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Harmonic curvature of the curve-surface pair under Möbius transformation

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The Möbius transformation is known as a part of analysis and mostly studied by the researchers and scientist who made their investigations on analysis. But in this paper it is worked together with differential geometry. It is well known that harmonic curvature H of the curve is found by Hacısalihoglu and also harmonic curvatures $\overline{H}$ and $\overline{H}^*$ of the curve-surface pair that we call the strip as shown as (α, M) is studied by Hacısalihoglu and Ertem Kaya. We observe the image of the harmonic curvatures $\overline{H}$ and $\overline{H}^*$ of the curve-surface pair (α, M) in differential geometry under Möbius transformation in analysis. We find the images of harmonic curvatures $\overline{H}$ and $\overline{H}^*$ of the strip (α, M) by the help of the Möbius transformation. Consequently, in this paper the curvatures and the harmonic curvature H^* of the curve-surface pair under Möbius transformation is studied and H^* is not invariant for Möbius transformation is obtained.

Key words: Curve-surface pair, Möbius transformation, curvature.

INTRODUCTION

In 3-dimensional Euclidean space, a regular curve is described by its curvatures k_1 and k_2 and also a curve - surface pair is described by its curvatures k_n, k_g and t_r. The relations between the curvatures of a curve-surface pair and the curvatures of the curve can be seen in many diferential books and papers. Möbius transformations are the automorphisms of the extended complex plane C∞:C∪{∞} that is the metamorphic bijections (Ozgür et al., 2005). M: C∞→C∞ a möbius transformation M has the form

$$M(z) = \frac{az+b}{cz+d}; a,b,c,d \in \mathrm{C} \text{ and } \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 .$$

The set of all Möbius transformations is a group under composition. The Möbius transformation with $c=0$ form the subgroup of similarities such transformations have the form $S(z) = Az + B; A, B \in \mathrm{C}, A \neq 0$. The transformation $J(z) = \frac{1}{z}$ is called an inversion. Every Möbius transformation M of the form $S(z) = Az + B$ is a composition of finitely many similarities and inversions (Ozgür, 2010). In this paper we investigate the harmonic curvatures of the curve-surface pair under Möbius transformations. H^* and H^{**} be the image of the harmonic curvatures of the curve-surface pair and a^*, b^* and c^* be the curvatures of the curve-surface pair under

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Möbius transformation. We obtain the harmonic curvature of the strip H^* is invariant for S transformation and H^* is not invariant for *J* transformation.

THE CURVE-SURFACE PAİR (STRIP)

Definition 1

Let M and α be a surface in E^3 and a curve in $M \subset E^3$. We define a surface element of M is the part of a tangent plane at the neighbour of the point. The locus of these surface element along the curve α is called a curve-surface pair and is shown as (α, M).

Definition 2

Let $\left\{\vec{t}, \vec{n}, \vec{b}\right\}$ and $\left\{\vec{\xi}, \vec{\eta}, \vec{\zeta}\right\}$ be the curve and curve-surface pair's vector fields. The curve-surface pair's tangent vector field, normal vector field and binormal vector field is given by $\vec{t} = \vec{\xi}$, $\vec{\zeta} = \vec{N}$ ($\vec{N} = \vec{n}$) and $\vec{\eta} = \vec{\zeta} \wedge \vec{\xi}$ (Hacısalihoglu, 1982).

Curvatures of the curve-surface pair and curvatures of the curve

Let $k_n = -b, k_g = c, t_r = a$ and $\left\{\vec{\xi}, \vec{\eta}, \vec{\zeta}\right\}$ be the normal curvature, the geodesic curvature, the geodesic torsion of the strip and the curve-surface pair's vector fields on α (Hacısalihoglu, 1982). Then we have

$$\vec{\xi}' = c\vec{\eta} - b\vec{\zeta}$$
$$\vec{\eta}' = -c\vec{\xi} + a\vec{\zeta}$$
$$\vec{\zeta}' = b\vec{\xi} - a\vec{\eta}.$$

We know that a curve α has two curvatures κ and τ. A curve has a strip and a strip has three curvatures k_n, k_g and t_r. Let k_n, k_g and t_r be the $-b, c$ and a. From last equations we have $\vec{\xi}' = c\vec{\eta} - b\vec{\zeta}$. If we substitude $\vec{\xi} = \vec{t}$ in last equation, we obtain

$$\vec{\xi}' = \kappa\vec{n}$$

And

$$b = -\kappa \sin\varphi$$
$$c = \kappa \cos\varphi$$

(Hacısalihoglu, 1982). From the last two equations we obtain,

$$\kappa^2 = b^2 + c^2.$$

This equation is a relation between the curvature κ of a curve α and normal curvature and geodesic curvature of a curve-surface pair. By using similar operations, we obtain a new equation as follows:

$$\tau = a + \frac{b'c - bc'}{b^2 + c^2}$$

(Ertem Kaya et al., 2010; Hacısalihoglu, 1982).

This equation is a relation between τ (torsion or second curvature of α and curvatures of a curve-surface pair that belongs to the curve α). And also we can write

$$a = \varphi' + \tau.$$

The special case: If φ is constant, then $\varphi' = 0$. So the equation is $a = \tau$. That is, if the angle is constant, then torsion of the curve-surface pair is equal to torsion of the curve.

Definition 3

Let α be a curve in $M \subset E^3$. If the geodesic curvature (torsion) of the curve α is equal to zero, then the curve-surface pair (α, M) is called a curvature curve-surface pair (strip) (Keleş, 1982).

Definition 4

Let $\overline{H}$ and $\overline{H}^*$ be the harmonic curvature of a strip in E^3. Then the harmonic curvature of the strip is the ratio of the torsion to the normal curvature of the strip and also the ratio of the torsion of the strip to the geodesic curvature of the strip, respectively (Ertem Kaya et al., 2010).

HARMONIC CURVATURES OF THE CURVE-SURFACE PAIR

In first case

If we take first curvature of a strip $k_n = -b$, then we obtain harmonic curvature of a strip is as follows;

$$\overline{H} = \frac{t_r}{k_n} = \frac{a}{-b}.$$

In second case

If we take first curvature of a strip $k_g = c$, then we obtain harmonic curvature of a strip is as follows;

$$\overline{H}^* = \frac{t_r}{k_g} = \frac{a}{c}.$$

RELATIONS BETWEEN HARMONIC CURVATURE OF A STRIP AND HARMONIC CURVATURE OF A CURVE

Lets take first curvature of the strip is $k_n = -b$ and torsion of the strip is $t_r = a$. If we write the equations of k_n, t_r and $\overline{H}$ into the harmonic curvature of the strip, then we obtain the equation

$$\overline{H} = \frac{t_r}{k_n} = \frac{a}{-b} = \frac{\varphi' + \tau}{\kappa \sin\varphi}$$

$$\overline{H} = \frac{\varphi'}{\kappa \sin\varphi} + \frac{\tau}{\kappa \sin\varphi}$$

$$\overline{H} = \frac{\varphi'}{\kappa \sin\varphi} + \frac{\tau}{\kappa}\frac{1}{\sin\varphi}$$

$$\overline{H} = \frac{\varphi'}{\kappa \sin\varphi} + H\frac{1}{\sin\varphi}$$

$$\overline{H} = \frac{\varphi'}{\kappa \sin\varphi} + H\csc\varphi.$$

The last equation is the relation between harmonic curvature of the strip and harmonic curvature of a curve in first case (Ertem Kaya, 2010; Ertem Kaya et al., 2010). Now let take first curvature of the strip is $k_g = c$ and torsion of the strip is $t_r = a$. If we write the equations of k_g, t_r and $\overline{H}^*$ into the harmonic curvature of the strip, then we obtain the equation:

$$\overline{H}^* = \frac{t_r}{k_g} = \frac{a}{c} = \frac{\varphi' + \tau}{\kappa \cos\varphi}$$

$$\overline{H}^* = \frac{\varphi'}{\kappa \cos\varphi} + \frac{\tau}{\kappa \cos\varphi}$$

$$\overline{H}^* = \frac{\varphi'}{\kappa \cos\varphi} + \frac{\tau}{\kappa}\frac{1}{\cos\varphi}$$

$$\overline{H}^* = \frac{\varphi'}{\kappa \cos\varphi} + H\frac{1}{\cos\varphi}$$

$$\overline{H}^* = \frac{\varphi'}{\kappa \cos\varphi} + H\sec\varphi.$$

The last equation is the relation between harmonic curvature of the strip and harmonic curvature of a curve in second case (Ertem Kaya, 2010; Ertem Kaya et al., 2010).

SPECİAL CASE

Theorem 1

Let $\overline{H}$ be the harmonic curvature of the strip. If the angle φ between normal vector field of the surface and binormal vector field of the curve is constant, $k_n = -b$ and $t_r = a$ is the first curvature of the strip and the torsion of the strip in E^3, we give the relation between harmonic curvatures of the strip and the curve in first case $\overline{H} = H\csc\varphi$.

Proof 1

In first case we know $\overline{H} = \frac{\varphi'}{\kappa \sin\varphi} + H\csc\varphi$ and ıf the angle φ is constant, then $\varphi' = 0$. So we take

$$\overline{H} = \frac{\varphi'}{\kappa \sin\varphi} + H\csc\varphi$$

$$\overline{H} = \frac{0}{\kappa \sin\varphi} + H\csc\varphi$$

$$\overline{H} = H\csc\varphi.$$

Theorem 2

Let $\overline{H}^*$ be the harmonic curvature of the strip. If the angle φ between normal vector field of the surface and binormal vector field of the curve is constant, $k_g = c$ and

$t_r = a$ is the first curvature of the strip and the torsion of the strip in E^3, we give the relation $\overline{H}^* = H \sec\varphi$ between harmonic curvatures of the strip and the curve in second case.

Proof 2

We know $\overline{H}^* = \dfrac{\varphi'}{\kappa\cos\varphi} + H\sec\varphi$ and $\varphi' = 0$. So we obtain;

$$\overline{H}^* = \frac{\varphi'}{\kappa\cos\varphi} + H\sec\varphi$$

$$\overline{H}^* = \frac{0}{\kappa\cos\varphi} + H\sec\varphi$$

$$\overline{H}^* = H\sec\varphi.$$

Theorem 3

Let the angle φ between normal vector field of the surface and binormal vector field of the curve be constant and (α, M) be strip in E^3. When the strip (α, M)'s curvatures a, b, c are not constant but harmonic curvature of strip is constant, the strip is called inclined strip $\Leftrightarrow \overline{H}^2 + \overline{H}^{*2} = cst.$ (Ertem Kaya, 2010).

Proof 3

$\Rightarrow$ Let $\overline{H}$ and $\overline{H}^*$ be the harmonic curvatures of (α, M). We should show that $\overline{H}^2 + \overline{H}^{*2}$ must be constant. We know that harmonic curvatures of the strip $\overline{H} = \dfrac{t_r}{k_n} = \dfrac{a}{-b}$ and $\overline{H}^* = \dfrac{t_r}{k_g} = \dfrac{a}{c}$. If we use these equations, we take

$$\overline{H}^2 + \overline{H}^{*2} = \left(\frac{a}{-b}\right)^2 + \left(\frac{a}{c}\right)^2$$

$$\overline{H}^2 + \overline{H}^{*2} = \left(\frac{\varphi' + \tau}{\kappa\sin\varphi}\right)^2 + \left(\frac{\varphi' + \tau}{\kappa\cos\varphi}\right)^2.$$

In here, φ is constant. So,

$\varphi' = 0$.

Since φ' is equal to zero, we obtain

$$\overline{H}^2 + \overline{H}^{*2} = \left(\frac{\tau}{\kappa}\right)^2 \frac{1}{(\sin\varphi\cos\varphi)^2}.$$

We know $\dfrac{\tau}{\kappa}$ and φ is constant. So $\overline{H}^2 + \overline{H}^{*2}$ must be constant $\Leftarrow$ Let $\overline{H}^2 + \overline{H}^{*2}$ be constant. In this case is we must prove (α, M) a helix strip. If we observe $\overline{H}^2 + \overline{H}^{*2}$, then we take

$$\overline{H}^2 + \overline{H}^{*2} = \left(\frac{\tau}{\kappa\sin\varphi}\right)^2 + \left(\frac{\tau}{\kappa\cos\varphi}\right)^2$$

$$\overline{H}^2 + \overline{H}^{*2} = \left(\frac{\tau}{\kappa}\right)^2 \frac{1}{(\sin\varphi\cos\varphi)^2}.$$

Since $\overline{H}^2 + \overline{H}^{*2}$ is constant, $\left(\dfrac{\tau}{\kappa}\right)^2 \dfrac{1}{(\sin\varphi\cos\varphi)^2}$ must be constant. We know φ is constant. So $\dfrac{\tau}{\kappa}$ must be constant. This means that the curve α is a helix and the strip (α, M) is a helix strip (Ertem Kaya, 2010; Ertem Kaya et al., 2010).

Theorem 4

Let (α, M) and $\overline{H}, \overline{H}^*$ be a curve-surface pair (strip) and harmonic curvatures of (α, M) in E^3. Let the angle φ between normal vector field of the surface and binormal vector field of the curve is constant. The strip (α, M) is inclined curve-surface pair $\Leftrightarrow \dfrac{\overline{H}^*}{\overline{H}} = \tan\varphi =$ constant (Ertem Kaya, 2010; Ertem Kaya et al., 2010).

Proof 4

$(\Rightarrow)$ We know $\overline{H} = \dfrac{H}{\sin\varphi}$ and $\overline{H}^* = \dfrac{H}{\cos\varphi}$ by the theorems 1 and 2. Thus

$$\frac{\overline{H}^*}{\overline{\overline{H}}}=\frac{\dfrac{H}{\cos\varphi}}{\dfrac{H}{\sin\varphi}}$$

$$\frac{\overline{H}^*}{\overline{\overline{H}}}=\tan\varphi.$$

Since φ is constant, $\frac{\overline{H}^*}{\overline{\overline{H}}}$ is constant.

$(\Leftarrow)$Let $\frac{\overline{H}^*}{\overline{\overline{H}}}$ be constant. Thus $\tan\varphi$ must be constant. So φ is constant. Since φ is constant, (α, M) is inclined (Ertem Kaya et al., 2010). Thus we take the strip (α, M) is inclined curve-surface pair $\Leftrightarrow \frac{\overline{H}^*}{\overline{\overline{H}}}=\tan\varphi=$ constant (Ertem Kaya, 2010; Ertem Kaya et al., 2010).

Example 1

Let take the curve $\alpha=(\cos s, \sin s, s)$ and be the angle $\varphi=\frac{\pi}{2}$ between normal vector field of the surface and binormal vector field of the curve be constant. Thus we find the curvatures, harmonic curvatures of (α, M) and the image of the Möbius transformations of these. Firstly, let find the curvatures of the Möbius transformations. We can compute the curvatures $\kappa=\frac{1}{2}$ and the torsion $\tau=\frac{1}{2}$ of the curve α. Thus we take the curvatures of the strip (α, M)

$$b=-\kappa\sin\varphi=-\frac{1}{2}\sin\frac{\pi}{2}=-\frac{1}{2},$$

$$c=\kappa\cos\varphi=\frac{1}{2}\cos\frac{\pi}{2}=0,$$

$$a=\varphi'+\tau=0+\frac{1}{2}.$$

Now we can find the harmonic curvature of curve as in the following

$$H=\tau/\kappa$$

$$H=1=const$$

We can say that the curve is an helix. Thus we have the harmonic curvatures of the strip.

In first case: If we take first curvature of a strip $k_n=-b$, then we obtain harmonic curvature of a strip is as follows;

$$\overline{H}=\frac{t_r}{k_n}=\frac{a}{-b}=\frac{1/2}{-(-1/2)}=1.$$

In second case: If we take first curvature of a strip $k_g=c$, then we obtain harmonic curvature of a strip is as follows;

$$\overline{H}^*=\frac{t_r}{k_g}=\frac{a}{c}=\frac{1/2}{0}=\infty..$$

We can not calculate harmonic curvature in second case.

THE IMAGE OF THE CURVE-SURFACE PAIRS UNDER THE MOBIUS TRANSFORMATION

Theorem 5

Let a_1, b_1, c_1 be the curvatures of the strip (α, M) and (α, M) transform (M $(\alpha), M$) under Möbius transformation. So a_2, b_2, c_2 are the curvatures of (M $(\alpha), M$). We know that ıf (α, M) is a curvature curve-surface pair, then $a_1=0$ (Keleş, 1982). So we can say that ıf (α, M) is a curvature curve-surface pair, then we find (M $(\alpha), M$) is a curvature curve-surface pair.

Proof 5

Let the image of the curve α and the angle φ under Möbius transformation be the curve β and the angle θ. We know that the curvatures of α belong to:

$$\kappa^2=b_1^2+c_1^2$$

$$\tau=a_1+\frac{b_1'c_1-b_1c_1'}{b_1^{\,2}+c_1^{\,2}},$$

$$\tau=a_1-\varphi'(a_1=0).$$

Before the Möbius transformation we have $\alpha, \varphi, \kappa, \tau, a_1, b_1$ and c_1, after Möbius transformation let take $\beta, \theta, \kappa^*, \tau^*, a_2, b_2$ and c_2. And the curvatures of β

belong to:

$$\kappa^{*2} = b_2^2 + c_2^2$$
$$\tau^* = a_2 + \frac{b_2' c_2 - b_2 c_2'}{b_2^{\ 2} + c_2^{\ 2}},$$
$$\tau^* = a_2 - \theta'.$$

We should proof a_2 is equal to zero. Every Möbius transformation M of the Form (2) is a composition of finitely many similarities and inversions (Ozgür, 2010). Such as we can write,

$$\mathrm{M}(\alpha) = S(\alpha) o J(\alpha)\ \alpha$$
$$\beta = \beta_1 o \beta_2.$$

Let A= (x_1, x_2) and $(\alpha_1, \alpha_2, \alpha_3)$ be a point and the coordinates of α on the curve α. So from the equation $S(z) = Az + B; A, B \in$ C, $A \neq 0$, we have

$$S(\alpha) = S(\alpha(t))$$
$$S(Z + t_j) = (AZ + B + |A| t_j) =$$
$$= (x_1\alpha_1 - x_2\alpha_2 + b_1, x_1\alpha_2 + x_2\alpha_2 + b_2, |A|\alpha_3)$$

And

$$J(Z + t_j) = \frac{\overline{Z} + t_j}{|Z|^2 + t_j^2}$$
$$J(\alpha(t)) = \left(\frac{\alpha_1(t)}{\|\alpha_1(t)\|^2}, \frac{-\alpha_2(t)}{\|\alpha_2(t)\|^2}, \frac{\alpha_3(t)}{\|\alpha_3(t)\|^2} \right).$$

For the transformation S, we have

$\kappa^* = \frac{1}{|A|}\kappa$ and $\tau^* = \frac{1}{|A|}\tau$,

$$a_2 = \tau^* + \theta'$$
$$a_2 = \frac{1}{|A|}\tau + \theta'$$
$$a_2 = \frac{1}{|A|}(-\varphi') + \theta'$$
$$a_2 = \frac{1}{|A|}\left(\frac{b_1' c_1 - b_1 c_1'}{b_1^{\ 2} + c_1^{\ 2}}\right) + \theta'$$
$$a_2 = \frac{1}{|A|}\left(\frac{b_1' c_1 - b_1 c_1'}{\kappa^2}\right) + \theta'$$
$$a_2 = \frac{1}{|A|}\left(\frac{b_1' c_1 - b_1 c_1'}{(|A|\kappa^*)^2}\right) + \theta'$$
$$a_2 = \frac{1}{|A|^3}\left(\frac{b_1' c_1 - b_1 c_1'}{b_2^{\ 2} + c_2^{\ 2}}\right) + \theta'.$$

We know that $a_1 = 0$ from the Theorem 5. We have

$$b_2 = \frac{1}{|A|}\kappa \sin\theta$$
$$b_2 = \frac{1}{|A|}\frac{b_1}{\sin\varphi}\sin\theta$$

and

$$c_2 = \frac{1}{|A|}\frac{c_1}{\cos\varphi}\cos\theta$$
$$c_1 = \frac{1}{|A|}\frac{c_2}{\cos\theta}\cos\varphi$$
$$b_1 = |A| b_2 \frac{\sin\varphi}{\sin\theta}.$$

If we apply b_1 to a_2, after some computaions, we find the equation

$$a_2 = \frac{1}{|A|}\left(\frac{b_2' c_2 - b_2 c_2'}{b_2^{\ 2} + c_2^{\ 2}}\right) + \theta'$$
$$a_2 = \frac{1}{|A|}(-\theta') + \theta'.$$

Now we have two cases:

Case 1. If $|A| = 1$, then we find $a_2 = 0$. That means (M $(\alpha), M$) is a curvature curve-surface pair (strip). This proves the theorem.

Case 2. If $|A| \neq 1$, then we find $a_2 = \frac{1}{|A|}\theta'(|A| - 1)$. Thus (M $(\alpha), M$) is not a curvature curve-surface pair.

Corollary 1

If $|A| = 1$, then the characterization of curvature strip is invariant under Möbius transformation.

Corollary 2

If $|A| \neq 1$, then the characterization of curvature strip is not invariant under Möbius transformation.

Example 2

Let us consider the curve $\alpha = (\cos s, \sin s, s)$ is an helix and find the image of the α and its curvatures under Möbius transformation. We know

$$J(Z + t_j) = \frac{\overline{Z} + t_j}{|Z|^2 + t_j^2}$$

Thus we have the image of α under J as

$$J(\cos s + (\sin s)i + sj) = \frac{\cos s - i\sin s + sj}{1+s^2},$$

Then the image of the α is $J(\alpha)$ has the form

$$\beta(s) = (\frac{\cos s}{1+s^2}, \frac{-\sin s}{1+s^2}, \frac{s}{1+s^2})$$

After some computations we find the curvatures κ^* of $\beta(s)$

$$\kappa^* = \left(\frac{(1+s^2)^3 \left[\begin{array}{c} (\sin s - 2\cos s + 2\sin s - s^2 \sin s)^2 \\ + (\cos s + 2s\sin s + 2\cos s - s^2\cos s)^2 + (3+s^2) \end{array} \right]}{\left[\begin{array}{c} (\sin s(1+s^2) + 2s\cos s)^2 \\ + \cos(1+s^2) - 2s\sin s)^2 + (1-s^2)^2 \end{array} \right]} \right)$$

$$\kappa^* = \left[\frac{(2s^4 + 4s^2 + 18)^{\frac{1}{2}}}{2^{\frac{3}{2}}} \right] = \frac{(s^4 + s^2 + 9)}{2}$$

κ^* can not be a constant. Similarly, the torsion of $\beta(s)$ is obtained as in the following

$$\tau^* = \frac{(s^6 - s^4 - 5s^2 - 3)}{2s^4 + 4s^2 + 18}$$

τ^* can not be a constant. Consequently $J(\alpha)$ is not a helix. We have

$$S(z) = Az + B$$

$\kappa^* = \frac{1}{|A|}\kappa$ and $\tau^* = \frac{1}{|A|}\tau$,

then $\kappa = const$ >0, $\tau = const \neq 0$. We see that $\kappa^* = const$ >0, $\tau^* = const \neq 0$. So $S(\alpha)$ is a helix (for more details (Ozgür et al., 2005).

Theorem 6

For the transformation J, let take $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in S_o^2 \Rightarrow \|\alpha\| = 1,\ \alpha$ is curvature strip and $J(\alpha) = \beta \in S_o^2$ [Möbius transformations take circles to circles (Ozgür, 2010)]. We have

$$a_1 = \tau + \varphi', a_1 = 0,$$
$$a_2 = \tau^* + \varphi'$$

And

$$\tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2},$$
$$\tau^* = \frac{\det(\beta', \beta'', \beta''')}{\|\beta' \wedge \beta''\|^2}.$$

So we have,

$$\beta = (\alpha_1, -\alpha_2, \alpha_3)$$
$$\beta' = (\alpha_1', -\alpha_2', \alpha_3')$$
$$\beta'' = (\alpha_1'', -\alpha_2'', \alpha_3'')$$
$$\beta''' = (\alpha_1''', -\alpha_2''', \alpha_3''')$$

And

$$\frac{\det(\beta', \beta'', \beta''')}{\|\beta' \wedge \beta''\|^2} = -\frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2},$$
$$\tau^* = -\tau.$$

We find $a_2 = -2\tau = 2\varphi'$ and $\varphi' = 0 \Rightarrow a_2 = 0$. Thus β is a curvature strip. This completes the proof of the theorem.

HARMONIC CURVATURE OF THE CURVE-SURFACE PAIR UNDER MOBIUS TRANSFORMATION

Let H^* and H^{**} be the image of the harmonic curvatures of the curve-surface pair and a^*, b^* and c^* be the curvatures of the curve-surface pair under Mobius transformation. In first case, we obtain the following theorem.

Theorem 7

We obtain the harmonic curvature of the strip H^* is invariant for S transformation.

Proof 7

We have

$$H^{*}=\frac{a^{*}}{-b^{*}}=\frac{\theta'+\tau^{*}}{\kappa^{*}\sin\theta}=\frac{\theta'+\frac{1}{|A|}\tau}{\frac{1}{|A|}\kappa\sin\theta}=\frac{|A|\theta'+\tau}{\kappa\sin\theta}.$$

If the angle θ is constant, then $\theta'=0$. Thus we take,

$$H^{*}=\frac{\tau}{\kappa}.$$

Remark 1

If the angle $\theta=k\pi(k=0,1,2...)$, we obtain $H^{**}=\infty$. So, we can take the harmonic curvature of the curve-surface pair H^{**} is not invariant for S transformation.

Theorem 8

In second case we can calculate

$$H^{**}=\frac{a^{*}}{c^{*}}=\frac{\frac{1}{|A|}\tau+\theta'}{\frac{1}{|A|}\kappa\cos\theta}.$$

If the angle θ is constant, then $\theta'=0$. Thus we take,

$$H^{**}=\frac{\frac{1}{|A|}\tau}{\frac{1}{|A|}\kappa\cos\theta}$$

$$H^{**}=\frac{\tau}{\kappa\cos\theta}$$

$$H^{**}=\frac{\tau}{\kappa}\sec\theta.$$

Theorem 9

If the angle $\theta=k\pi(k=0,1,2...)$ we can take the harmonic curvature of the curve-surface pair H^{**} is invariant for S transformation.

Proof 9

If $k=0$, then we have $\cos\theta=\cos 0=1$. So we observe

$$H^{**}=\frac{\tau}{\kappa},$$

If $k=1$, then
$\cos\theta=\cos\pi=-1$. So

$$H^{**}=-\frac{\tau}{\kappa},$$

If $k=2$, then
$\cos\theta=\cos 2\pi=1$. So

$$H^{**}=\frac{\tau}{\kappa},$$

We can do same operations for every k. Thus we have $\cos\theta$ is constant. So H^{**} is invariant for S transformation. This completes the proof of the theorem.

Theorem 10

If the curvatures a,b,c of the strip (α,M) are not constant but harmonic curvatures a^{*},b^{*} and c^{*} of the strip under Möbius transformation and the angle θ are constant, then the strip is called inclined strip $\Leftrightarrow H^{*2}+H^{**2}=cst.$

Proof 10

$(\Rightarrow)$ Let H^{*} and H^{**} be the harmonic curvatures of (α,M) under Mobius transformation. We should show that $H^{*2}+H^{**2}$ must be constant. We know that harmonic curvatures of the strip

$$H^{*}=\frac{a^{*}}{-b^{*}}=\frac{\theta'+\frac{1}{|A|}\tau}{\frac{1}{|A|}\kappa\sin\theta}\text{ and }H^{**}=\frac{a^{*}}{c^{*}}=\frac{\frac{1}{|A|}\tau+\theta'}{\frac{1}{|A|}\kappa\cos\theta}.$$

If we use these equations, we take

$$H^{*2}+H^{**2}=\left(\frac{a^*}{-b^*}\right)^2+\left(\frac{a^*}{c^*}\right)^2$$

$$H^{*2}+H^{**2}=\left(\frac{\theta'+\frac{1}{|A|}\tau}{\frac{1}{|A|}\kappa\sin\theta}\right)^2+\left(\frac{\frac{1}{|A|}\tau+\theta'}{\frac{1}{|A|}\kappa\cos\theta}\right)^2$$

Since θ' is equal to zero, we obtain

$$H^{*2}+H^{**2}=\left(\frac{\tau}{\kappa}\right)^2\frac{1}{(\sin\theta\cos\theta)^2}.$$

We know $\frac{\tau}{\kappa}$ and θ is constant. So $H^{*2}+H^{**2}$ must be constant.

$\Leftarrow$Let $H^{*2}+H^{**2}$ be constant. So we must prove (α,M) a helix strip. If we observe $H^{*2}+H^{**2}$, then we take

$$H^{*2}+H^{**2}=\left(\frac{\tau}{\kappa\sin\theta}\right)^2+\left(\frac{\tau}{\kappa\cos\theta}\right)^2$$

$$H^{*2}+H^{**2}=\frac{\tau^2}{\kappa^2\sin^2\theta}+\frac{\tau^2}{\kappa^2\cos^2\theta}$$

$$H^{*2}+H^{**2}=\frac{\tau^2\cos^2\theta+\tau^2\sin^2\theta}{\kappa^2\sin^2\theta\cos^2\theta}$$

$$H^{*2}+H^{**2}=\frac{\tau^2(\cos^2\theta+\sin^2\theta)}{\kappa^2\sin^2\theta\cos^2\theta}$$

$$H^{*2}+H^{**2}=(\frac{\tau}{\kappa})^2\frac{1}{(\sin\theta\cos\theta)^2}$$

Since

$$H^{*2}+H^{**2}$$

is constant,

$$(\frac{\tau}{\kappa})^2\frac{1}{(\sin\theta\cos\theta)^2}$$

must be constant. We know θ is constant. So

$$\frac{\tau}{\kappa}$$

must be constant. This means that the curve is helix and the strip is a helix strip.

Theorem 11

The harmonic curvatures of curve-surface pair is invariant under mobius transformation.

Proof 11

It is seen obviosly from the proof of the theorem 2 and proof of the theorem 3.

Theorem 12

Let (α,M), and H^* and H^{**} be a curve-surface pair (strip) and harmonic curvatures of (α,M) under Mobius transformation in E^3. Let the angle φ between normal vector field of the surface and binormal vector field of the curve is constant. Let θ be the image of the under Mobius transformation. The curve-surface pair (α,M) is inclined curve-surface pair under Mobius transformation $\Leftrightarrow \frac{H^*}{H^{**}}=\tan\theta=$ constant.

Proof 12

We know $H^*=\frac{\tau}{\kappa\sin\theta}=\frac{H}{\cos\theta}$, Thus we obtain

$$\frac{H^*}{H^{**}}=\frac{\frac{H}{\cos\theta}}{\frac{H}{\sin\theta}}$$

$$\frac{H^*}{H^{**}}=\tan\theta.$$

Since $\tan\theta$ is constant, $\frac{H^*}{H^{**}}$ is constant.

$(\Leftarrow)$ Let $\frac{H^*}{H^{**}}$ be constant. So $\tan\theta$ must be constant. So θ is constant. Since θ is constant, (α,M) is inclined under Mobius transformation.

Theorem 13

We obtain the harmonic curvature of the strip H^* is not invariant for J transformation.

Proof 13

If (α, M) is a helix curve-surface pair (but not curvature curve-surface pair) under möbius transformation and $J(z)=\frac{1}{z}$ then the image $J(\alpha, M)$ is not a a helix curve-surface pair. So H^* is not invariant for J transformation.

Corollary 3

Harmonic curvature of the strip H^* is not invariant for Mobius transformation.

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