved and the variance of each group is obtained. The initial weights can be selected as $P_1^{(0)} = P_2^{(0)} = ... = P_i^{(0)} = 1$ for all groups. After each iteration, the weights of the groups are computed as in Equation (24). When $\sigma_i^2 = 1$ for all groups, the iteration is stopped.

CRITERIA FOR EXAMINING THE STOCHASTIC MODELS

Resulting time

The observations vector and the functional model are kept fixed in the adjustment computations for the comparison of the stochastic models. Naturally, the stochastic models, that give meaningful and true results in an acceptable time, are superior to the others, which do not give meaningful and true results in an acceptable time.

Global test and gross error localization and elimination

To test the compatibility of the estimated a posterior variance factor $\hat{\sigma}_0^2$ with the selected a priori variance factor σ_0^2, the global test is applied first. In this test, σ_0^2 is compared with $\hat{\sigma}_0^2$. If the global test fails and some residuals show excessive magnitude, a gross error localisation and elimination technique (Baarda's data snooping or Pope tau-test) is employed. Considering the elimination of a minimum number of observations with gross error during the adjustment, the stochastic model which proves the null hypothesis is accepted to be superior to those that reject the null hypothesis.

Homogeneity test

The homogeneity of a network depends on the stochastic model under the condition that the other parameters of the adjustment are fixed. Because of error ellipses are used to get two dimensional mean square errors for the adjusted network points, the simple arithmetical mean and the standard deviation of the sum of the major (a) and the minor (b) semi-axes of the computed error ellipses, related to the network points after the adjustment, have been taken as a criterion to determine the superior model from the point of homogeneity in this study,. The homogeneity condition for this network is as follows:

$$a_i = a_j \quad , \quad i = 1,2,....,u-1 \quad i \neq j$$

$$b_i = b_j \quad , \quad j = 2,3,....,u$$

or

$$a_i + b_i = a_j + b_j \tag{29}$$

Equation (29) can be used for the comparison of the adjustment results, obtained by using the stochastic models, from the point of homogeneity. The sums $c_i = a_i + b_i \quad (i = 1,2,....,u)$ for each stochastic model are accepted as a measurement set. The mean values and the standard deviations of these mean values are:

$$c_0 = \frac{\sum_{i=1}^{u} c_i}{u} \, , \; \sigma^{(0)} = \sqrt{\frac{\sum_{i=1}^{u} (c_0 - c_i)^2}{u(u-1)}} \tag{30}$$

If the homogeneity condition Equation (29) is exactly valid

$$c_0 = c_i \text{ and } \sigma^{(0)} = 0$$

then the standard deviations $\sigma^{(0)}$ calculated from Equation (30) are homogeneity criteria for the stochastic models. In the homogeneity test, standard deviations $\sigma^{(0)}$ related to each stochastic models are compared in pairs.

If $\sigma_j^{(0)}$ *and* $\sigma_k^{(0)}$ are the standard deviations of the means c_0 belonging to the j-th, and k-th stochastic models and the number of stochastic model is m, the null hypothesis.

$$H_0\{E(\sigma_j^{(0)2} - \sigma_k^{(0)2})\} = 0 \, ,$$

$$j = 1,....,m-1 \; ; \; k = 2,....,m \; , \; j \neq k \tag{31}$$

and the test statistic

$$T_{j,k} = \frac{\sigma_j^{(0)2}}{\sigma_k^{(0)2}} \, , \sigma_j^{(0)} > \sigma_k^{(0)} \tag{32}$$

If $T_{j,k} \leq F_{1-\alpha, f_j, f_k}$ (33)

Where:
$f_j = f_k = u - 1$

then the null hypothesis is true. In this case, we cannot decide which one of two models at hand are superior.
If

$$T_{j,k} > F_{1-\alpha, f_j, f_k} \quad (34)$$

it is accepted that the stochastic model having the value of σ_k is superior to the model having the value of σ_j.
If null hypothesis is true, we cannot decide the model superiority. In this case, mean accuracy test is applied and the order of superiority of the stochastic models is made according to result of this test.
Null hypothesis for mean accuracy test,

$$H_0\{E(c_{0,j}) = E(c_{0,k})\}, \; j = 1,2,.....,m-1 \quad (35)$$
$$k = 1,2,.....,m, \; j \neq k$$

and the test statistic,

$$t_{j,k} = \frac{c_{0,j} - c_{0,k}}{\sqrt{\sigma_j^{(0)2} + \sigma_k^{(0)2}}}, \; c_{0,j} > c_{o,k} \quad (36)$$

The test value is compared with the value of $F_{1-\alpha, f}$.
Where: $f=2(u-1)$

If

$$t_{j,k} \leq F_{1-\alpha, f} \quad (37)$$

Null hypothesis is true. In this case, it is decided that there is no superiority between two stochastic models taken in hand.
If

$$t_{j,k} > F_{1-\alpha, f} \quad (38)$$

it is accepted that the stochastic model having the value $c_{o,k}$ is superior than the others (Equation 36).

Isotropy test

Isotropy is a characteristic concerning with the accuracy criteria, which is related to the positions of the points, being independence from the direction in horizontal control networks. Error ellipses take the form of circle in full isotrop nets. Mathematical form of this speciality is as follows:

$a_i = b_i$ or
$$a_i - b_i = 0 \;, \; i = 1,2,....,u \quad (39)$$

Equation (39) can be used to compare the results of the adjustments, obtained from the stochastic models, from the point of isotropy. To carry out this, following equation belonging to each stochastic models is thought as a measurement set.

$$e_i = a_i - b_i \qquad i = 1,2,....,u \quad (40)$$

Then, the mean values and the standard deviations of these measurement sets are computed as follows:

$$e_0 = \frac{\sum_{i=1}^{u} e_i}{u} \quad (41)$$

$$\overline{\sigma} = \sqrt{\frac{\sum_{i=1}^{u} (e_0 - e_i)^2}{u(u-1)}} \quad (42)$$

These values are accepted as isotropy criteria for the comparison of stochastic models. Comparison is made similarly as expressed in section 3.3 using the Equations from (28) to (34) or Equation (38) if needed. But in the equations mentioned previously, instead $\sigma^{(0)}$ and $\overline{\sigma}$, c_0 and e_0 must be used.

EXPERIMENTS

The part of Istanbul Metropolitan Triangulation Network, surveyed in 1987, has been selected as application network. 448 direction and 208 distance observations were made in the net. Observations are separated into two groups like directions and distances for Models 2, 3, 4 and 5.

Comparison of the models according to criteria 1

Needed time to determinate the weights are taken as criteria 1. It continued < 1 min for Model 1. For the other models, needed time for an iteration is as follows (Table 1):

The final adjustment computation step is finished in 4

Model 1.

Model 2	Model 3	Model 4	Model 5
$1^{m}22^{s}$	$32^{m}15^{s}$	$16^{m}20^{s}$	$6^{m}20^{s}$

Table 1. Total iteration times to calculate the weights of the models (σ^2 = 1).

Model no.	Total iteration number	Total time	Calculated weights	
			First group	Second group
Model 2	7	$0^{h}9^{m}34^{s}$	0.2575835929	0.1689553333
Model 3	7	$3^{h}45^{m}45^{s}$	0.2575839782	0.1689548039
Model 4	10	$2^{h}43^{m}20^{s}$	0.2575828403	0.1689563675
Model 5	11	$1^{h}09^{m}40^{s}$	0.2575834582	0.1689555168

Model 2.

Model 1	Model 2	Model 5	Model 4	Model 3
$\sim 5^{m}$	$0^{h}13^{m}34^{s}$	$1^{h}13^{m}40^{s}$	$2^{h}47^{m}20^{s}$	$3^{h}49^{m}45^{s}$

min for both models. As a result, considering the total computation times, the superiority arrangement of the models according to criteria 1 can be given as Model 2.

Comparison of the models according to criteria 2

Global test on the posterior variance factor is applied to the adjustment results, and the following results are obtained (from F distribution table $F_{431,\infty,0.95} = 1.172$):

For Model 1,

$$T = \frac{1.848^2}{1.136^2} = 2.646$$ > 1.172 null hypothesis is rejected.

For Models 2, 3, 4 and 5,

$$T = \frac{1.084^2}{1^2} = 1.175$$ > 1.172 null hypothesis is rejected.

After the elimination of four observations with gross error from the observations set, the adjustment is remade using all of the models and the global test on the posterior variance factor is reapplied to the adjustment results:

For Model 1,

$$T = \frac{1.568^2}{1.136^2} = 2.460$$ > 1.173 null hypothesis is rejected

For Model 2, 3, 4 and 5,

$$T = \frac{1^2}{0.941^2} = 1.129$$ < 1.173 null hypothesis is true.

The results mentioned previously prove that a lot of measurements must be eliminated from the observations heap for Model 1. But, because of the elimination of 28 observations with gross error (Table 2) might weaken the net geometry, this is not done. As a result, the superiority arrangement of the models according to Criteria 2 can be given as:

Models 2, 3, 4 and 5
Model 1

Comparison of the models according to criteria 3

Superiority arrangement of the models according to Criteria 3 can be given as:

Model 2, 3, 4 and 5
Model 1

Comparison of the models according to criteria 4

Isotropy test results have been given in Table 3 and 4.

Table 2. Gross error detection test results.

	Gross error detection test	
	Direction observations with gross error	**Distance observations with gross error**
Model 1 Baarda-Data snooping	22 directions	6 distances
Model 2, 3, 4 and 5 Pope Tau Baarda-Data snooping	34138-34105 34138-34098 34098-34138	34069-34070

Table 3. Homogeneity and mean accuracy test results; $F_{75,75,0.95} = 1.4656$, $F_{150,0.95} = 1.6551$.

Model no.	1 $\sigma^{(0)} = 0.0939$	2, 3, 4, 5 $\sigma^{(0)} = 0.0689$	Superior model according to homogeneity	Mean accuracy test value	Superior model according to mean accuracy
1		T=1.8573	2, 3, 4, 5		

Table 4. Homogeneity and mean accuracy test results; $F_{75,75,0.95} = 1.4656$, $F_{150,0.95} = 1.6551$.

Model no.	1 $\bar{\sigma} = 0.0184$	2, 3, 4, 5 $\bar{\sigma} = 0.0156$	Superior model according to isotropy	Mean accuracy test value	Superior model according to mean accuracy
1		T=1.3912	equivalent	t=1.5298	equivalent

Table 5. The general superiority arrangement of the models.

Model no.	Criteria 1	Criteria 2	Criteria 3	Criteria 4	Total	General arrangement
1	1	5	5	5	16	5
2	2	1	1	1	5	1
3	5	1	1	1	8	4
4	4	1	1	1	7	3
5	3	1	1	1	6	2

As a result, the superiority arrangement of the models can be given as:

Models 2, 3, 4 and 5
Model 1

The evaluation of the results related to the adjustment of Asiatic Site net considering the whole criterion.

The general superiority arrangement of the models, according to the whole criterion in the adjustment of Asiatic Site net, are summarised.

According to Table 5, the models except Model 1 are equivalents according to three criteria. Only calculation times are different.

GENERAL REMARKS AND CONCLUSION

The stochastic models determined by the methods Helmert, Minque, AUE, Förstner, proved the superiorities against the conventional model by the aid of theoretical researches and practical applications are done in this study, is to be used especially in the adjustment of geodetic nets, which require higher accuracy and precision, in private and public sector applications. Since the Helmert model has taken the first plase in the general superiority arrangement of the models, we propose the Helmert stochastic model for using in horizontal control networks. These aforementioned results determined, are transferable to other networks. We did not find any researches related to similar comparisons of the stochastic models made by the other researchers.

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