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Full Length Research Paper

An efficient two step Laplace decomposition algorithm for singular Volterra integral equations

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The aim of this article is to introduce a new reliable algorithm, namely two-step Laplace decomposition algorithm (TSLDA). This new algorithm provides us with a convenient way to find exact solution with less computation as compared with standard Laplace decomposition algorithm (LDA). The proposed algorithm is use to solve Abel's second kind integral equations efficiently.

Key words: Two step Laplace decomposition algorithm, Laplace decomposition algorithm, Abel's second kind integral equations.

INTRODUCTION

Singular integral equation that has enormous applications in applied problems including fluid mechanics, biomechanics, electromagnetic theory and chemistry applications such as heat conduction, crystal growth and electrochemistry. An integral equation is called a singular integral equation if one or both limits of integration become infinite, or if the kernel of the equation becomes infinite at one or more points in the interval of integration. Norwegian mathematician Niels Abel who invented them in 1823, in his research of mathematical physics (Jerri, 1999; Rahman, 2007). There are many numerical and analytical schemes such as finite element method, finite difference method and perturbation methods can be used to obtain an approximate solution for the model problem. However, there exist many difficulties such as a mesh refinement, a stability condition and selection of small and large parameters, etc. To avoid these difficulties, decomposition method was introduced (Adomian, 1994; Jafari and Gejji, 2006a, b, c) which is a very powerful method for solving linear and non-linear problems in many fields. Recently, a modification of Laplace decomposition algorithm (LDA) was proposed (Majid et al., 2011; Hussain and Khan, 2010). The modified decomposition algorithm needs only a slight variation from the standard LDA and has been shown to be computationally efficient. The modified LDA (MLDA) was established and based on the assumption that the function f can be divided into two parts and the success of the modified algorithm depends on the proper choice of the parts f_1 and f_2. The TSLDA overcomes this difficulty and explains how we can choose f_1 and f_2 properly without having noise terms. The LDA is used by different scientists for solving different equations arises in different physical phenomena (Majid et al., 2010, 2011; Hosseinzadeh et al., 2010).

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TWO STEP LAPLACE DECOMPOSITION ALGORITHM FOR ABEL'S INTEGRAL EQUATIONS

The weakly-singular Volterra-type integral equations in terms of Abel's integral equation can be written as (Jerri, 1999; Rahman 2007):

$$\begin{aligned} u(x) &= f(x) + \int_0^x K(x,t)u(t)\,dt, \\ &= f(x) + \int_0^x \frac{u(t)}{\sqrt{x-t}}\,dt, \end{aligned} \qquad (1)$$

where $f(x)$ is non-homogeneous term and $K(x,t)$ kernel of the equation that approaches infinity as $x \to t$ which is a singular behaviour of the kernel. The solution of this integral is attributed by the convolution theorem of Laplace transform. Taking the Laplace transform of both sides of the equation yields:

$$£[u(x)] = £[f(x)] + £\left[\int_0^x \frac{u(t)}{\sqrt{x-t}} dt\right]. \quad (2)$$

Using convolution theorem of Laplace transform, we have:

$$£[u(x)] = £[f(x)] + £\left[\frac{1}{\sqrt{x}}\right]£[u(x)]. \quad (3)$$

Operating inverse Laplace transform on both sides of Equation 3, we have:

$$u(x) = f(x) + £^{-1}\left[\frac{\sqrt{\pi}}{\sqrt{s}} £[u(x)]\right]. \quad (4)$$

The Laplace decomposition algorithm assumes the solution u can be expanded into infinite series as:

$$u = \sum_{m=0}^{\infty} u_m. \quad (5)$$

By substituting Equation 5 in Equation 4, the solution can be written as:

$$\sum_{m=0}^{\infty} u_m(x) = f(x) + £^{-1}\left[\frac{\sqrt{\pi}}{\sqrt{s}} £\left[\sum_{m=0}^{\infty} u_m\right]\right]. \quad (6)$$

In general, the recursive relation is given by:

$$u_0(x) = f(x),$$
$$u_{m+1}(x) = £^{-1}\left[\frac{\sqrt{\pi}}{\sqrt{s}} £[u_m]\right], \quad m \geq 0. \quad (7)$$

where $f(x)$ represents the source term. Now we illustrate TSLDA after applying the inverse operator, we have $f(x)$ which can be denoted by another function Ψ as follows:

$$\Psi = f(x). \quad (8)$$

By using TSLDA we set:

$$\Psi = f_0(x) + f_1(x) + f_2(x) + ... + f_m(x), \quad (9)$$

where $f_0, f_1, f_2, ..., f_m$, are the terms arising from applying inverse Laplace transform on the source term $f(x)$. We define:

$$u_0 = f_k(x) + ... + f_{k+s}(x), \quad (10)$$

where $k = 0,1...,m,\quad s = 0,1,...m-k$. Then we verify that u_0 satisfies the original Equation 1 and by substituting, once the exact solution is obtained we finish. Otherwise we go to step two. In second step we set $u_0 = f(x)$ and continue with the standard LDA:

$$u_{m+1}(x) = £^{-1}\left[\frac{\sqrt{\pi}}{\sqrt{s}} £[u_m]\right], \quad m \geq 0. \quad (11)$$

By comparison with LDA and TSLDA, it is clear that TSLDA may provide the solution by using one iteration only and does not have the difficulties arising in the modified method. Further, the number of terms in Ψ namely m, is small in many practical problems. TSLDA is less time consuming. Our purpose in this paper is combining the LDA and TSLDA. We divide $f(x)$ into its components and check the required conditions for proper choice of $u_0(x)$. After applying inverse transform, by TSLDA criterion, we can find the exact solution of our equation after one iteration.

APPLICATIONS

Here, some examples are given in order to demonstrate the effectiveness of TSLDA. For all examples, the exact solutions are obtained by TSLDA.

Example

Consider second kind Volterra equation in terms of Abel's integral equation is given by (Rahman, 2007):

$$u(x) = \sqrt{x} + \frac{\pi x}{2} - \int_0^x \frac{u(t)}{\sqrt{x-t}} dt. \quad (12)$$

Applying Laplace transform algorithm and using convolution theorem of Laplace transform we have:

$$£[u(x)] = \frac{\Gamma(3/2)}{s^{\frac{3}{2}}} + \frac{\pi}{s^2} - \frac{\Gamma(1/2)}{s^{\frac{1}{2}}} £[u(x)]. \quad (13)$$

Applying inverse Laplace transform to Equation 13 we have:

$$u(x) = \sqrt{x} + \frac{\pi x}{2} - £^{-1}\left[\frac{\Gamma(1/2)}{s^{\frac{1}{2}}} £\left[u(x)\right]\right]. \quad (14)$$

The Laplace decomposition method (LDA) assumes a series solution of the function $u(x)$ given by:

$$u = \sum_{n=0}^{\infty} u_n(x). \quad (15)$$

Using Equation 15 into 14 yields:

$$\sum_{n=0}^{\infty} u_n(x) = \sqrt{x} + \frac{\pi x}{2} - £^{-1}\left[\frac{\Gamma(1/2)}{s^{\frac{1}{2}}} £\left[\sum_{n=0}^{\infty} u_n(x)\right]\right]. \quad (16)$$

From Equation our required recursive relation is given as follows:

$$u_0(x) = \sqrt{x} + \frac{\pi x}{2}, \quad (17)$$

$$u_{n+1}(x) = -£^{-1}\left[\frac{\Gamma(1/2)}{s^{\frac{1}{2}}} £\left[u_n(x)\right]\right], \qquad n \geq 0. \quad (18)$$

The first few components of $u_n(x)$ by using recursive relation of Equation 18 follow immediately as:

$$u_1(x) = -\frac{\pi x}{2} - \frac{2\pi}{3} x^{3/2} \quad (19)$$

As it can be seen from the original Equation 12, non-homogeneous Abel's second kind integral equation and exact solution is in the zeroth component u_0, there is the phenomena of the noise term (Khan and Gondal, 2010a, b, c; Khan and Gondal, 2011a, b; Khan and Hussain, 2011). By examining u_0 and u_1, we can easily observe the appearance of the noise term $\pi x/2$ in u_0. Therefore, by cancelling the noise term in u_0, the remaining non-cancelled terms provide the exact solution. But there are many problems in which the zeroth component does not contain the exact solution. Thus, the approximation by the standard LDA is compared with the exact solution.

The two-step Laplace decomposition algorithm

By using TSLDA we decompose the function $f(x)$ as follows:

$$f(x) = f_0(x) + f_1(x), \quad (20)$$

where $f_0(x) = \sqrt{x}, \quad f_1(x) = \frac{\pi x}{2},$ (21)

where f_1 does not satisfy Equation 12. By choosing $u_0 = f_0$ and by verifying that u_0 satisfy Equation 12, the exact solution will be obtained immediately and we have:

$$u_0(x) = \sqrt{x}, \quad (22)$$

$$u_1(x) = \frac{\pi x}{2} - £^{-1}\left[\frac{\Gamma(1/2)}{s^{\frac{1}{2}}} £\left[u_0(x)\right]\right] = 0, \quad (23)$$

$$u_{n+1}(x) = 0, \qquad n \geq 1. \quad (24)$$

Therefore, solution by TSLDA is:

$$u = \sum_{n=0}^{\infty} u_n(x) = \sqrt{x}. \quad (25)$$

Example

Considering the second kind Volterra singular integral equation given by (Rahman, 2007):

$$u(x) = x + \frac{4x^{\frac{3}{2}}}{3} - \int_0^x \frac{u(t)}{\sqrt{x-t}} dt. \quad (26)$$

Applying Laplace transform and convolution theorem yields:

$$£\left[u(x)\right] = \frac{1}{s^2} + \frac{4\Gamma(5/2)}{3s^{\frac{5}{2}}} - \frac{\Gamma(1/2)}{s^{\frac{1}{2}}} £\left[u(x)\right]. \quad (27)$$

Applying inverse Laplace transform to Equation 27, we have:

$$u(x) = x + \frac{4x^{\frac{3}{2}}}{3} - £^{-1}\left[\frac{\Gamma(1/2)}{s^{\frac{1}{2}}} £\left[u(x)\right]\right]. \quad (28)$$

The Laplace decomposition method assumes that the solution function $u(x)$ can be decomposed as an infinite series as follows:

$$u = \sum_{n=0}^{\infty} u_n(x). \quad (29)$$

Using Equation 29 into Equation 28 yields:

$$\sum_{n=0}^{\infty} u_n(x) = x + \frac{4x^{\frac{3}{2}}}{3} - \pounds^{-1}\left[\frac{\Gamma(1/2)}{s^{\frac{1}{2}}}\pounds\left[\sum_{n=0}^{\infty} u_n(x)\right]\right]. \quad (30)$$

From Equation 30, our required recursive relation is given as follows:

$$u_0(x) = x + \frac{4x^{\frac{3}{2}}}{3}, \quad (31)$$

$$u_{n+1}(x) = -\pounds^{-1}\left[\frac{\Gamma(1/2)}{s^{\frac{1}{2}}}\pounds\left[u_n(x)\right]\right], \qquad n \geq 0. \quad (32)$$

The first few components of $u_n(x)$ by using recursive relation of Equation 32 follow immediately as:

$$u_1(x) = -\frac{4x^{\frac{3}{2}}}{3} - \frac{\pi}{2}x^2, \quad (33)$$

Similar, we examining u_0 and u_1, can easily observe the appearance of the noise term $\frac{4x^{\frac{3}{2}}}{3}$ in u_0. Therefore, by cancelling the noise term in u_0, the remaining non-cancelled terms provide the exact solution. But, there are many problems in which the zeroth component does not contain the exact solution. Thus, the approximation by the standard LDA is compared with the exact solution.

The two-step Laplace decomposition algorithm

By using TSLDA we decompose the function $f(x)$ as follows:

$$f(x) = f_0(x) + f_1(x), \quad (34)$$

where

$$f_0(x) = x, \quad f_1(x) = \frac{4x^{\frac{3}{2}}}{3}. \quad (35)$$

It is obvious that f_1 does not satisfy Equation 26. By choosing $u_0 = f_0$ and by verifying that u_0 satisfy Equation 26, the exact solution will be obtained immediately and we have:

$$\begin{aligned} u_0(x) &= x, \\ u_1(x) &= \frac{4x^{\frac{3}{2}}}{3} - \pounds^{-1}\left[\frac{\Gamma(1/2)}{s^{\frac{1}{2}}}\pounds\left[u_0(x)\right]\right] = 0, \\ &\vdots \\ u_{n+1}(x) &= 0, \qquad n \geq 1. \end{aligned} \quad (36)$$

Therefore, solution by TSLDA is given as:

$$u = \sum_{n=0}^{\infty} u_n(x) = x. \quad (37)$$

Conclusion

In this work, we proposed new modification in standard Laplace decomposition algorithm. In the two illustrated examples, we showed that TSLDA consists of three steps: the first step is applying Laplace transform on our equation and then inverse transform, the second step is verifying that the zeroth component of the series solution includes the exact solution. If yes we finish, otherwise we should go to third step where we continue with the standard LDM. The obtained results in examples indicate that TSLDA is feasible, effective and do not have the "noise terms". The TSLDA overcomes the difficulties arising in the modified decomposition method. The power of TSLDA depends on the proper choice of u_0 and u_1 and the occurrence of the exact solution in the zeroth term. If the exact solution exists in the zeroth component, TSLDM requires less calculation in comparison with LDA. This article is the first step to apply transforms methods to solve singular integral equations, and will be an interesting area of research in a near future.

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