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Full Length Research Paper

Application of He's energy balance method for pendulum attached to rolling wheels that are restrained by a spring

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We consider periodic solution for nonlinear free vibration of conservative, single degree of freedom systems. A new analytical technique called Energy Balance Method (EBM) is applied to calculate approximations in order to achieve the nonlinear frequency of the system. Comparing with numerical solution using Runge-Kutta method, just one iteration leads us to high accuracy of solutions which are valid for a wide range of vibration amplitudes as indicated in this study. The EBM is a novel method which alleviates drawbacks of the traditional numerical techniques.

Key words: Runge-kutta method, conservative oscillations, energy balance method.

INTRODUCTION

Most of the engineering problems especially vibrating systems, are often described by their governing equations in the form of differential equations, either linear or nonlinear. Many researchers have worked on linear and nonlinear systems having single or multi degrees of freedom, but hardly any of those problems have analytical solution. Surveys of the literature with numerous references have been given by many authors utilizing various analytical methods for solving nonlinear oscillation systems. Some of these well-known analytical methods such as perturbation techniques (He, 1999) (traditional perturbation methods) contain many shortcomings. They are not useful for strongly nonlinear equations, so for overcoming the shortcomings, many new techniques have been appeared in open literature, for instance: Homotopy perturbation (Bayat et al., 2010; Bayat et al., 2011a), parameter–expansion (Kimiaeifar et al., 2010), parameterized perturbation (He,1999), energy balance (Bayat et al., 2011b,c; Bayat and Pakar, 2011a; He, 2002), variational approach (Bayat et al., 2011d; Pakar et al., 2011b; He, 2007) and other analytical and numerical methods (Bayat et al., 2011e, f, g; Bayat and Abdollahzadeh, 2011i, j; Bayat and Pakar, 2011k; Ghasemi et al., 2011; Shahidi et al., 2011; Soleimani et al., 2011; Ehidiamhen, 2009; Yang, 2010).

Among this method, EBM has been considered to solve the nonlinear systems in this paper. In this paper, we use EBM for pendulum attached in rolling wheels that are restrained by a spring. This method can be seen as a Ritz-like method and leads to a very high convergence of the solution and can be easily extended to other nonlinear oscillations. In short, this method yields extended scope of applicability, simplicity, flexibility in application and avoidance of complicated numerical and analytical integration as compared to others among the previous approaches such as the perturbation methods and so could prove widely applicable in engineering and science. To illustrate the accuracy and application of this method some comparisons are presented. The EBM seems very easy to study the behavior of dynamical systems and also calculate the natural frequency.

A SINGLE-DEGREE-OF-FREEDOM CONSERVATIVE SYSTEM

An example of a single-degree-of-freedom conservative system has been considered that is described by an equation as follows. A rigid rod is rigidly attached to the

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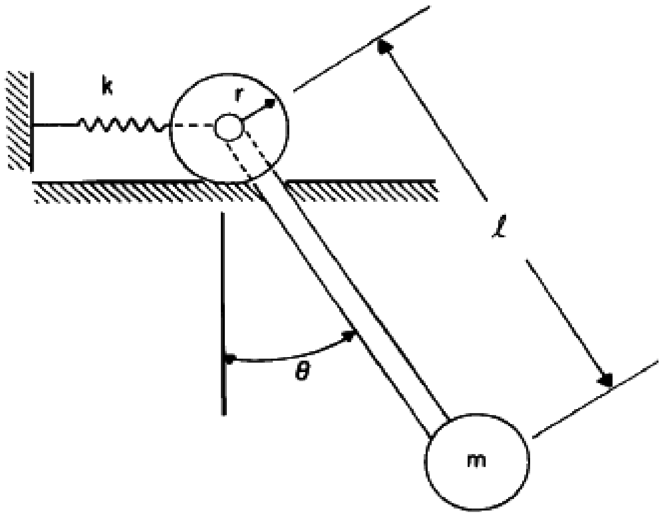

Figure 1. Pendulum attached to rolling wheels that are restrained by a spring (Nayfeh, 1979).

axle as shown in Figure 1. The wheels roll without slip as the pendulum swings back and forth. The wheel is restrained by a spring which is fixed to a wall on the other side .Only the ball on the end of the pendulum has appreciable mass and it may be considered a particle. The governing equation of the motion is:

$$\theta_{tt} m\left(l^2+r^2-2rl\cos(\theta)\right)+mrl\sin(\theta)\theta_t^{\,2}+mgl\sin(\theta)+kr^2\theta=0 \tag{1}$$

With initial conditions

$$\theta(0)=A, \qquad \theta_t(0)=0. \tag{2}$$

BASIC IDEA OF EBM

In the present paper, we consider a general nonlinear oscillator in the Form (He, 2002):

$$\ddot{u}+f\,(u(t))=0 \tag{3}$$

In which u and t are generalized dimensionless displacement and time variables, respectively. Its variational principle can be easily obtained:

$$J(u)=\int_0^t(-\frac{1}{2}\dot{u}^2+F(u))\,dt \tag{4}$$

Where $T=\frac{2\pi}{\omega}$ is period of the nonlinear oscillator,

$$F(u)=\int f\,(u)du.$$

Its Hamiltonian, therefore, can be written in these forms:

$$H=\frac{1}{2}\dot{u}^2+F(u)+F(A) \tag{5}$$

Or

$$R(t)=-\frac{1}{2}\dot{u}^2+F(u)-F(A)=0 \tag{6}$$

Oscillatory systems contain two important physical parameters, that is, the frequency ω and the amplitude of oscillation. A . So let us consider such initial conditions:

$$u(0)=A, \quad \dot{u}(0)=0 \tag{7}$$

We use the following trial function to determine the angular frequency ω

$$u(t)=A\cos\omega\,t \tag{8}$$

Substituting (8) into u term of Equation (6), yield:

$$R(t)=\frac{1}{2}\omega^2A^2\sin^2\omega\,t+F\left(A\cos\omega\,t\right)-F\left(A\right)=0 \tag{9}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω. Since Equation (8) is only an approximation to the exact solution, R cannot be made zero everywhere. Collocation at $\omega\,t=\frac{\pi}{4}$ gives:

$$\omega=\sqrt{\frac{2\left(F\left(A\right)\right)-F\left(A\cos\omega\,t\right)}{A^2\sin^2\omega\,t}} \tag{10}$$

Its period can be written in the form:

$$T=\frac{2\pi}{\sqrt{\frac{2\left(F\left(A\right)\right)-F\left(A\cos\omega\,t\right)}{A^2\sin^2\omega\,t}}} \tag{11}$$

RUNGE-KUTTA METHOD

For the numerical approach to verify the analytic solution, the fourth Runge-Kutta method has been used. This iterative algorithm is written in the form of the following formulae for the second-order differential equation:

$$\dot{u}_{i+1} = \dot{u}_i + \frac{\Delta t}{6}\left(h_1 + 2h_2 + 2h_3 + k_4\right)$$
$$u_{i+1} = u_i + \Delta t\left(\dot{u}_i + \frac{\Delta t}{6}\left(h_1 + h_2 + k_3\right)\right) \tag{12}$$

Where, Δt is the increment of the time and h_1, h_2, h_3 and h_4 are determined from the following formulae:

$$h_1 = f\ (\dot{u}, u_i, \dot{u}_i)k\ ,$$
$$h_2 = f\ \left(t_i + \frac{\Delta t}{2},\ u_i + \frac{\Delta t}{2}\dot{u}_i\ ,\ \dot{u}_i + \frac{\Delta t}{2}h_1\right),$$
$$h_3 = f\ \left(t_i + \frac{\Delta t}{2},\ u_i + \frac{\Delta t}{2}\dot{u}_i\ ,\ \frac{1}{4}\Delta t^2 h_1,\ \dot{u}_i + \frac{\Delta t}{2}h_2\right),$$
$$h_4 = f\ \left(t_i + \Delta t,\ u_i + \Delta t\,\dot{u}_i\ ,\ \frac{1}{2}\Delta t^2 h_2,\ \dot{u}_i + \Delta t\ h_3\right). \tag{13}$$

The numerical solution starts from the boundary at the initial time, where the first value of the displacement function and its first-order derivative are determined from initial condition. Then, with a small time increment Δt, the displacement function and its first-order derivative at the new position can be obtained using Equation (12). This process continues to the end of the time limit.

APPLICATIONS

Its variational formulation can be readily obtained from Equation (1) and is as follows:

$$J(\theta) = \int_0^t \left(\frac{1}{2}ml^2\theta_t^2 + \frac{1}{2}mr^2\theta_t^2 - \theta_t^2\cos(\theta)mrl - mgl\cos(\theta) + \frac{1}{2}kr^2\theta^2\right)dt\ . \tag{14}$$

Its Hamiltonian, therefore, can be written in these forms:

$$H = \left(\frac{1}{2}ml^2\theta_t^2 + \frac{1}{2}mr^2\theta_t^2 - \theta_t^2\cos(\theta)mrl - mgl\cos(\theta) + \frac{1}{2}kr^2\theta^2\right) \tag{15}$$

And

$$H_{t=0} = -mgl\cos(A) + \frac{1}{2}kr^2A^2, \tag{16}$$

$$H_t - H_{t=0} = \left(\frac{1}{2}ml^2\theta_t^2 + \frac{1}{2}mr^2\theta_t^2 - \theta_t^2\cos(\theta)mrl - mgl\cos(\theta) + \frac{1}{2}kr^2\theta^2\right) - \left(-mgl\cos(A) + \frac{1}{2}kr^2A^2\right) \tag{17}$$

We use the trial function to determine the angular frequency ω, that is,

$$\theta(t) = A\cos\omega\,t \tag{18}$$

If we substitute (18) into (17), it will results to the following residual equation:

$$H_t - H_{t=0} = \begin{pmatrix} \frac{1}{2}ml^2(A^2\omega^2\sin(\omega t)) + \frac{1}{2}mr^2(A^2\omega^2\sin(\omega t)) \\ -(A^2\omega^2\sin(\omega t))\cos(\theta)mrl \\ -mgl\cos((A\cos(\omega t))) + \frac{1}{2}kr^2((A^2\cos(\omega t))) \end{pmatrix} - \left(-mgl\cos(A) + \frac{1}{2}kr^2A^2\right) = 0 \tag{19}$$

If we collocate at $\omega t = \frac{\pi}{4}$ we obtain:

$$\frac{1}{4}ml^2A^2\omega^2 + \frac{1}{4}mr^2A^2\omega^2 - \frac{1}{2}A^2\omega^2\cos(\frac{\sqrt{2}}{2}A)mrl - mgl\cos(\frac{\sqrt{2}}{2}A) - \frac{1}{4}kr^2A^2 + mgl\cos(A) = 0 \tag{20}$$

This leads to the following result:

$$\omega = \frac{\sqrt{m\left(l^2 + r^2 - 2\cos(\frac{\sqrt{2}}{2}A)rl\right)\left(-4mgl\cos(A) + 4mgl\cos(\frac{\sqrt{2}}{2}A) + k\ r^2A^2\right)}}{m\left(l^2 + r^2 - 2\cos(\frac{\sqrt{2}}{2}A)rl\right)A} \tag{21}$$

According to Equations (21) and (18), we can obtain the following approximate solution:

$$\theta(t) = A\cos\left(\frac{\sqrt{m\left(l^2 + r^2 - 2\cos(\frac{\sqrt{2}}{2}A)rl\right)\left(-4mgl\cos(A) + 4mgl\cos(\frac{\sqrt{2}}{2}A) + k\ r^2A^2\right)}}{m\left(l^2 + r^2 - 2\cos(\frac{\sqrt{2}}{2}A)rl\right)A}t\right) \tag{22}$$

RESULTS AND DISCUSSION

To show the efficiency and the accuracy of the EBM, the procedures explained in previous sections are applied to obtain natural frequency and corresponding displacement of a conservative single degree of freedom system. Comparisons of results for different parameters via numerical and EBM is presented in Figures 2 to 5.From Figures 2 and 3, it is obvious that the motion of the

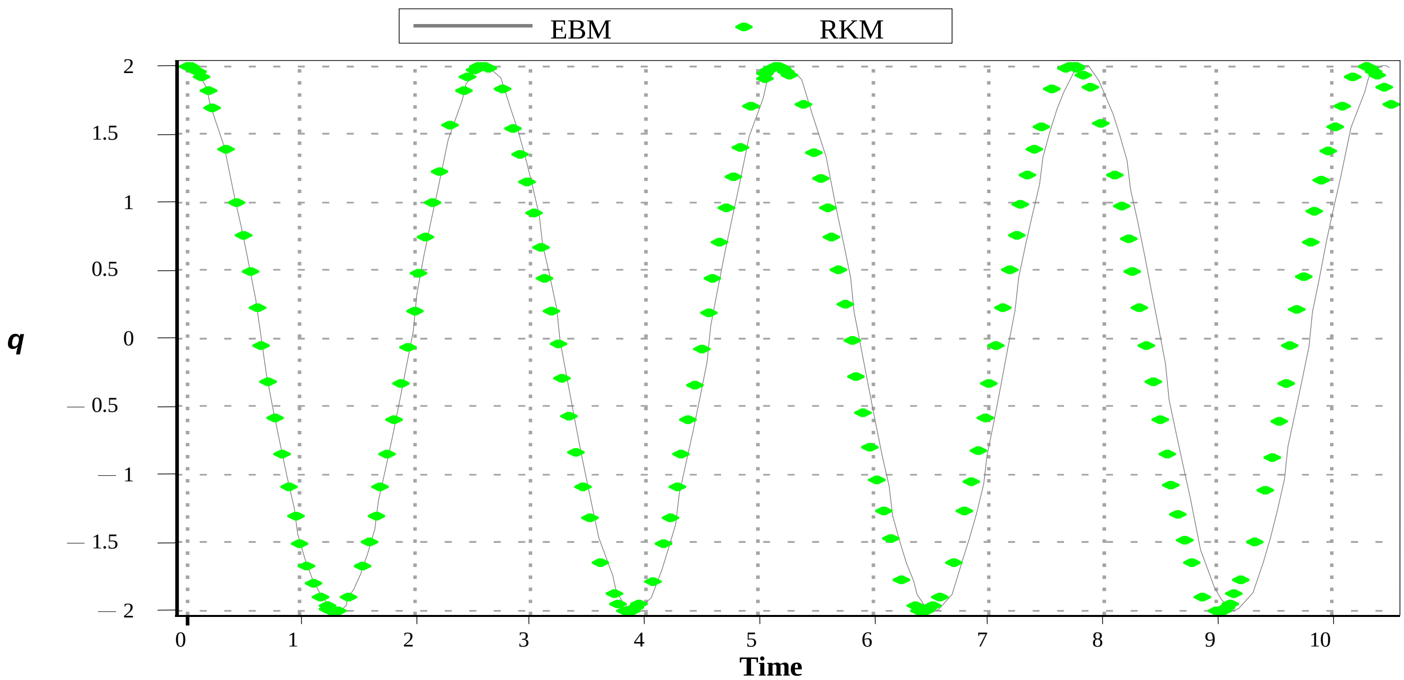

Figure 2. Comparison of analytical solution of θ based on time with the numerical solution for $m = 5$, $l = 1$, $r = 0.2$, $g = 9.81$, $k = 5$, $A = 2$.

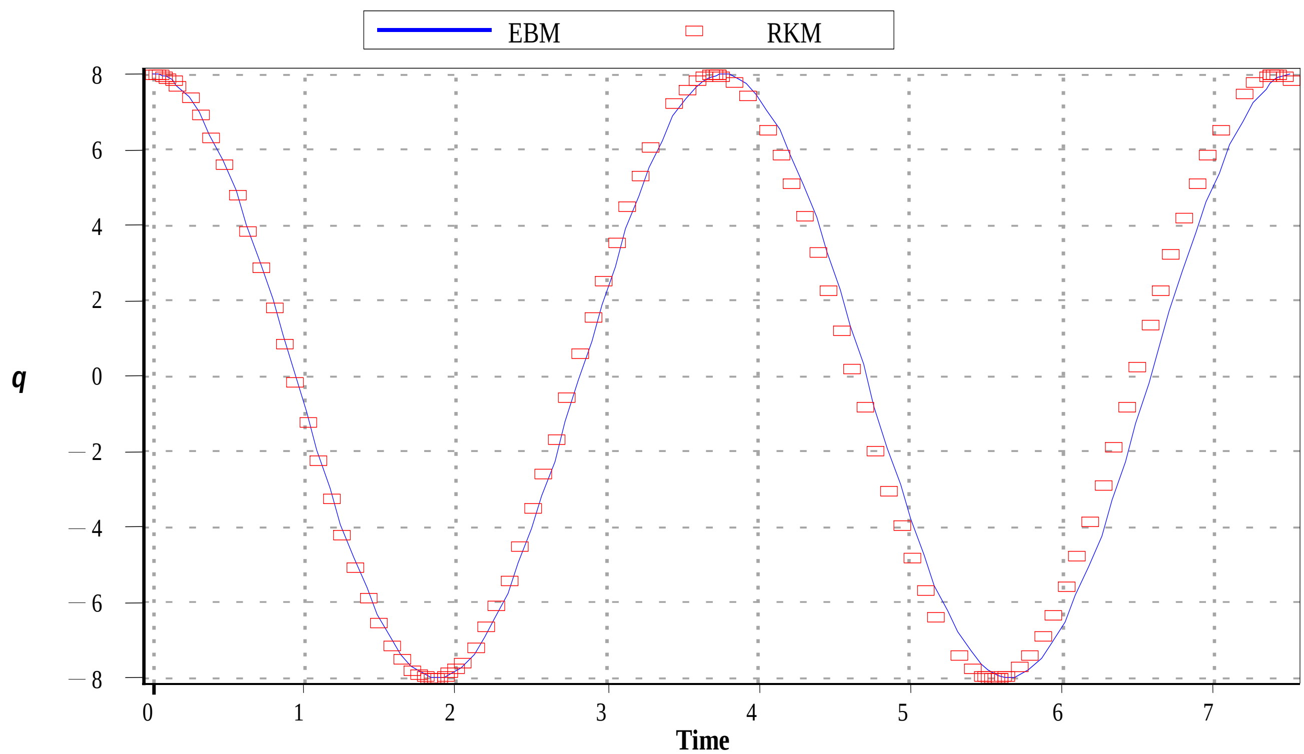

Figure 3. Comparison of analytical solution of θ based on time with the numerical solution for $m = 10$, $l = 0.6$, $r = 0.3$, $g = 9.81$, $k = 10$, $A = 8$.

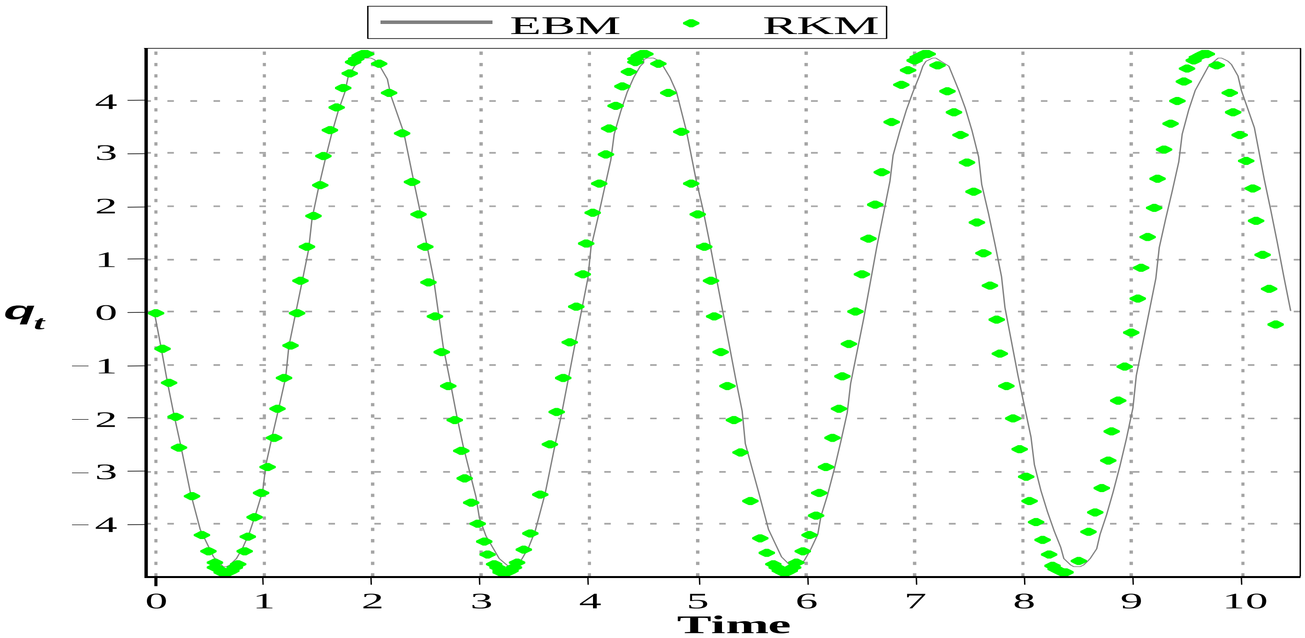

Figure 4. Comparison of analytical solution of θ_t based on time with the numerical solution for $m = 5,\quad l = 1,\quad r = 0.2,\quad g = 9.81,\quad k = 5,\quad A = 2$.

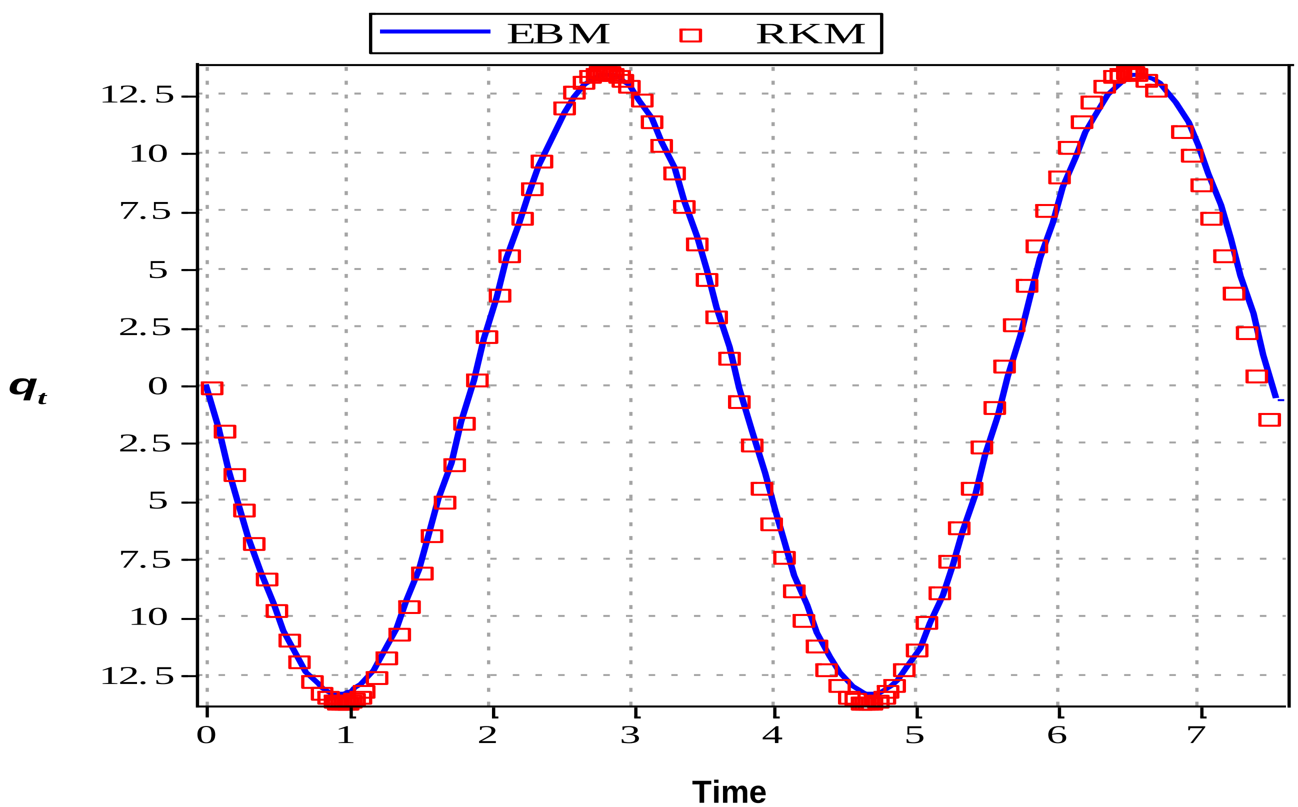

Figure 5. Comparison of analytical solution of θ_t based on time with numerical solution for $m = 10,\quad l = 0.6,\quad r = 0.3,\quad g = 9.81,\quad k = 10,\quad A = 8$.

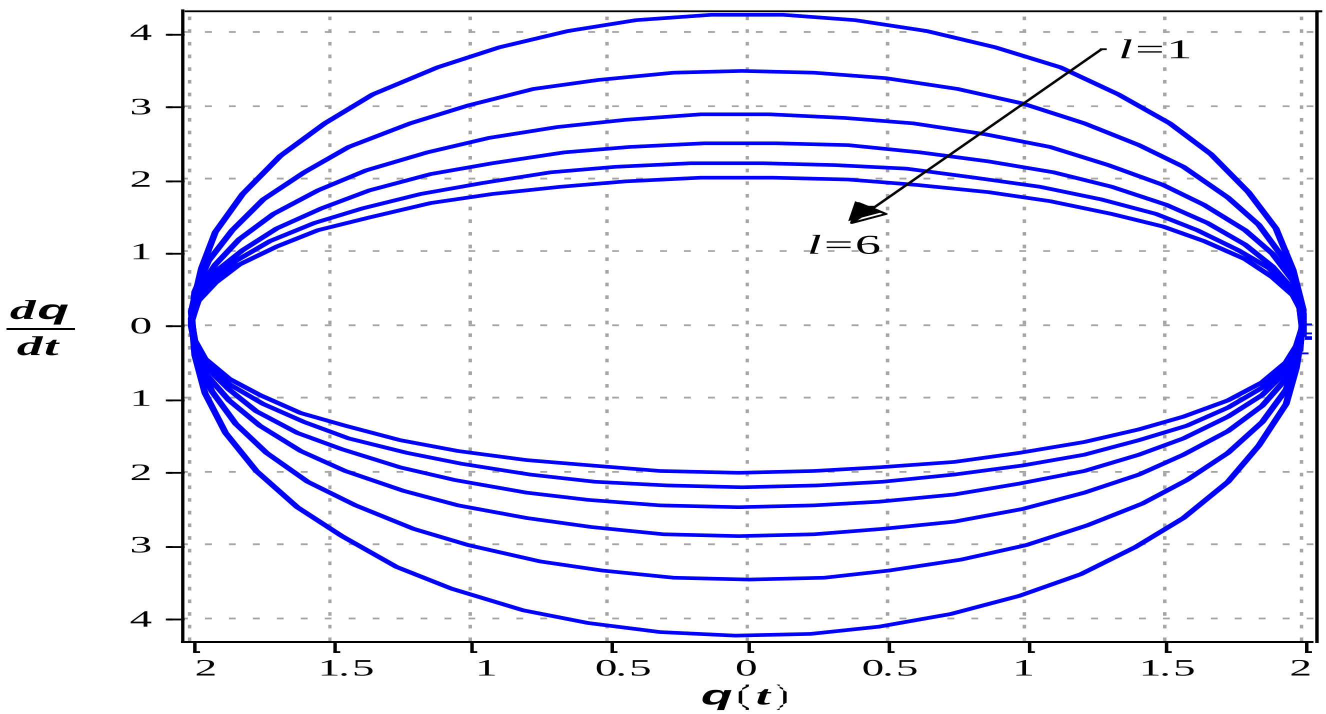

Figure 6. The phase plane, $m = 5$, $r = 1$, $g = 9.81$, $k = 10$, $A = 2$.

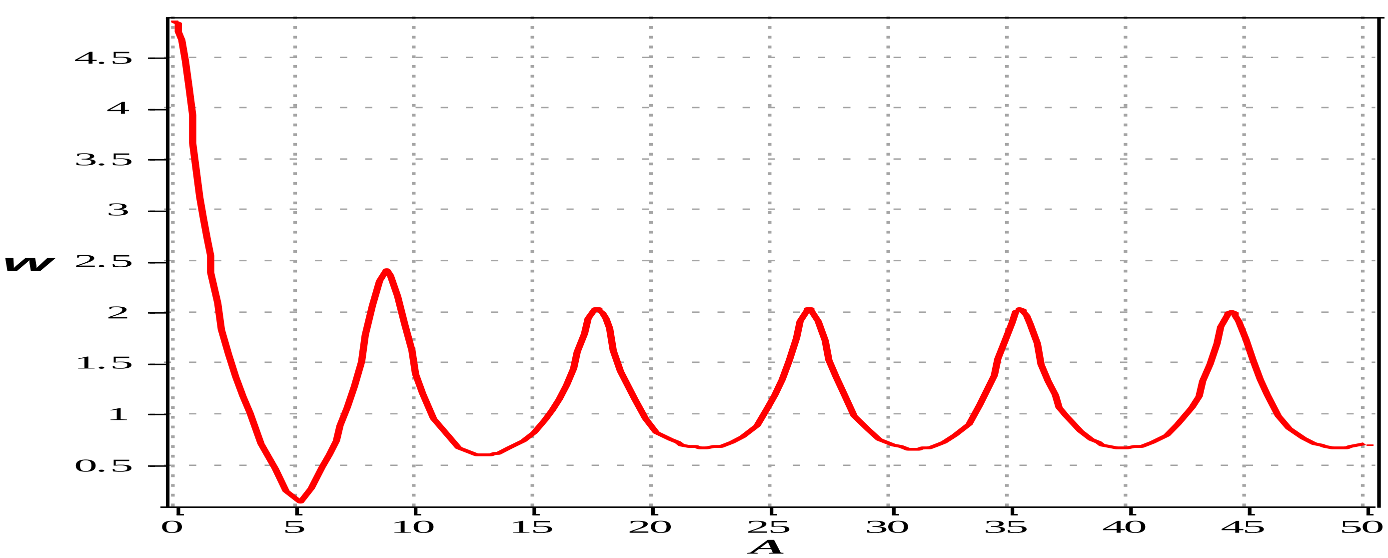

Figure 7. Variation of frequency respect to various parameters of amplitude (A) for $m = 10$, $l = 0.5$, $r = 1$, $g = 9.81$, $k = 10$.

system is periodic. Figures 4 and 5 represents comparison of analytical solution of θ_t based on time with the numerical solution for different parameters of the system. As shown in Figures 2 to 5, it is apparent that the EBM has an excellent agreement with the numerical solution using Rung-Kutta and these expressions are valid for a wide range. Figure 6 shows the phase plan of the problem and Figure 7 to 9 represent the variation of frequency respect to various parameters of amplitude (A), (r) and (l) and Figure 10 is the sensitivity analysis of

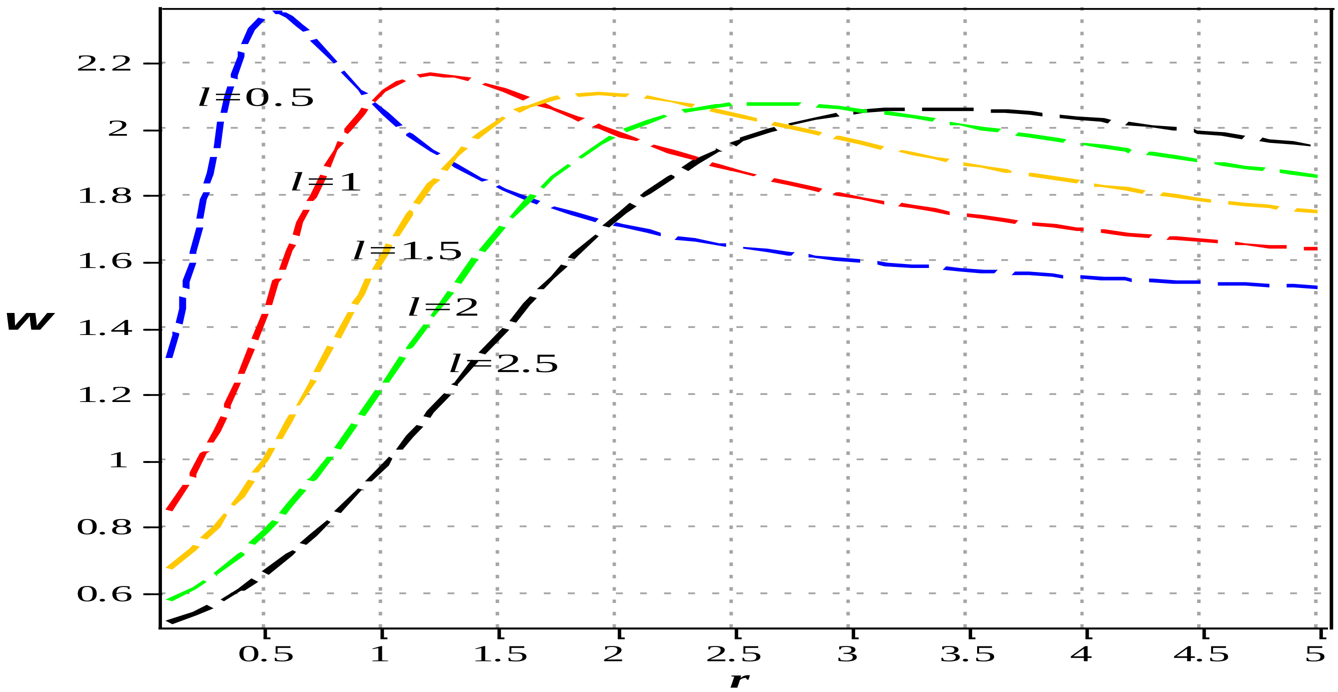

Figure 8. Variation of frequency respect to various parameters of (*r*) for $m = 5, \quad g = 9.81, \quad k = 10, \quad A = 10$.

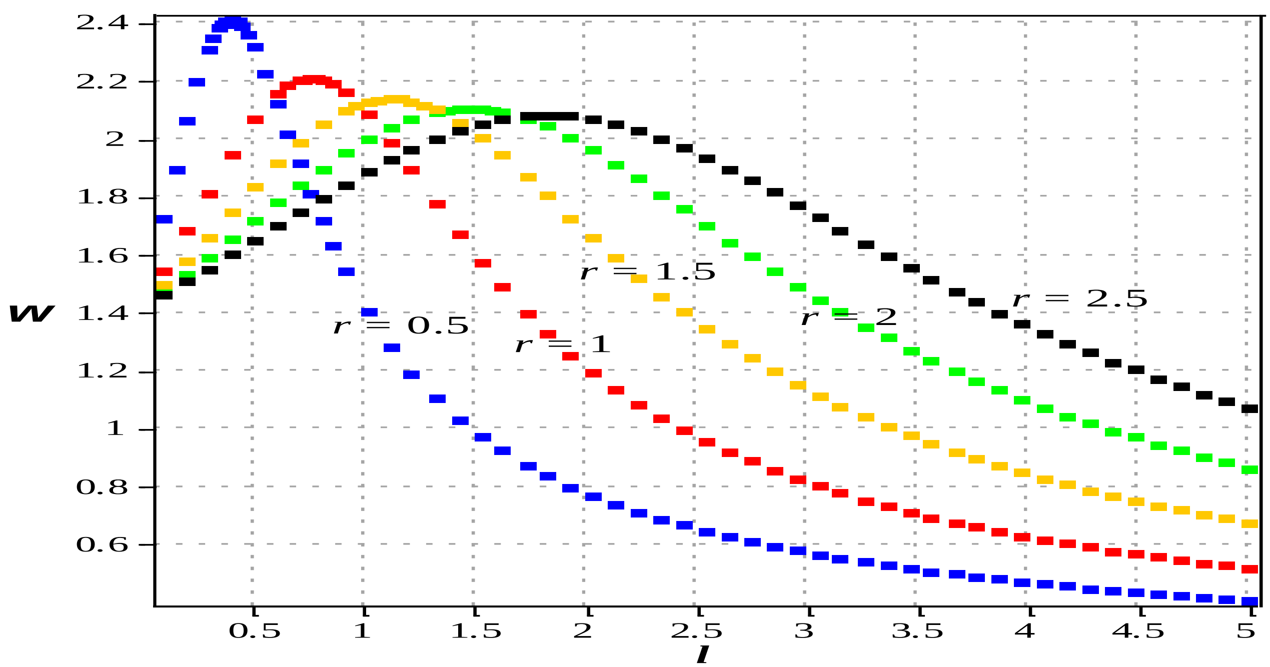

Figure 9. Variation of frequency respect to various parameters of (*l*) for $m = 5, \ g = 9.81, \ k = 10, \quad A = 10$.

frequency. To further illustrate and verify the accuracy for this approximate analytical approach, the results obtained with the EBM and the Runge-Kutta are tabulated in Table 1 and the maximum relative error is less than 2.9376%.

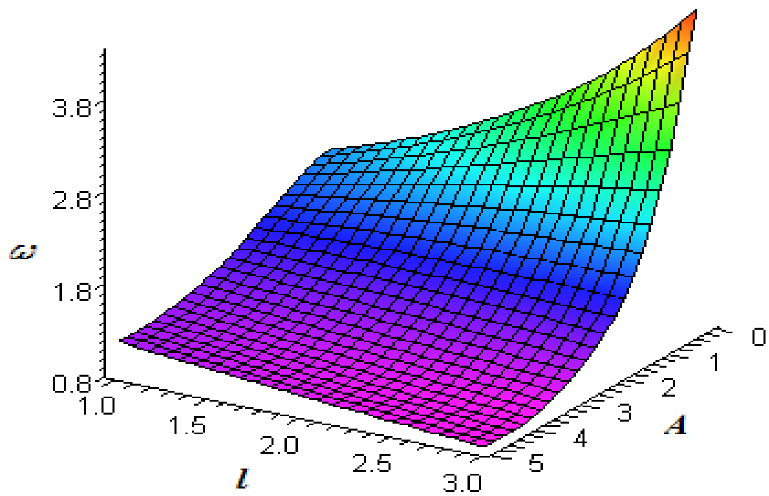

Figure 10. Sensitivity analysis of frequency for $m=5,\ r=5,\ g=9.81,\ k=10,\ A=2$.

Table 1. Comparison of frequency corresponding to various parameters of system.

A	*m*	*l*	*r*	*g*	*k*	ω_{EBM}	ω_{NM}	Error (%)
0.1	5	2	0.5	9.81	2	2.9554	2.8685	2.9376
0.5	5	2	0.5	9.81	2	2.8376	2.7566	2.8542
2	5	1	0.2	9.81	5	2.4045	2.3406	2.6587
3	8	0.5	0.3	9.81	10	1.5081	1.4867	1.4222
8	8	0.5	0.3	9.81	10	2.0451	2.0059	1.9172
10	10	3	2	9.81	20	1.4712	1.4518	1.3190
15	10	3	2	9.81	20	0.6835	0.6782	0.7790
20	10	3	2	9.81	20	0.7767	0.7698	0.8870

Conclusion

In this study we have considered the nonlinear vibration of a single degree of freedom system. It has been proved that EBM is clearly effective, convenient and does not require any linearization or small perturbation and adequately accurate to both linear and nonlinear problems in physics and engineering. These cases study demonstrates that the results of EBM are exactly equal and in excellent agreement with numerical solution obtained by the Runge-Kutta method. EBM is a novel method for the analysis of nonlinear systems. The results indicated that EBM is extremely speedy, light, with high accuracy. EBM provides an easy and direct procedure for determining approximations of periodic solutions.

Nomenclature: θ, Rotation angle of the pendulum; m, mass of pendulum; r, radius of rolling wheel; l, distance between center of rolling wheel and center of mass of pendulum; K, spring stiffness; g, gravity of earth; A, amplitude; ω, system frequency; **EBM,** energy balance method.

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