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Full Length Research Paper

Characterizations of a helix in the pseudo - Galilean space G_3^1

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In this manuscript, we obtained some characterizations for a helix in the pseudo-Galilean space G_3^1.

Key words: Pseudo-Galilean space, helix, position vector.

INTRODUCTION

A Galilean space may be considered as the limit case of a pseudo-Euclidean space in which the isotropic cone degenerates to a plane. This limit transition corresponds to the limit transition from the special theory of relativity to classical mechanics. On the other hand, Galilean space-time plays an important role in nonrelativistic physics. The fact that the fundamental concepts such as velocity, momentum, kinetic energy, etc and principles such as laws of motion and conservation laws of classical physics are expressed in terms of Galilean space (Yaglom, 1979).

The geometry of the pseudo-Galilean space is similar (but not the same) to the Galilean space which is presented in Röschel (1984). This paper is written from the pseudo-Galilean point of view.

The pseudo-Galilean geometry is one of the real Cayley-Klein geometries (of projective signature (0,0,+,-). The absolute of the pseudo-Galilean geometry is an ordered triple $\{w, f, I\}$ where w is the ideal (absolute) plane, f is a line in w and I is the fixed hyperbolic involution of points of f (Divjak and Sipus, 2004).

The pseudo-Galilean length of the vector $x=(x, y, z)$ is given by

$$|x| = \begin{cases} x & , \quad x \neq 0 \\ \sqrt{|y^2 - z^2|} & , \quad x = 0 \end{cases} \tag{1.1}$$

The group of motions of G_3^1 is a six-parameter group given (in affine coordinates) by

$$\bar{x} = a + x,$$

$$\bar{y} = b + cx + ych\varphi + zsh\varphi,$$

$$\bar{z} = d + ex + ysh\varphi + zch\varphi.$$

It leaves invariant the pseudo-Galilean length (1.1) of the vector (Divjak and Sipus, 2004).

A vector $X(x, y, z)$ is said to be non isotropic if $x \neq 0$. All unit non-isotropic vectors are of the form $(1, y, z)$. For isotropic vectors $x = 0$ holds. There are four types of isotropic vectors: space-like $(y^2 - z^2 > 0)$, time-like $(y^2 - z^2 < 0)$ and two types of lightlike $(y = \pm z)$ vectors. A non-lightlike isotropic vector is a unit vector if $y^2 - z^2 = 0$.

A trihedron $(T_o; e_1, e_2, e_3)$ with a proper origin $T_o(x_o, y_o, z_o) : (1 : x_o : y_o : z_o)$; is orthonormal in pseudo-Galilean sense if the vectors e_1, e_2, e_3 are of the following form: $e_1 = (1, y_1, z_1)$, $e_2 = (0, y_2, z_2)$, $e_3 = (0, \varepsilon z_2, \varepsilon y_2)$ with $y_2^2 - z_2^2 = \delta$, where ε, δ is $+1$ or -1. Such trihedron $(T_o; e_1, e_2, e_3)$ is called positively oriented if for its vectors $\det(e_1, e_2, e_3)$ holds; that is if $y_2^2 - z_2^2 = \varepsilon$ (Bektaş, 2005).

In differential geometry, curves can be defined as a geometric set of points. Heuristically, we are thinking of a curve as the path traced out by a particle moving in E^3. So, investigating position vectors of the curves is a classical goal to determine behavior of the curve. Because, a position vector is a vector which describes the position of a point P in space in relation to an arbitrary reference origin O. It is equivalent to an imaginary displacement from O to P. There exists a vast literature on this subject, for instance (Ali and Turgut, 2010; Chen, 2003; and İlarslan and Boyacıoğlu, 2008).

A curve of constant slope or general helix is defined by the property that the tangent lines make a constant angle with a fixed direction. A necessary and sufficient condition for a curve to be general helix in pseudo - Galilean space is that ratio of curvature to torsion be constant (Öğrenmiş et al., 2007). Indeed, a helix is a special case of the general helix. If both curvature and torsion are non-zero constants, it is called a helix (Öğrenmiş et al., 2007).

In this paper, firstly, basic information about pseudo - Galilaen space was given. Frenet equations of curve which is parametrized with arc parameter were introduced. Later, the definition of helix was given in pseudo - Galilean space and equation which gives position vector of a helix in pseudo - Galilean space was obtained. Besides, necessary and sufficient conditions of this helix which lies on a pseudo - Galilean sphere were given with a theorem and results. We hope these results will be helpful to mathematicians.

BASIC NOTIONS AND PROPERTIES

Let r be a spatial curve given first by

$$r(t)=(x(t),y(t),z(t)), \tag{2.1}$$

where $x(t),y(t),z(t)\in C^3$ (the set of three-times continuously differentiable functions) and t run through a real interval (Divjak, 1998).

Definition 1

A curve r given by (2.1) is admissible if

$$\dot{x}(t)\neq 0. \tag{2.2}$$

Then the curve r can be given by

$$r(x)=(x,y(x),z(x)) \tag{2.3}$$

and we assume in addition that

$$y''^2(x)-z''^2(x)\neq 0. \tag{2.4}$$

From now on, we will denote the derivation by x by upper prime (Divjak, 1998).

Definition 2

For an admissible curve given by (2.1) the parameter of arc length is defined by

$$ds=\left|\dot{x}(t)dt\right|=\left|dx\right|. \tag{2.5}$$

For simplicity we assume $ds=dx$ and $s=x$ as the arc length of the curve r (Divjak, 1998).

The vector $\mathbf{t}(x)=r'(x)$ is called the tangential unit vector of an admissible curve r in a point $\mathbf{P}(x)$. Further, we define the so called osculating plane of r spanned by the vectors $r'(x)$ and $r''(x)$ in the same point. The absolute point of the osculating plane is

$$H(0:0:y''(x):z''(x)). \tag{2.6}$$

We have assumed in (2.4) that H is not lightlike. H is a point at infinity of a line which direction vector is $r''(x)$. Then the unit vector

$$\mathbf{n}(x)=\frac{r''(x)}{\sqrt{\left|y''^2(x)-z''^2(x)\right|}} \tag{2.7}$$

is called the principal normal vector of the curve r in the point P.

Now the vector

$$\mathbf{b}(x)=\frac{(0,\varepsilon z''(x),\varepsilon y''(x))}{\sqrt{\left|y''^2(x)-z''^2(x)\right|}} \tag{2.8}$$

is orthogonal in pseudo-Galilean sense to the osculating plane and we call it the binormal vector of the given curve in the point $\mathbf{P}$. Here $\varepsilon=+1$ or -1 is chosen by the criterion $\det(\mathbf{t},\mathbf{n},\mathbf{b})=1$. That means

$$\left|y''^2(x)-z''^2(x)\right|=\varepsilon(y''^2(x)-z''^2(x)). \tag{2.9}$$

By the above construction the following can be summarized (Divjak, 1998).

Definition 3

In each point of an admissible curve in G_3^1, the associated orthonormal (in pseudo-Galilean sense) trihedron $\{\mathbf{t}(x), \mathbf{n}(x), \mathbf{b}(x)\}$ can be defined. This trihedron is called pseudo-Galilean Frenet trihedron.

If a curve is parametrized by the arc length, that is given by (2.3), then the tangent vector is non-isotropic and has the form of

$$\mathbf{t}(x) = r'(x) = (1, y'(x), z'(x)). \qquad (2.10)$$

Now we have

$$\mathbf{t}'(x) = r''(x) = (0, y''(x), z''(x)). \qquad (2.11)$$

According to the clasical analogy we write (2.7) in the form

$$r''(x) = \kappa(x)\mathbf{n}(x), \qquad (2.12)$$

and so the curvature of an admissible curve r can be defined as follows

$$\kappa(x) = \sqrt{\left| y''^2(x) - z''^2(x) \right|}. \qquad (2.13)$$

Remarks

For the pseudo-Galilean Frenet trihedron of an admissible curve r given by (2.3) the following derivative Frenet formulas are true (Divjak, 1998).

$$\mathbf{t}'(x) = \kappa(x)\mathbf{n}(x)$$

$$\mathbf{n}'(x) = \tau(x)\mathbf{b}(x) \qquad (2.14)$$

$$\mathbf{b}'(x) = \tau(x)\mathbf{n}(x)$$

Where $\mathbf{t}(x)$ is a spacelike, $\mathbf{n}(x)$ is a spacelike and $\mathbf{b}(x)$ is a timelike vektor, $\kappa(x)$ is the pseudo-Galilean curvature given by (2.13) and $\tau(x)$ is the pseudo-Galilean torsion of r defined by

$$\tau(x) = \frac{y''(x)z'''(x) - y'''(x)z''(x)}{\kappa^2(x)}. \qquad (2.15)$$

The formula (2.15) can be written as

$$\tau(x) = \frac{\det(r'(x), r''(x), r'''(x))}{\kappa^2(x)}. \qquad (2.16)$$

The unit Pseudo - Galilean sphere is defined by

$$S_{\mp}^2 = \{u \in G_3^1 : g(u,u) = \mp 1\}.$$

More about the PSEUDO - Galilean geometry can be found in (Divjak, 1998) and (Divjak, 1997).

POSITION VECTOR OF A HELIX IN THE PSEUDO-GALILEAN SPACE G_3^1

In this section, we obtained position vector of a helix in the Pseudo - Galilean space G_3^1.

If $\alpha(s)$ is a helix, then we can write its position vector as follows:

$$\alpha(s) = \lambda(s)\mathbf{t}(s) + \mu(s)\mathbf{n}(s) + \gamma(s)\mathbf{b}(s) \qquad (3.1)$$

For some differentiable functions λ, μ and γ of $s \in I \subset R.$ These functions are called component functions of the position vector.

Differentiating (3.1) with respect to s and by using (2.14), we find

$$\begin{aligned} &\lambda'(s) = 1 \\ &\lambda(s)\kappa(s) + \mu'(s) + \gamma(s)\tau(s) = 0 \\ &\mu(s)\tau(s) + \gamma'(s) = 0. \end{aligned} \qquad (3.2)$$

From (3.2), we get the following differential equation:

$$\mu''(s) - \mu(s)\tau^2 + \kappa = 0. \qquad (3.3)$$

The solution of Equation (3.3) is

$$\mu(s) = (c_1 + c_2 s)e^{\tau s} + \frac{\kappa}{\tau^2}, \qquad (3.4)$$

where $c_1, c_2 \in R.$ From $\lambda'(s) = 1,$ we find the solution of this equation as follows:

$$\lambda(s) = s + c_3, \qquad (3.5)$$

where $c_3 \in R.$ By using $\gamma'(s) = -\mu(s)\tau$ and Equation (3.4), we find the solution of this equation as follows:

$$\gamma(s) = \kappa s - \tau c_1 s - (c_2 s - \frac{1}{\tau})e^{\tau s}. \quad (3.6)$$

Thus we find the position vector as

$$\alpha(s) = (s + c_3)\mathbf{t}(s) + [(c_1 + c_2 s)e^{\tau s} + \frac{\kappa}{\tau^2}]\mathbf{n}(s) + [\kappa s - \tau c_1 s - (c_2 s - \frac{1}{\tau})e^{\tau s}]\mathbf{b}(s). \quad (3.7)$$

Corollary 1

Let $\alpha = \alpha(s)$ be a unit speed helix in the Galilean space G_3 with $\kappa \neq 0,\ \tau \neq 0$ for each $s \in I \subset R$. Then the position vector of the curve $\alpha = \alpha(s)$ is given by Equation (3.7).

HELICES ON THE PSEUDO-GALILEAN SPACE G_3^1

In this section, we give some characterizations for the helix whose image lies on a pseudo Galilean sphere $S_{\mp}^2$.

Remark 1

In Öğrenmiş and Ergüt (2009), let r be an admissible curve in G_3^1. If its pseudo-Galilean trihedron $\{\mathbf{t}(x), \mathbf{n}(x), \mathbf{b}(x)\}$, then the center of osculating sphere of r at the point $r(x)$ is given by

$$a(x) = r(x) + m_2(x)\mathbf{n}(x) + m_3(x)\mathbf{b}(x)$$

where

$$m_2(x) = \frac{1}{\kappa(x)}, \quad m_3(x) = -\frac{m_2'(x)}{\tau(x)}.$$

Remark 2

In Öğrenmiş and Ergüt (2009), let $S_{\mp}^2$ be a pseudo-Galilean sphere centered origin and a curve r on $S_{\mp}^2$ be given. In this case, if admissible curve's parameter is x, then the following equations is valid

$$g(\mathbf{n}(x), r(x)) = -m_2(x)$$

$$g(\mathbf{b}(x), r(x)) = m_3(x).$$

Theorem 1

Let $\alpha(s)$ be a unit speed helix in the Pseudo - Galilean space G_3^1 with $\kappa \neq 0,\ \tau \neq 0.$ The image of the curve lies on a pseudo - Galilean sphere $S_{\mp}^2$ if and if only for each $s \in I \subset R$ the curvatures satisfy the following equality:

$$\begin{aligned} & s + c_3 = 0 \\ & (c_1 + c_2 s)e^{\tau s} + \frac{\kappa}{\tau^2} = -\frac{1}{\kappa} \\ & \kappa s - \tau c_1 s - (c_2 s - \frac{1}{\tau})e^{\tau s} = 0 \end{aligned} \quad (4.1)$$

where $c_1, c_2, c_3 \in R.$

Proof

By assumption we have

$$g(\alpha, \alpha) = r^2$$

for every $s \in I \subset R$ and r is radius of the Galilean sphere. Differentiation in s gives

$$g(\mathbf{t}, \alpha) = 0. \quad (4.2)$$

By a new differentiation, we find that

$$g(\mathbf{n}, \alpha) = -\frac{1}{\kappa}. \quad (4.3)$$

Then one more differentiation in s gives

$$g(\mathbf{b}, \alpha) = 0. \quad (4.4)$$

By using Equations (4.2 - 4.4) in Equation (3.7), we find Equation (4.1). Conversely, we assume that Equation (4.1) holds for each $s \in I \subset R,$ then from (3.7) we find the position vector of curve $\alpha = -\frac{1}{\kappa}\mathbf{n}$ which satisfy the equation $g(\alpha, \alpha) = r^2$ which means that the curve lies on a pseudo - Galilean sphere $S_{\mp}^2$.

Corollary 1

Let $\alpha(s)$ be a unit speed helix in the pseudo - Galilean space G_3^1 with $\kappa \neq 0,\ \tau \neq 0$ for each $s \in I \subset R.$ The image of the curve lies on a pseudo - Galilean sphere $S_{\mp}^2$ of radius $r \in R^+$ and with the center at origin, if and only if α is a normal curve, that is, curves with position vector always lying in its normal plane.

Corollary 2

Let $\alpha(s)$ be a unit speed helix in the pseudo - Galilean space G_3^1 with $\kappa \neq 0,\ \tau \neq 0$ for each $s \in I \subset R.$ If α is a pseudo - Galilean spherical curves then the radius of $S_{\mp}^2$ is $r = -\frac{1}{\kappa}.$

Corollary 3

Let $\alpha(s)$ be a unit speed helix in the pseudo - Galilean space G_3^1 with $\kappa \neq 0,\ \tau \neq 0$ for each $s \in I \subset R.$ The image of the curve lies on a pseudo - Galilean sphere $S_{\mp}^2$ of radius $r \in R^+$ and with the center at origin, if and only if its position vector is constant.

CONCLUSİON

Consider a curve in a space where the curve is sufficiently smooth so that the Frenet-Serret frame adapted to it is defined the curvature κ and the torsion τ. Then provide a complete characterization of the curve. If the curvature and torsion are non-zero constant, then the curve is called a helix.

Helix is one of the most fascinating curves in science and nature. Scientists have long held a fascination, sometimes bordering on mystical obsession, for helical structures in nature.

In this paper, we obtained the position vector of a helix in pseudo - Galilean space G_3^1. Also, by using the position vector of the curve, we gave some characterizations for the helix whose image lies on a pseudo - Galilean sphere $S_{\mp}^2$. Furthermore, we showed that such curves are normal curves G_3^1.

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